# Lecture 2  The Krein-Rutman Theorem and Principal eigenvalue

**Def** (Cones) Let $K$ be a subset of a Banach space $X$.

1. $K$ is a cone if $K$ is closed, convex, $\mu K \subseteq K$ $\forall \mu > 0$, and $K \cap (-K) = \{0\}$.
2. $K$ is a solid cone if it has nonempty interior.
3. $K$ is a total cone if $X = K - K$. ($K$ is solid $\Rightarrow$ $K$ is total.)

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**Def** (ordering) We say that $X$ is an ordered Banach space with order induced by the cone $K$ and write

$$x \leq \bar{x} \qquad (\text{resp. } x < \bar{x} \qquad x \ll \bar{x})$$

if $\bar{x} - x \in K$ (resp. $\bar{x} - x \in K \setminus \{0\}$ $\quad \bar{x} - x \in \operatorname{Int} K$ )

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**Examples** $X = \mathbb{R}^k$, $K = [0,\infty)^k$ is a solid and total cone.

$X = C(\bar{\Omega})$, $K = \{\phi \in C(\bar{\Omega}) : \phi \geq 0 \text{ in } \bar{\Omega}\}$ is solid and total cone.

$X = L^p(\Omega)$, $K = \{\phi \in L^p(\Omega) : \phi \geq 0 \text{ in } \Omega\}$ is total but not solid.

<u>Def</u> Let $X$ be a real Banach space and $T: X \to X$ be a linear operator.

1. Denote the <u>spectral radius</u> by
$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}, \quad \text{where } \|T^n\| = \sup_{\|x\|\le 1} \|T^n x\|.$$

2. Define the complexification of $X$ by $\tilde{X} = \{x+iy : x, y \in X\}$ with
$$\|x+iy\| = \sup_{0\le\theta\le 2\pi} \|(\cos\theta)x + (\sin\theta)y\|;$$
and $\tilde{T}: \tilde{X} \to \tilde{X}$ to be the usual extension of $T$ to $\tilde{X}$.

<u>Rmk 1</u> (exercise) $\|\tilde{T}^n\| = \|T^n\|$ $\forall n$ so that $r(T) = r(\tilde{T})$.

<u>Rmk 2</u> $r(T) = \sup\{|z| : z \in \sigma(\tilde{T})\}$, where $\sigma(\tilde{T})$ is the complement of resolvent set of $\tilde{T}$.

**Def** Let $X$ be an ordered Banach space with other cone $K \subseteq X$.

1. Let $C$ be a subset of $X$. A map $f: C \to C$ is *monotone* if

$$x \leq y \implies f(x) \leq f(y)$$

It is *strongly monotone* if $x < y \implies f(x) \ll f(y)$.

2. A map $f: K \to K$ is *homogeneous* if

$$f(tx) = t f(x) \quad \text{for } t > 0 \text{ and } x \in K.$$

and we define $\|f\|_K = \sup_{\substack{x \in K \\ \|x\| \leq 1}} \|f(x)\|$.

3. The *Bonsall cone spectral radius* is $\tilde{r}_K(f) = \limsup_{n \to \infty} \|f^n\|_K^{1/n}$

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In particular $T: X \to X$ is *monotone* (resp. *str. monotone*) if

$$T(K) \subseteq K \qquad (\text{resp. } T(K \setminus \{0\}) \subseteq \operatorname{Int} K).$$

Note: "homogeneous" is short hand for "positively homogeneous of degree one".

## The classical Krein-Rutman Theorem (weak form)

Let $\mathbb{X}$ be an ordered Banach space with a total cone $K$, and let $T: \mathbb{X} \to \mathbb{X}$ be a compact linear operator such that $T(K) \subseteq K$.

If $r(T) > 0$, then there exist $x_0 \in K \setminus \{0\}$ and $f_0 \in K^* \setminus \{0\}$ such that

$$T x_0 = r(T) x_0 \quad \text{and} \quad T^* f_0 = r(T) f_0 .$$

where $T^*$ is the adjoint operator, and $K^* = \{ f \in \mathbb{X}^* : f(x) \geq 0 \ \forall x \in K \}$.

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## The classical Krein-Rutman Theorem (Strong form)

Let $\mathbb{X}$ be an ordered Banach space with a solid cone $K$, and let $T: \mathbb{X} \to \mathbb{X}$ be a compact linear operator such that $T(K \setminus \{0\}) \subseteq \operatorname{Int} K$.

Then
1. $r(T) > 0$ and it is a geometrically simple eigenvalue of $T$.
2. There exists $\varepsilon > 0$ such that $|\lambda| < r(T) - \varepsilon$ for all eigenvalue $\lambda \neq r(T)$.
3. If $T x' = \lambda x'$ for some $\lambda \geq 0$ and $x' \in K \setminus \{0\}$, then $\lambda = r(T)$ and $x' \in \operatorname{span}\{x_0\}$.

Detailed proof in Appendix B

## § Existence of principal eigenvalue for periodic-parabolic problems.

**Theorem** The problem
$$\begin{cases} \varphi_t + \mathcal{L}\varphi = \lambda\varphi & \text{in } \Omega_T \\ B\varphi = 0 & \text{on } S\Omega_T \\ \varphi(x,0) = \varphi(x,T) \end{cases}$$

$a^{ij}, b^{j}, c$ can be time dependent

has a principal eigenvalue $\lambda_1$, in the sense that

1. $\lambda_1 \in \mathbb{R}$ is simple with a positive eigenfcn $\phi_1$ in $\overline{\Omega_T}$.
2. $\lambda_1 < \inf_{\lambda \neq \lambda_1} \mathrm{Re}\,\lambda$ for all other eigenvalues $\lambda$
3. If $(\lambda, \phi)$ is an eigenpair and $\phi \geq 0$ in $\overline{\Omega_T}$, then $\lambda = \lambda_1$ and $\phi \in \mathrm{span}\{\phi_1\}$.

Pf. Let $X = C(\overline{\Omega})$, $K = C(\overline{\Omega}; \mathbb{R}_+)$.

For each $t > 0$, consider the evolution operator $\Phi_t : X \to X$ such that $u(\cdot, t) = \Phi_t(u_0)$ satisfies
$$\begin{cases} u_t + \mathcal{L}u = 0 & \text{in } \Omega_T \\ Bu = 0 & \text{on } S\Omega_T \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

then $\Phi_t$ is compact, linear and strongly positive w.r.t. $K$.

$\Longrightarrow$ $r(\Phi_T) > 0$ and $\lambda_1 = -\frac{1}{T}\log r(\Phi_T)$.

## § Existence of principal eigenvalue for elliptic problems.

**Theorem** The problem $(P_e)$ $\begin{cases} \mathcal{L}\varphi = \mu\varphi & \text{in } \Omega \\ B\varphi = 0 & \text{on } \partial\Omega \end{cases}$ $a^{ij}, b^{j}, c$ are indep. of time

has a principal eigenvalue $\mu_1$, in the sense that

1. $\mu_1 \in \mathbb{R}$ is simple with a positive eigenfcn $\phi_1$ in $\overline{\Omega}$.
2. $\mu_1 < \inf_{\lambda \neq \lambda_1} \operatorname{Re}\mu$ for all other eigenvalues $\lambda$
3. If $(\mu, \phi)$ is an eigenpair and $\phi \geq 0$ in $\overline{\Omega}$, then $\mu = \mu_1$ and $\phi \in \operatorname{span}\{\phi_1\}$.

**Pf.** Suppose $(P_e)$ has a real eigenvalue $\mu_1$ with eigenfcn $\phi_1 \geq, \not\equiv 0$ in $\overline{\Omega}$,

$$\Rightarrow \quad \Phi_t[\phi_1] = e^{-\mu_1 t}\phi_1 \qquad \forall t > 0.$$

previous theorem $\Rightarrow$ $$r(\Phi_t) = e^{-\mu_1 t} \qquad \forall t > 0$$

and $\mu_1$ has properties 1–3. It remains to show existence.

Let $\Phi_1 = \Phi_t \big|_{t=1}$. By KRT, $r(\Phi_1) > 0$ and has p.e.f. $\phi_1 > 0$ in $\overline{\Omega}$.

Define $\mu_1 = -\log r(\Phi_1)$.

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Define $u(\cdot, t) = \Phi_t[\phi_1]$. It remains to show that

(*) $\quad u(x,t) = e^{-\mu_1 t}\phi_1(x) \quad$ for $x \in \Omega$, $t \geq 0$.

To show (*), observe that $u(x,n) = e^{-\mu_1 n}\phi_1(x) \quad \forall n \in \mathbb{N}$.

Next, consider $\Phi_{1/2}$.

Let $\Phi_1 = \Phi_t \big|_{t=1}$. By KRT, $r(\Phi_1) > 0$ and has p.e.f. $\phi_1 > 0$ in $\overline{\Omega}$.

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Define $u(\cdot, t) = \Phi_t[\phi_1]$. It remains to show that

$$(*) \qquad u(x,t) = e^{-\mu_1 t}\phi_1(x) \qquad \text{for } x \in \Omega,\ t > 0.$$

To show $(*)$, observe that $u(x,n) = e^{-\mu_1 n}\phi_1(x) \quad \forall n \in \mathbb{N}$.

Next, consider $\Phi_{1/2}$. By KRT, $\Phi_{\frac{1}{2}}(\hat{\phi}) = r(\Phi_{\frac{1}{2}})\hat{\phi}$ $\exists\, \hat{\phi} > 0$ in $\overline{\Omega}$.

$$\Rightarrow \quad \Phi_1(\hat{\phi}) = (\Phi_{\frac{1}{2}}) \circ (\Phi_{\frac{1}{2}})(\hat{\phi}) = r(\Phi_{\frac{1}{2}})^2 \hat{\phi} \Rightarrow \boxed{\hat{\phi} \text{ is an eigenvector of } \Phi_1}$$

Let $\Phi_1 = \Phi_t\big|_{t=1}$. By KRT, $r(\Phi_1)>0$ and has p.e.f. $\phi_1>0$ in $\overline{\Omega}$.

Define $\mu_1 = -\log r(\Phi_1)$.

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Define $u(\cdot,t) = \Phi_t[\phi_1]$. It remains to show that

$$(*) \qquad u(x,t) = e^{-\mu_1 t}\phi_1(x) \quad \text{for } x\in\Omega,\ t>0.$$

To show $(*)$, observe that $u(x,n) = e^{-\mu_1 n}\phi_1(x)$ $\forall n\in\mathbb{N}$.

Next, consider $\Phi_{1/2}$. By KRT, $\Phi_{\frac12}(\hat\phi) = r(\Phi_{\frac12})\hat\phi$ $\exists\,\hat\phi>0$ in $\overline{\Omega}$.

$$\Rightarrow \quad \Phi_1(\hat\phi) = (\Phi_{\frac12})\circ(\Phi_{\frac12})(\hat\phi) = r(\Phi_{\frac12})^2\hat\phi \Rightarrow \boxed{\hat\phi \text{ is an eigenvector of } \Phi_1}$$

By uniqueness, this implies $r(\Phi_{\frac12})^2 = e^{-\mu_1}$ and $\hat\phi\in\operatorname{span}\{\phi_1\}$.

$$\Rightarrow \quad \Phi_{1/2}(\phi_1) = e^{-\mu_1/2}\phi_1 \quad \text{and} \quad u(x,\tfrac{n}{2}) = (e^{-\frac12\mu_1})^n\phi_1$$

$\Rightarrow$ $(*)$ holds for $t=\frac{n}{2}$ for $n\in\mathbb{N}$.

Let $\Phi_1 = \Phi_t \big|_{t=1}$. By KRT, $r(\Phi_1) > 0$ and has p.e.f. $\phi_1 > 0$ in $\overline{\Omega}$.

Define $\mu_1 = -\log r(\Phi_1)$.

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Define $u(\cdot, t) = \Phi_t[\phi_1]$. It remains to show that

$$(*) \qquad u(x,t) = e^{-\mu_1 t} \phi_1(x) \qquad \text{for } x \in \Omega,\ t > 0.$$

To show $(*)$, observe that $u(x,n) = e^{-\mu_1 n} \phi_1(x)$ $\forall n \in \mathbb{N}$.

Next, consider $\Phi_{1/2}$. By KRT, $\Phi_{\frac{1}{2}}(\hat{\phi}) = r(\Phi_{\frac{1}{2}})\hat{\phi}$ $\exists \hat{\phi} > 0$ in $\overline{\Omega}$.

$$\Rightarrow \quad \Phi_1(\hat{\phi}) = (\Phi_{\frac{1}{2}}) \circ (\Phi_{\frac{1}{2}})(\hat{\phi}) = r(\Phi_{\frac{1}{2}})^2 \hat{\phi} \Rightarrow \boxed{\hat{\phi} \text{ is an eigenvector of } \Phi_1}$$

By uniqueness, this implies $r(\Phi_{\frac{1}{2}})^2 = e^{-\mu_1}$ and $\hat{\phi} \in \operatorname{span}\{\phi_1\}$.

$$\Rightarrow \quad \Phi_{1/2}(\phi_1) = e^{-\mu_{1}/2} \phi_1 \quad \text{and} \quad u(x, \tfrac{n}{2}) = (e^{-\frac{1}{2}\mu_1})^n \phi_1$$

$\Rightarrow$ $(*)$ holds for $t = \frac{n}{2}$ for $n \in \mathbb{N}$.

By induction, $(*)$ holds for $t = \frac{n}{2^m}$ for any $n, m \in \mathbb{N}$

By continuity, $(*)$ holds for $t > 0$.

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§ Characterization of elliptic maximum principle (Berestycki-Nirenberg-Varadhan)

Def. Given $(\mathcal{L}, B)$, we say that (MP) holds if:

$\mathcal{L} u \leq 0$ in $\Omega$ and $Bu \leq 0$ on $\partial\Omega$ implies $u \leq 0$ in $\Omega$.

e.g. Let $\mathcal{L} = -\Delta - c$, $B = n \cdot \nabla + p^0$ with $p^0 \geq 0$, and $c \leq 0$,

Then MP holds provided $(c, p^0) \not\equiv (0,0)$.

Proposition MP holds if and only if there exists a strict positive supersol.

i.e. $\exists\, w > 0$ in $\overline{\Omega}$ such that $\mathcal{L} w \gneq 0$ in $\Omega$ and $Bw \geq 0$ on $\partial\Omega$

Corollary MP holds if and only if $\mu_1 > 0$.

Proof of Proposition By strong max prin, $w > 0$ in $\overline{\Omega}$.

Suppose to the contrary that $u > 0$ somewhere in $\overline{\Omega}$, then we can choose $k > 0$ s.t. $v = u - kw$ satisfies

$\sup_{\Omega} v = 0$ and $\mathcal{L} v \leq 0$ in $\Omega$, $Bv \leq 0$ on $\partial\Omega$.

By strong max. prin., $v \equiv 0 \Rightarrow u \equiv kw \Rightarrow w$ is not a strict supersol.

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<u>Application</u> Consider the semilinear parabolic equation

$$\begin{cases} u_t - d\Delta u = u f(x,u) & \text{in } \Omega \times (0,\infty) \\ n\cdot \nabla u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

<u>Proposition</u> Let $\mu_1$ be the p.e.v. of $\begin{cases} -d\Delta\phi = f(x,0)\phi + \mu_1\phi & \text{in } \Omega \\ n\cdot\nabla\phi = 0 & \text{on } \partial\Omega \end{cases}$

1. $\mu_1 < 0 \Rightarrow$ The trivial equilibrium is (globally) asymp. stable
2. $\mu_1 > 0 \Rightarrow$ the trivial equilibrium is unstable.

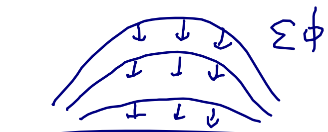
<u>Pf</u>. For $\varepsilon > 0$ small

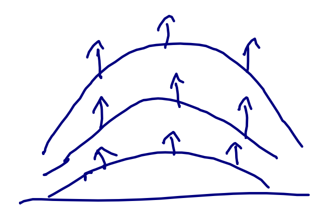
$$-d\Delta(\varepsilon\phi) - (\varepsilon\phi) f(x,\varepsilon\phi) = \varepsilon\left(-d\Delta\phi - \left(f(x,0) + o(1)\right)\phi\right)$$

$$= \varepsilon(\mu_1 + o(1))\,\phi_1$$

$\Rightarrow \forall \varepsilon > 0$ small, $\varepsilon\phi$ is strict supersol. if $\mu_1 > 0$

subsol. $\mu_1 < 0$.



In particular, if $\mu_1 < 0$, then there exists $\varepsilon_0 > 0$ such that every nonnegative, nontrivial solution $u(x,t)$ satisfies

$$\liminf_{t\to\infty} \left[\inf_{x\in\Omega} u(x,t)\right] \geq \varepsilon_0$$

(uniformly strongly persistent [Smith-Thieme, AMS Monograph 2011])

Devoir

Moreover, $\varepsilon_0$ can be chosen independently of initial data $u(x,0)$.

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Similarly, one define an equilibrium solution $\theta(x)$ to be linearly stable (resp. unstable) if the p.e.v. $\tilde{\mu}_1$ of

$$\begin{cases} -d\,\Delta\phi = f(x,\theta)\,\phi + \theta f_\theta(x,\theta)\,\phi + \tilde{\mu}\,\phi & \text{in } \Omega \\ n\cdot\nabla\phi = 0 & \text{on } \partial\Omega \end{cases}$$

is positive (resp. negative).

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Dependence on diffusion rate: Consider the diffusive logistic equation,

$$(DLE)\begin{cases} u_t - d\Delta u = u(m(x)-u) & \text{in } \Omega\times(0,\infty), \\ n\cdot\nabla u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x) & \text{in } \Omega. \end{cases}$$

such that $m(x)\not\equiv \text{const.}$

Prop 1. If $\int_\Omega m \geq 0$, then $\mu_1<0$ and the population persists $\forall d>0$.

2. If $\sup_\Omega m \leq 0$, then $\mu_1>0$ and $u(x,t)\longrightarrow 0$ as $t\to\infty$, $\forall d>0$.

3. If $\int_\Omega m \leq 0 < \sup_\Omega m$, then there exists $\hat{d}>0$ such that

the population persists if $d\in(0,\hat{d})$

the population goes to extinction if $d\in[\hat{d},\infty)$.

Pf Let $\mu_1$ be p.e.v. of $-d\Delta\phi = m(x)\phi+\mu_1\phi$ in $\Omega$, $n\cdot\nabla\phi=0$ on $\partial\Omega$.

1. Divide eqn. of $\phi$ by $\phi$ and integrate by parts.

$$0 > -d\int_\Omega \frac{|\nabla\phi|^2}{\phi^2} - \int_\Omega m = -d\int_\Omega\frac{\Delta\phi}{\phi} - \int_\Omega m = \mu_1$$

m and $\phi$ are nonconstant.

$\int_\Omega m \geq 0$.

<u>Dependence on diffusion rate</u>: Consider the diffusive logistic equation,

$$(\mathrm{DLE})\begin{cases} u_t - d\Delta u = u(m(x)-u) & \text{in } \Omega\times(0,\infty), \\ n\cdot\nabla u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x) & \text{in } \Omega. \end{cases}$$

such that $\underline{m(x)\not\equiv \text{const.}}$

<u>Prop</u> 1. If $\int_\Omega m \geq 0$, then $\mu_1<0$ and the population persists $\forall d>0$.

2. If $\sup_\Omega m \leq 0$, then $\mu_1>0$ and $u(x,t)\to 0$ as $t\to\infty$

3. If $\int_\Omega m \leq 0 < \sup_\Omega m$, then there exists $\hat{d}>0$ such that

the population persists if $d\in(0,\hat{d})$

the population goes to extinction if $d\in[\hat{d},\infty)$.

<u>Pf</u> Let $\mu_1$ be p.e.v. of $-d\Delta\phi = m(x)\phi+\mu_1\phi$ in $\Omega$, $n\cdot\nabla\phi=0$ on $\partial\Omega$.

2. Integrate over $\Omega$

$$0< -\int_\Omega m\phi = d\int_\Omega \Delta\phi - \int_\Omega m\phi = \mu_1\int_\Omega\phi \Rightarrow \mu_1>0.$$

By previous theorem (Lecture 1), the population persists.

**Dependence on diffusion rate**: Consider the diffusive logistic equation,

$$(DLE)\begin{cases} u_t - d\Delta u = u(m(x)-u) & \text{in } \Omega\times(0,\infty), \\ n\cdot\nabla u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

such that $\underline{m(x) \not\equiv \text{const.}}$

**Prop** 1. If $\int_\Omega m \geq 0$, then $\mu_1 < 0$ and the population persists $\forall d > 0$.

2. If $\sup_\Omega m \leq 0$, then $\mu_1 > 0$ and $u(x,t) \to 0$ as $t \to 0$

3. If $\int_\Omega m \leq 0 < \sup_\Omega m$, then there exists $\hat{d} > 0$ such that

the population persists if $d \in (0, \hat{d})$

the population goes to extinction if $d \in [\hat{d}, \infty)$.

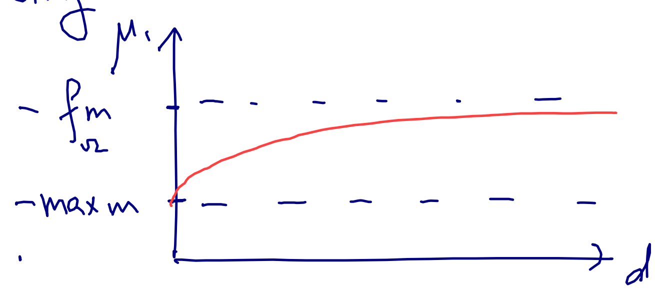
**Pf** Let $\mu_1$ be p.e.v. of $-d\Delta\phi = m(x)\phi + \mu_1\phi$ in $\Omega$, $n\cdot\nabla\phi = 0$ on $\partial\Omega$.

3. It suffices to show that

(i) $d \longmapsto \mu_1(d)$ is strictly increasing in $d$

(ii) $\lim_{d\to 0+} \mu_1(d) = -\max_\Omega m$

(iii) $\lim_{d\to\infty} \mu_1(d) = \frac{-1}{|\Omega|}\int_\Omega m\,dx$



To show monotonicity of $d \longmapsto \mu_1(d)$, differentiate $d\Delta\phi + m\phi + \mu_1\phi = 0$

$$(\spadesuit)\ \begin{cases} d\Delta\phi' + m\phi' + \mu_1\phi' + \Delta\phi = -\mu_1'\phi, & \text{in } \Omega \\ n\cdot\nabla\phi' = 0 & \text{on } \partial\Omega \end{cases}$$

where $'$ denotes derivative w.r.t. diffusion rate.

Observe that $\int_\Omega \phi\,\Delta\phi' = \int_\Omega \phi'\,\Delta\phi$ thanks to the Neumann b.c.

Multiply $(\spadesuit)$ by $\phi$ and

$$-\mu_1'\int_\Omega |\phi|^2 = \int_\Omega \phi\,(d\Delta\phi' + m\phi' + \mu_1\phi') + \int_\Omega \phi\Delta\phi$$

integration by parts using the Neumann b.c..

$$= \int_\Omega \phi'(d\Delta\phi + m\phi + \mu_1\phi) - \int_\Omega |\nabla\phi|^2$$

$$= -\int_\Omega |\nabla\phi|^2 < 0,$$

whe the strict inequality is thanks to $m, \phi$ being nonconstant in $\Omega$.

$\Longrightarrow \mu_1' > 0$ for all $d > 0$

$\Longrightarrow d \longmapsto \mu_1$ is strictly increasing. This proves (i).

(ii) and (iii) are proved in the lecture notes.

We briefly discuss <u>KRT for homogeneous maps</u>. with applications in biology

<u>Def</u> Let $X$ be an ordered Banach space with other cone $K \subseteq X$.

1. Let $C$ be a subset of $X$. A map $f: C \to C$ is <u>monotone</u> if

$$x \leq y \implies f(x) \leq f(y)$$

It is <u>strongly monotone</u> if $x < y \implies f(x) \ll f(y)$.

2. A map $f: K \to K$ is <u>homogeneous</u> if

$$f(tx) = t f(x) \quad \text{for } t > 0 \text{ and } x \in K.$$

and we define $\|f\|_K = \sup_{\substack{x \in K \\ \|x\| \leq 1}} \|f(x)\|$.

3. The <u>Bonsall cone spectral radius</u> is $\tilde{r}_K(f) = \limsup_{n \to \infty} \|f^n\|_K^{1/n}$

Theorem (KRT for maps, weak) Let $X$ be a Banach space ordered by a cone $K$. Let $f: K \to K$ be compact, continuous, homogeneous and monotone. If $r_K(f) > 0$, then there exists $\tilde{x} \in K$ with $\|\tilde{x}\| = 1$ such that

$$f(\tilde{x}) = r_K(f)\tilde{x}.$$

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Theorem (KRT for maps, strong) Let $X$ be a Banach space ordered by a solid cone $K$. Let $f: K \to K$ be compact, continuous, homogeneous and strongly monotone. Then $r_K(f) > 0$ and there is $\tilde{x} \in \operatorname{int} K$ such that

$$f(\tilde{x}) = r_K(f)\tilde{x}.$$

Furthermore, whenever $f(x') = r'x'$ for some $r' > 0$ and $x' \in K \setminus \{0\}$, we have $r' = r_K(f)$, $x' \in \operatorname{span}\{\tilde{x}\}$.

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In Appendix B, we follow the approach of Nussbaum to prove the KRT for maps via fixed point index arguments, and then derive the KRT for linear operators.

# Mating / Pair-formation Functions

In order to consider a nonlinear homogeneous operator on cones of $C(\bar{\Omega})$, we connect to the concept of pair formation due to [Hadeler, J.M.B. (2012)]

**Def** We say that $F:[0,\infty)^2 \to [0,\infty)$ is a mating function if

(i) $u_1 \le u_2,\ v_1 \le v_2 \Rightarrow F(u_1,v_1) \le F(u_2,v_2)$ (monotone)

(ii) $F(tu,tv) = tF(u,v)$ for $t,u,v > 0$ (homogeneous)

(iii) $F \in Lip([0,\infty)^2)$ and $F(1,1) > 0$.

(iv) $F(u,0) = F(0,v) = 0$ for $u,v \ge 0$.

**Examples** ① Harmonic mean: $F(u,v) = \rho \dfrac{uv}{\beta v + (1-\beta)u}$ for $\rho > 0$, $0 < \beta < 1$.

② Minimum function: $F(u,v) = \rho \min\{\beta u, (1-\beta)v\}$

The mating function is key in models of two-sex dynamics and sexually-transmited diseases.

Here we consider a toy model of a sexual reproduction population.

$$(S)\quad \begin{cases} u_t = d_1 \Delta u + p_1(x) F(u,v) - a_1 u - b_1 u^2 & \text{in } \Omega \times (0,\infty) \\ v_t = d_2 \Delta v + p_2(x) F(u,v) - a_2 v - a_2 v^2 & \text{in } \Omega \times (0,\infty) \\ n \cdot \nabla u = n \cdot \nabla v = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) \quad \text{and} \quad v(x,0) = v_0(x) & \text{in } \Omega \end{cases}$$

where $u$ and $v$ are the male/female population, with diffusion rate $d_i > 0$.
$p_i(x) > 0$ is the probability for an offspring to be male/female,
$a_i, b_i > 0$ are positive constants.

We will consider the persistence of population today. (Global dynamics later)

Let $\lambda_1$ be the p.e.v. of $\begin{cases} 0 = d_i \Delta \psi_i + p_i F(\psi_1,\psi_2) - a_i \psi_i + \lambda_1 \psi_i & \text{in } \Omega \\ n \cdot \nabla \psi_i = 0 & \text{on } \Omega \end{cases}$

Pf. Let $K = C(\bar{\Omega}; \mathbb{R}^2_+)$ and $\Phi_t : K \to K$ be the semiflow of

$\partial_t \psi_i = d_i \Delta \psi_i + p_i F(\psi_1,\psi_2) - a_i \psi_i$ in $\Omega$, $n \cdot \nabla \psi_i = 0$ on $\partial\Omega$, $i = 1,2$.

Then for each $t > 0$, $\Phi_t$ is compact, homogeneous and strongly monotone.

By KRT, $r(\Phi_t)$ is the p.e.v. of $\Phi_t$

Finally, note that $\lambda_1 = \frac{1}{t} \log r(\Phi_t)$ is indep. of $t > 0$. #

<u>Theorem</u> If $\lambda_1 \geq 0$, then every solution satisfies $(u,v) \to 0$ as $t \to \infty$.

If $\lambda_1 < 0$, then every nonnegative, nontrivial solution strongly persists, i.e.

$$\liminf_{t\to\infty} \min\left\{ \inf_{x\in\Omega} u(x,t), \inf_{x\in\Omega} v(x,t) \right\} > 0.$$

<u>Pf</u> Suppose $\lambda_1 \geq 0$, and let $(\psi_1, \psi_2)$ be the corresponding eigenfunctions.

① For each $M > 0$, $(\bar{u}_M, \bar{v}_M) = (M\psi_1, M\psi_2)$ is a strict supersol.

$$-d_i \Delta (M\psi_i) - p_i F(M\psi_1, M\psi_2) + a_i M\psi_i + b_i (M\psi_i)^2$$

$$= M\lambda_1 \psi_i + b_i (M\psi_i)^2 = M\psi_i(\lambda_1 + b_i M\psi_i) > 0$$

② (S) has no nontrivial equilibrium.

<u>Reason</u> Suppose $(\theta_1, \theta_2)$ is a positive equilibrium, $d_i \Delta\theta_i + p_i(\theta_1,\theta_2) - a_i\theta_i = b_i\theta_i^2 > 0$

$\Rightarrow \Phi_t(\theta_1, \theta_2) > (\theta_1, \theta_2) \quad \forall t > 0 \Rightarrow r(\Phi_t) > 1 \ \forall t > 0 \Rightarrow \lambda_1 < 0$ !!!

③ For any $M > 0$, the solution $(u_M, v_M)$ of (S) with initial data $(\bar{u}_M, \bar{v}_M)$ is decreasing in time and $(u_M, v_M) \to 0$ as $t \to \infty$.

④ By comparison, any nonneg. sol. satisfies $(u,v) \to 0$ as $t \to \infty$.

<u>Theorem</u> If $\lambda_1 \geq 0$, then every solution satisfies $(u,v) \longrightarrow 0$ as $t \to \infty$.

If $\lambda_1 < 0$, then every nonnegative, nontrivial solution strongly persists, i.e.

$$\liminf_{t\to\infty} \min \left\{ \inf_{x\in\Omega} u(x,t), \inf_{x\in\Omega} v(x,t) \right\} > 0.$$

<u>pf</u> Suppose $\lambda_1 < 0$, and let $(\varphi_1, \varphi_2)$ be the corresponding eigenfunctions.

① For $\varepsilon > 0$ small, $(\underline{u}_\varepsilon, \underline{v}_\varepsilon) = (\varepsilon\varphi_1, \varepsilon\varphi_2)$ is strict subsol.

② For $M > 1$ large, $(\bar{u}_\varepsilon, \bar{v}_\varepsilon) = (M, M)$ is strict supersol.

③ Any nonneg. sol. is uniformly bounded in time. (by comparison).

④ Each nonnegative, nontrivial sol satisfies a "linear" parabolic eqn.

$$\partial_t u - d_1 \Delta u = \underbrace{\left( p_1 \frac{F(u,v) - F(0,v)}{u - 0} - a_1 - b_1 u \right)}_{\text{bounded in } L^\infty} u$$

By strong maximum principle, $u > 0$, $v > 0$ for $x \in \bar{\Omega}$ and $t > 0$.

⑤ Choose $\varepsilon > 0$, s.t. $(\varepsilon\varphi_1, \varepsilon\varphi_2) \leq (u(\cdot,1), v(\cdot,1))$

then $(\varepsilon\varphi_1, \varepsilon\varphi_2) \leq (u(\cdot,t), v(\cdot,t)) \quad \forall t \geq 1$.

This proves strong persistence.

#

Application: Diffusing population model with two sexes and short reproductive season.

[Reference: W. Jin and H.R. Thieme, DCDS-B (2014)]

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^k$, $X = C(\bar{\Omega};\mathbb{R})$, $K = C(\bar{\Omega};\mathbb{R}_+)$.

Define $S: K \to K$ by $S = F \circ \Phi$

$$u_0 \xmapsto{\Phi_1} (v_1(\cdot,1), v_2(\cdot,1)) \xmapsto{F} F(u_1(\cdot,1), u_2(\cdot,1))$$

within-year dispersion

Mating and Reproduction

$$\begin{cases} \partial_t v_i - d_i \Delta v_i = g_i(x,t,v_1,v_2)v_i & \text{in } \Omega \times [0,1] \\ n \cdot \nabla v_i = 0 & \text{on } \partial\Omega \times [0,1] \\ v_i(x,0) = \underbrace{p_i(x)}_{} u_0 & \text{in } \Omega \end{cases}$$

probability that offspring is male/female

$$F(u_1,u_2) = \beta(x)\frac{u_1u_2}{u_1+u_2} \quad \text{(harmonic mean)}$$

or

$$F(u_1,u_2) = \min\{\beta_1(x)u_1, \beta_2(x)u_2\}$$

Then the persistence is determined by the "linearization" at zero.

$$f : K \to K, \qquad f = F \circ D\Phi(0).$$

$$\psi_0 \longmapsto^{D\Phi_1(0)} \left(\phi_1(\cdot,1), \phi_2(\cdot,1)\right) \stackrel{F}{\longmapsto} F\left(\phi_1(\cdot,1), \phi_2(\cdot,1)\right)$$

where $F$ is as before (continuous, homogeneous), and

$(\phi_1, \phi_2)$ satisfies
$$\begin{cases} \partial_t \phi_i - d_i \Delta \phi_i = g_i(x,t,0)\phi_i & \Omega \times [0,1] \\ n \cdot \nabla \phi_i = 0 & \partial\Omega \times [0,1] \\ \phi_i(x,0) = p_i(x)\psi_0(x) & \end{cases}$$

i.e. $D\Phi_1(0) : X \to X$ is linear, compact and strongly monotone

$\Rightarrow$ $f = F \circ D\Phi(0)$ is compact, continuous, and homogeneous. $D\Phi_1(0)(K \setminus \{0\}) \subseteq \mathrm{Int}(K \times K)$

By the Krein-Rutman Theorem, $f : K \to K$ has a p.e.v. $\mu_1 > 0$.

such that $\mu_1 > 1 \Rightarrow$ the population is strongly persistent.

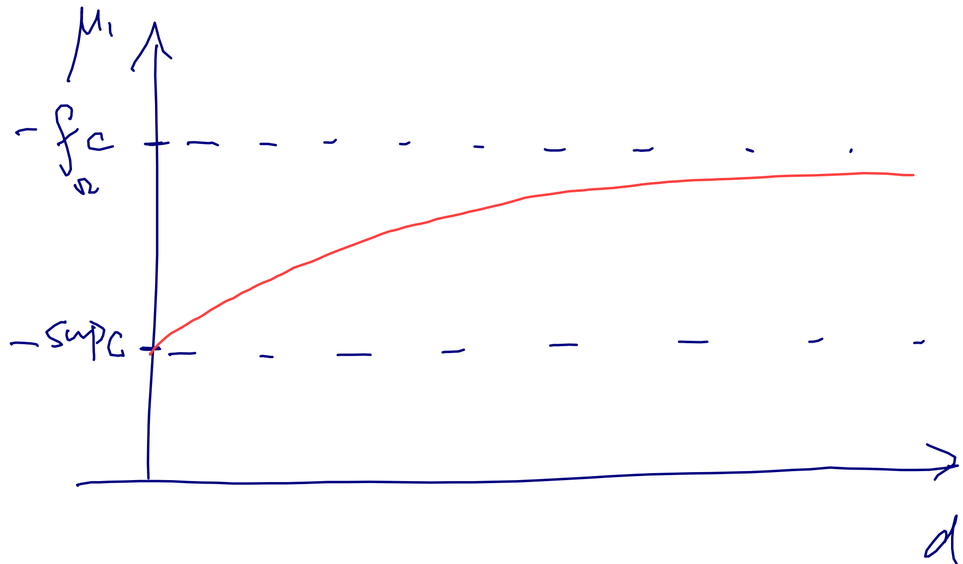
$0 < \mu_1 < 1 \Rightarrow$ the trivial fixed point is locally asymptotically stable.

# Asymptotics of $\mu_1$ when $d \to 0, \infty$

$$\begin{cases} -d\Delta\phi - c\phi = \mu_1 \phi & \text{in } \Omega \\ n\cdot\nabla\phi + p^0\phi = 0 & \text{on } \partial\Omega \\ \text{where } p^0 \geq 0. \text{ or } \equiv 0. \end{cases}$$

<u>Prop 1</u> (a) $\mu_1 \geq -\sup_{\Omega} c \quad \forall d > 0$

(b) $\lim_{d\to 0} \mu_1 = -\sup_{\Omega} c$




<u>Prop 2</u> Normalize $\phi_1$ by $\sup \phi_1 = 1$

$\phi_1 \to 0$ in $C_{loc}(\bar{\Omega}\setminus\Gamma)$

where $\Gamma = \{x \in \bar{\Omega} : c(x) = \sup_{\Omega} c\}$

<u>Prop 3</u> (a) $\mu_1 \leq \frac{1}{|\Omega|}\int_\Omega c$

(b) If, in addition, $p^0 \equiv 0$, then $\lim_{d\to\infty} \mu_1 = \frac{-1}{|\Omega|}\int_\Omega c$.

# Asymptotics of $\mu_1$ when $d \to 0$

$$(**)\quad \begin{cases} -d\Delta\phi - c\phi = \mu_1\phi & \text{in } \Omega \\ n\cdot\nabla\phi + p^0\phi = 0 & \text{on } \partial\Omega \\ \text{where } p^0 \geq 0. \text{ or } \equiv 0. \end{cases}$$

**Prop 1** (a) $\mu_1 \geq -\sup_{\Omega} c \qquad \forall d > 0$

(b) $\lim_{d\to 0} \mu_1 = -\sup_{\Omega} c$

**Pf** Divergence thm $\Rightarrow \quad -\int_{\Omega} \Delta\phi_1 = -\int_{\partial\Omega} n\cdot\nabla\phi_1 = \int_{\partial\Omega} p^0\phi_1 \geq 0$

Integrate $(**)$, $\quad -d\int_{\Omega}\Delta\phi_1 - \int c\phi_1 = \mu_1\int\phi_1$

$$-\int c\phi_1 \leq \mu_1\int\phi_1$$

$$\Rightarrow \mu_1 \geq -\sup_{\Omega} c.$$

This proves (a).

## Asymptotics of $\mu_1$ when $d \to 0$

$$(**)\quad \begin{cases} -d\Delta\phi - c\phi = \mu_1 \phi & \text{in } \Omega \\ n\cdot\nabla\phi + p^0\phi = 0 & \text{on } \partial\Omega \\ \text{where } p^0 \geq 0 \text{. or } \equiv 0. \end{cases}$$

**Prop 1** (a) $\mu_1 \geq -\sup_\Omega c \quad \forall d > 0$

(b) $\boxed{\lim_{d\to 0} \mu_1 = -\sup_\Omega c}$

**Pf of (b)** Fix $x_0 \in \Omega$ and $0 < R < \mathrm{dist}(x_0, \partial\Omega)$.

Let $(\lambda_R, \phi_R)$ be the eigenpair $\boxed{-\Delta\phi_R = \lambda_R\phi_R \text{ in } B_R, \quad \phi_R|_{\partial B_R} = 0}$

We may arrange that $\phi_1 \geq \phi_R$ in $B_R(x_0)$ and $\phi_1(x_0') = \phi_R(x_0')$ $\exists x_0' \in B_R(x_0)$.

$\Rightarrow \Delta(\phi_1 - \phi_R)(x_0') \geq 0$

$\Rightarrow d\lambda_R\phi_R(x_0') = -d\Delta\phi_R(x_0') \geq -d\Delta\phi_1(x_0') = (c(x_0') + \mu_1)\,\phi_R(x_0')$

$\Rightarrow \mu_1 \leq d\lambda_R - c(x_0') \leq d\lambda_R - \inf_{B_R(x_0)} c$

Letting $d \to 0$, then $R \to 0$, $\limsup_{d\to 0+} \mu_1 \leq -c(x_0)$.

Since $x_0 \in \Omega$ is arbitrary, we are done.

#

# Asymptotics of $\mu_1$ when $d \to \infty$

$$(**)\quad \begin{cases} -d\Delta\phi - c\phi = \mu_1\phi & \text{in } \Omega \\ n\cdot\nabla\phi + p^0\phi = 0 & \text{on } \partial\Omega \\ \text{where } p^0 \equiv 0 & \end{cases}$$

**Prop 1** (a) $\mu_1 \le \frac{-1}{|\Omega|}\int_\Omega c \qquad \forall d > 0$

(b) $\lim_{d\to\infty} \mu_1 = -\frac{1}{|\Omega|}\int_\Omega c$

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**Pf of (a)**

$$-\frac{\Delta\phi_1}{\phi_1} - c(x) = \mu_1$$

$$|\Omega|\mu_1 = -\int_\Omega \frac{\Delta\phi_1}{\phi_1} - \int_\Omega c$$

$$= \int_\Omega \nabla\phi_1 \cdot \nabla\frac{1}{\phi_1} - \int_\Omega c$$

$$= \int_\Omega \frac{-|\nabla\phi_1|^2}{\phi_1^2} - \int_\Omega c \le -\int_\Omega c. \quad \forall d > 0.$$

This proves (a).

## Asymptotics of $\mu_1$ when $d \to \infty$

$$(**)\quad \begin{cases} -d\Delta\phi - c\phi = \mu_1\phi & \text{in } \Omega \\ n\cdot\nabla\phi + p^0\phi = 0 & \text{on } \partial\Omega \\ \text{where } p^0 \equiv 0. \end{cases}$$

**Prop 1** (a) $\mu_1 \le \frac{-1}{|\Omega|}\int_\Omega c \quad \forall d > 0$

(b) $\lim_{d\to\infty} \mu_1 = -\frac{1}{|\Omega|}\int_\Omega c$

**Pf of (b)** Normalize $\sup_\Omega \phi_1 = 1$.

$$(***)\qquad -\Delta\phi_1 = \frac{1}{d}(c + \mu_1)\phi_1 = O\left(\frac{1}{d}\right).$$

Multiply by $\phi_1$, integrate $\Longrightarrow \int_\Omega |\nabla\phi_1|^2 \le O\left(\frac{1}{d}\right)$.

Poincaré's Ineq. $\Longrightarrow \int_\Omega \left|\phi_1 - \frac{1}{|\Omega|}\int_\Omega \phi_1\right|^2 \le o\left(\frac{1}{d}\right)$.

Hence $\phi_1 \to 1$ in $L^2(\Omega)$.

Using $p^0 \equiv 0$, we may integrate $(**) \Rightarrow -\int_\Omega c\phi_1 = \mu_1 \int_\Omega \phi_1$.

Letting $d \to \infty$, then $-\int_\Omega c = |\Omega| \lim_{d\to\infty} \mu_1$ #.

## Asymptotic behavior of principal eigenfunction.

- ecology $\to$ spatial profile of invasive species, when rare.

- epidemiology $\to$ disease "hot spots"
  - See [Tien-Shuai-Eisenberg-van den Driessche JMB (2015)] for disease invasion on community networks (discrete Laplacian).
  - In the diffusive SIS model [Allen-Bolker-Lou-Nevai DCDS-A (2008)]

$$\begin{cases} S_t = d_S \Delta S - \beta(x)\frac{SI}{S+I} + \gamma(x) I, \quad I_t = d_I \Delta I + \beta(x)\frac{SI}{S+I} - \gamma(x) I & \text{in } \Omega \times \mathbb{R}_+ \\ n \cdot \nabla S = n \cdot \nabla I = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ S(x,0) = S_0(x), \quad I(x,0) = I_0(x), \end{cases}$$

The stability of the disease free equilibrium is determined by the p.e.v. of $\begin{cases} d_I \Delta \phi + (\beta - \gamma)\phi + \mu_1 \phi = 0 & \Omega \\ n \cdot \nabla \phi = 0 & \Omega \end{cases}$

The profile of $\phi_1$ determines the disease hot spots.

By Prop 3, $\phi_1$ concentrates at max. pt of $\beta - \gamma$ in $\overline{\Omega}$.

**Prop 3** Normalize $\phi_1$ by $\sup_\Omega \phi_1 = 1$

$$(**)\begin{cases} -\varepsilon^2\Delta\phi_1 - c\phi_1 = \mu_1\phi_1 & \Omega \\ n\cdot\nabla\phi_1 + p^0\phi_1 = 0 & \partial\Omega \end{cases}$$

Then $\phi_1 \to 0$ in $C_{loc}(\overline{\Omega}\setminus\Gamma)$ as $d\to 0$,

where $\Gamma = \{x\in\overline{\Omega} : c(x) = \sup_\Omega c.\}$.

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Pf. Set $w_\varepsilon = -\varepsilon\log\phi_1$, then

$$\begin{cases} -\varepsilon\Delta w_\varepsilon + |\partial_x w_\varepsilon|^2 + c(x) + \mu_1 = 0 \text{ in } \Omega, & n\cdot\nabla w_\varepsilon \geq 0 \text{ on } \partial\Omega \\ \inf_\Omega w_\varepsilon = 0 \end{cases}$$

$+\infty$ is allowed.

Define the semi-relaxed limit $w_*: \overline{\Omega} \to [0,+\infty]$

$$w_*(x) = \liminf_{\substack{\varepsilon\to 0\\ x'\to x}} w_\varepsilon(x') \quad \text{for } x\in\overline{\Omega}.$$

which is nonnegative and lower semi-continuous.

Since $\phi_1(x) = e^{-\frac{w_*(x)+o(1)}{\varepsilon}}$,

it suffices to show that $w_*(x) > 0$ when $c(x) < \sup_\Omega c$.

$$\begin{cases} -\varepsilon\Delta w_\varepsilon + |\partial_x w_\varepsilon|^2 + c(x) + \mu_1 = 0 & \text{in } \Omega \\ n\cdot\nabla w_\varepsilon \geq 0 & \text{on } \partial\Omega \\ \inf_\Omega w_\varepsilon = 0 \end{cases}$$

<u>Claim</u> $w_*(x) > 0$ at $x$ if $c(x) < \sup_\Omega c$.

Suppose $w_*(x_0) = 0$ for some $x_0 \in \overline{\Omega}$,

then $w_* + |x - x_0|^2$ has a strict local min at $x_0$.

To show $c(x_0) \geq \sup_\Omega c$.

<u>Case ①</u> $x_0 \in \operatorname{Int}\Omega$.

$w_\varepsilon(x) + |x - x_0|^2$ has a local min $x_\varepsilon \to x_0$, $x_\varepsilon \in \operatorname{Int}\Omega$.

$\Rightarrow \nabla(w_\varepsilon + |x-x_0|^2) = 0$ and $\Delta(w_\varepsilon + |x - x_0|^2) \geq 0$ at $x = x_\varepsilon$.

Hence,
$$\begin{aligned} c(x_\varepsilon) + \mu_1 &= -|\partial_x w_\varepsilon(x_\varepsilon)|^2 + \varepsilon\Delta w_\varepsilon(x_\varepsilon) \\ &\geq -|2(x_\varepsilon - x_0)|^2 - 2N\varepsilon \end{aligned}$$

Taking $\varepsilon \to 0$, $c(x_0) + (-\sup_\Omega c) \geq 0 \Rightarrow c(x_0) \geq \sup_\Omega c$.

<u>Case ②</u> $x_0 \in \partial\Omega$, consider $w_\varepsilon(x) + |x - x_0 + \varepsilon n_0|^2$,
where $n_0$ is the outer unit normal vector at $x_0$.

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