Tuesday § Subhomogeneous Dynamical Systems.

Last time, we considered the logistic eqn. for a single species:

$$(*)\begin{cases} u_t + \mathcal{L} u = u(m(x)-u) & \Omega\times(0,\infty) \\ \mathcal{B} u = 0 & \partial\Omega\times(0,\infty) \\ u(x,0)=u_0(x) & \Omega \end{cases}$$

and showed a dichotomy result:

Either (*) has no positive equilibria and 0 is g.a.s.

or (*) has a unique positive equilibria θ and θ is g.a.s.

Let $0<\lambda<1$, we claim that $\lambda u(x,t) \leq u_\lambda(x,t)$,

where $u(x,t)$ is the solution to (*) with initial data u_0

and $u_\lambda(x,t)$ is the solution to (*) with initial data λu_0.

Pf: $(\lambda u)_t - d\Delta(\lambda u) = \lambda(u_t - d\Delta u) = \lambda u(m-u) < \lambda u(m-\lambda u)$

$\Rightarrow \lambda u$ is subsol to (*) with initial data λu_0. $\Rightarrow \lambda u \leq u_\lambda$. #

In fact, (*) belongs to the class of "subhomogeneous systems".

Let $\mathbb{X}$ be a Banach space, ordered by a cone K, and $\operatorname{Int} K \neq \emptyset$.

$$x \leq \bar{x} \iff \bar{x} - x \in K$$

$$x < \bar{x} \iff \bar{x} - x \in K \setminus \{0\}$$

$$x \ll \bar{x} \iff \bar{x} - x \in \operatorname{Int} K.$$

Let X be a Banach space, ordered by a cone K, and $\operatorname{Int} K \neq \phi$.

Let $S : K \to K$ be a continuous map satisfying:

(S1) $S(0) = 0$ and S is strongly monotone: $x < \bar{x} \Rightarrow S(x) \ll S(\bar{x})$.

(S2) $\{S^n(x)\}_{n=1}^{\infty}$ is precompact for each $x \in K$.

(S3) S is strictly subhomogeneous, i.e.

$$\lambda S(x) < S(\lambda x) \qquad \text{for } 0 < \lambda < 1 \text{ and } x \in \operatorname{Int} K.$$

(S4) There exists a compact map $f : K \to K$ such that

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} S(\lambda x) = f(x) \qquad \text{for each } x \in K.$$

(S4) can be replaced by $\boxed{\text{(S5)} \quad S : K \to K \text{ is compact and is Fréchet differentiable at } 0.}$

<u>Rmk</u> (S5) $\Rightarrow$ (S4), by noting that

S compact $\Rightarrow$ $DS(0)$ compact.

For $(*)\begin{cases} u_t + \mathcal{L}u = u(m(x,t)-u) & \Omega\times(0,\infty) \\ Bu = 0 & \partial\Omega\times(0,\infty) \\ u(x,0) = u_0(x) & \Omega \end{cases}$

If $m(x,t)$ is T-periodic in t or $m(x,t)=m(x)$,

Take $X = C(\overline{\Omega})$, $K = C(\overline{\Omega}; [0,\infty))$,

Define $S: K \to K$ by $S(u_0) = u(\cdot, T)$.

Claim S satisfies $(S1)-(S5)$.

$(S1)$ $(S2)$ are clear. $(S3)$ is verified on page 1.

For $(S4)$: Let u_λ be sol. to $(*)$ with initial data λu_0, and $v_\lambda = \frac{1}{\lambda} u_\lambda$.

$$\partial_t u_\lambda + \mathcal{L} u_\lambda = (m - u_\lambda) u_\lambda \text{ in } \Omega\times(0,\infty), \qquad u_\lambda(x,0) = \lambda u_0.$$

$$\partial_t v_\lambda + \mathcal{L} v_\lambda = (m - u_\lambda) v_\lambda \qquad v_\lambda(x,0) = u_0$$

Let $\lambda \to 0$, then $u_\lambda \to 0$ and $v_\lambda \to v$

where $\begin{cases} \partial_t v + \mathcal{L} v_\lambda = m v_\lambda & \text{in } \Omega\times(0,\infty) \\ \mathcal{B} v = 0 & \text{on } \partial\Omega\times(0,\infty) \\ v(x,0) = u_0 & \text{in } \Omega. \end{cases}$

In fact, S is Fréchet diff at 0 and $DS(0)[u_0] = v(\cdot, T)$.

Another example is the sexual reproduction model:

$$\begin{cases} u_t = d_1 \Delta u + a_1 F(u,v) + b_1 u + c_1 v - u^2 & \Omega \times (0,\infty) \\ v_t = d_2 \Delta v + a_2 F(u,v) + c_2 u + b_2 v - v^2 & \Omega \times (0,\infty) \\ n \cdot \nabla u = n \cdot \nabla v = 0 & \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \Omega \end{cases}$$

where $d_i, a_i, b_i, c_i \in C^\alpha(\bar{\Omega} \times \mathbb{R})$ and T-periodic in t.

$d_i > 0, \quad a_i \geq 0, \quad c_i \geq 0$

$F(u,v)$ is monotone and homogeneous: $F(tu, tv) = tF(u,v) \quad \forall t > 0$.

e.g. $F(u,v) = \dfrac{uv}{u+v}$, $\min\{\beta u, (1-\beta)v\}$.

Take $X = C(\bar{\Omega}; \mathbb{R}^2)$, $K = C(\bar{\Omega}; [0,\infty)^2)$

Define $S : K \to K$ by $S(u_0, v_0) = (u(\cdot, T), v(\cdot, T))$

then (S1) – (S4) holds but S is not differentiable at 0.

Dichotomy / Threshold Result.

Thm A Suppose $S: K \to K$ is a continuous map s.t. (S1) - (S4) hold.

1. If $\tilde{r}_K(f) \le 1$, then $S^n(x) \to 0$ $\forall x \in K$;
2. If $\tilde{r}_K(f) > 1$, then S has a unique fixed point $x^* \in \operatorname{Int} K$ such that $S^n(x) \to x^*$ for each $x \in K \setminus \{0\}$;

where $\tilde{r}_K(f) > 0$ is the Bonsall cone spectral radius of the mapping f given in (S4).

Rmk If (S4) is replaced by (S5), then the Conclusion holds by replacing $\tilde{r}_K(f)$ by the spectral radius $r(DS(0))$.
Note that the Fréchet derivative $DS(0)$ is a linear operator.

Dichotomy / Threshold Result.

Thm A Suppose $S: K \to K$ is a continuous map s.t. (S1)–(S4) hold.

1. If $\tilde{r}_K(f) \le 1$, then $S^n(x) \to 0$ $\forall x \in K$;
2. If $\tilde{r}_K(f) > 1$, then S has a unique fixed point $x^* \in \mathrm{Int}\, K$ such that $S^n(x) \to x^*$ for each $x \in K \setminus \{0\}$;

where $\tilde{r}_K(f) > 0$ is the Bonsall cone spectral radius of the mapping f given in (S4).

Cor B Suppose (S1)–(S3) and one of (S4), (S5) holds, then exactly one of the following holds:

1. S has no fixed points in $K \setminus \{0\}$. In such a case $S^n(x) \to 0$ $\forall x \in K$.
2. There exists $x_0 \in K \setminus \{0\}$ such that $S(x_0) \ge x_0$. In such a case, S has a unique fixed point $x^* \in \mathrm{Int}\, K$ such that $S^n(x) \to x^*$ $\forall x \in K \setminus \{0\}$.

Cor B Suppose (S1)-(S3) and one of (S4), (S5) holds, then exactly one of the following holds:

1. S has no fixed points in $K\setminus\{0\}$. In such a case $S^n(x)\to 0$ $\forall x\in K$.
2. There exists $x_0\in K\setminus\{0\}$ such that $S(x_0)\geq x_0$. In such a case, S has a unique fixed point $x^*\in \operatorname{Int} K$ such that
$$S^n(x)\to x^* \qquad \forall x\in K\setminus\{0\}.$$

Pf • S has no fixed point implies alternative 1 of Thm A holds.

• Suppose $S(x_0)\geq x_0$ for some $x_0\in K\setminus\{0\}$, then
$$S^n(x_0)\geq S^{n-1}(x_0)\geq \cdots \geq S(x_0)\gg 0$$
$$\Longrightarrow S^n(x_0)\not\longrightarrow 0.$$
$\Longrightarrow$ alterative 2 of Thm A holds.

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Note. [M. Hirsch (1984)], [H. Smith (1986)] proved convergence for concave maps.
Generalizations by P. Takáč, J. Jiang, Y. Wang, X.-Q. Zhao....

Preliminaries

Lemma (Monotone convergence) Let X be an ordered Banach space with cone K.
If $\{x_n\}_{n=1}^{\infty} \subseteq X$ is precompact and $x_n \leq x_{n+1}$ $\forall n$.
then it is convergent.

Pf. Define the set of subsequential limit:

$$A = \{ y \in X : x_{n_k} \to y \quad \exists\, n_k \to \infty \}.$$

By compactness, A is non-empty.

It remains to show that A is a singleton set.

Suppose $y, \bar{y} \in A$, i.e. $x_{n_k} \to y$ and $x_{m_k} \to \bar{y}$

WLOG, assume $n_k \leq m_k \leq n_{k+1}$.

then $x_{n_k} \leq x_{m_k} \leq x_{n_{k+1}}$

Letting $k \to \infty$ $\quad y \leq \bar{y} \leq y$

$\Rightarrow \quad y - \bar{y} \in K \cap (-K) \Rightarrow y - \bar{y} = 0$ $\quad \#$

Lemma C Assume (S1)–(S4), then

1. f is homogeneous and monotone.
2. $f(x) > S(x)$ $\quad \forall x \in K \setminus \{0\}$
3. $f(x) \gg 0$ $\quad \forall x \in K \setminus \{0\}$
4. $\tilde{r}_K(f) > 0$ and there exists $\tilde{x} \in \mathrm{Int}\, K$ s.t. $f(\tilde{x}) = r_K(f)\tilde{x}$.

Pf (homogeneity) For $t > 0$,

$$f(tx) = \lim_{\lambda \to 0} \frac{1}{\lambda} S(\lambda t x) = t \lim_{\lambda \to 0} \frac{1}{\lambda t} S(\lambda t x) = f(x).$$

(monotonicity) For $x \leq \bar{x}$,

$$S(\lambda x) \leq S(\lambda \bar{x})$$

$$\frac{1}{\lambda} S(\lambda x) \leq \frac{1}{\lambda} S(\lambda \bar{x}), \text{ then take } \lambda \to 0.$$

Next, we claim that $\lambda \longmapsto \frac{1}{\lambda} S(\lambda x)$ is monotone in $0 < \lambda < 1$.

Indeed, for $0 < \lambda < \lambda' < 1$,

$$S(\lambda x) = S\left(\frac{\lambda}{\lambda'} \lambda' x\right) > \frac{\lambda}{\lambda'} S(\lambda' x) \Rightarrow \frac{1}{\lambda} S(\lambda x) > \frac{1}{\lambda'} S(\lambda' x).$$

Taking $\lambda \to 0, 1$, we obtain $f(x) > \frac{1}{\lambda} S(\lambda x) > S(x) \gg 0 \quad \forall 0 < \lambda < 1$.

Lemma C Assume (S1) − (S4), then

1. f is homogeneous and monotone.
2. $f(x) > S(x) \quad \forall x \in K \setminus \{0\}$
3. $f(x) \gg 0 \quad \forall x \in K \setminus \{0\}$
4. $\tilde{r}_K(f) > 0$ and there exists $\tilde{x} \in \operatorname{Int} K$ s.t. $f(\tilde{x}) = r_K(f)\,\tilde{x}$.

Pf (cont.) It remains to prove 4.

By 3., $\exists\, c_1 > 0$ and $x_1 \in \operatorname{Int} K$ such that $f(x_1) \geq c_1 x_1$.

Claim $\hat{r}_K(f) > 0$.

Otherwise, $\hat{r}_K(f/c_1) = \frac{1}{c_1}\tilde{r}_K(f) = 0$

$$\Rightarrow \quad x_1 \leq (f/c_1)^n (x_1) \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad x_1 \in -K.$$

Since $K \cap (-K) = \{0\}$, we deduce $x_1 \notin K \setminus \{0\}$. contradictions.

4. follows by Krein-Rutman Theorem for maps.

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<u>Pf of Thm A</u> Suppose $r_K(f) \le 1$, then by Lemma C(1,2,4), there exists eigenvector $\tilde{x} \in \operatorname{Int} K$ of f such that

(C.2) $$S(t\tilde{x}) < f(t\tilde{x}) = r_K(f)t\tilde{x} \le t\tilde{x} \qquad \forall t > 0$$

<u>Claim</u> S has no fixed points in $K \setminus \{0\}$.

Suppose not, then $S(x_0) = x_0$ $\exists x_0 \in K \setminus \{0\}$.

By (C.2), $x_0 \notin \operatorname{span}\{\tilde{x}\}$.

$\tilde{x} \in \operatorname{Int} K$, $-x_0 \notin K \Rightarrow t\tilde{x} - x_0 \in \operatorname{Int} K$ for $t > 0$ large.

$\Rightarrow \exists$ minimal $t_0 > 0$ such that $t_0 \tilde{x} - x_0 \in \partial K \setminus \{0\}$.

$\Rightarrow x_0 = S(x_0) \ll S(t_0\tilde{x}) < t_0 \tilde{x}$ <u>contradictions</u>.

By (C.2) again, for each fix $t > 0$, $S^n(t\tilde{x})$ is decreasing in n.

By monotone convergence lemma, $S^n(t\tilde{x})$ converges to a fixed point y of S.

(since $S^{n+1}(t\tilde{x}) = S(S^n(t\tilde{x})) \Rightarrow y = S(y)$)

Pf of Thm A (cont.) Since S has no nonzero fixed points,

$$S^n(t\tilde{x}) \longrightarrow 0 \quad \text{for each fixed } t > 0.$$

Given $x \in K$, choose $t > 0$ s.t.

$$0 \leq x \leq t\tilde{x}.$$

$$\Rightarrow \quad 0 \leq S^n(x) \leq S^n(t\tilde{x}). \quad \forall n$$

$$\Rightarrow \quad S^n(x) \longrightarrow 0 \quad \text{as } n \to \infty.$$

(Precisely, one argues that the set of subsequential limits of $\{S^n(x)\}_{n=1}^{\infty}$ is nonempty subset of $\{0\}$.).

This proves the first part of Thm A.

Pf of Thm A (conti). Suppose now $\tilde{r}_K(f) > 1$.

Lemma C $\Longrightarrow$ $f(\tilde{x}) = \tilde{r}_K(f)\,\tilde{x} \gg \tilde{x}$ $\exists\, \tilde{x} \in \operatorname{Int} K$.

By def. of f, $\exists\; 0 < \hat{\lambda} < 1$ such that

(C.3) $$S(\hat{\lambda}\hat{x}) \geq \tilde{\lambda}\hat{x}.$$

In particular, $S^n(\tilde{\lambda}\hat{x})$ is increasing in n.

By (S3) and monotone convergence, $S^n(\hat{\lambda}\hat{x}) \longrightarrow x^*$ $\exists\, x \in \operatorname{Int} K$.

Claim x^* is the unique fixed point of S in $K \setminus \{0\}$.

Suppose $S(\hat{x}) = \hat{x}$ $\exists\; \hat{x} \in K \setminus \{0\}$, $\hat{x} \neq x^*$, then $\hat{x} \in \operatorname{Int} K$.

Also, we cannot have $\hat{x} \leq x^*$ and $x^* \leq \hat{x}$.

By exchanging $x^*, \hat{x}$ if necessary, $\exists$ a maximal $\lambda^* < 1$ s.t. $\lambda^* x^* \leq \hat{x}$.

$\Longrightarrow \lambda^* x^* = \lambda^* S(x^*) < S(\lambda^* x^*) \ll S(\hat{x}) = \hat{x}$. Contradictions.

So x^* is the unique fixed point of S in $K \setminus \{0\}$.

Pf of Thm A (cont.) Let $0 < \lambda < 1$

Claim $S^n(\lambda x^*)$ is increasing in n; $S^n(\frac{1}{\lambda}x^*)$ is decreasing in n

Indeed, $S(\lambda x^*) > \lambda S(x^*) = \lambda x^*$.

$x^* = S(x^*) = S(\lambda \cdot \frac{1}{\lambda} x^*) > \lambda S(\frac{1}{\lambda} x^*) \Rightarrow \frac{1}{\lambda} x^* > S(\frac{1}{\lambda} x^*)$.

Since $S(x) \gg 0$ $\forall x \in K \setminus \{0\}$, it suffices to show

Claim $S^n(x) \longrightarrow x^*$ $\forall x \in \operatorname{Int} K$.

Given $x \in \operatorname{Int} K$, choose $0 < \lambda < 1$ s.t. $\lambda \hat{x} \leq x \leq \frac{1}{\lambda} \hat{x}$.

$\Longrightarrow$ $S^n(\lambda \tilde{x}) \leq S^n(x) \leq S(\frac{1}{\lambda} \tilde{x})$

By monotone convergence, the left and right hand side are convergent to x^*.

Hence $S^n(x) \longrightarrow x^*$ as well. #

See Appendix C of lecture notes for the formulation and results for subhomogeneous semiflow $\Phi_t : K \to K$.