Lecture 5

Competition Systems of two-species

$$(1)\begin{cases} u_t - d_1 \Delta u = u(a_1 - b_1 u - c_1 v) & \Omega \times (0,\infty) \\ v_t - d_2 \Delta v = v(a_2 - b_2 u - c_2 v) & \Omega \times (0,\infty) \\ n\cdot\nabla u = 0 = n \cdot \nabla v & \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \end{cases}$$

where $a_i, b_i, c_i \in C^\alpha(\bar{\Omega})$, strictly positive.

Setting $v \equiv 0$, we obtain

$$(2)\begin{cases} u_t - d_1 \Delta u = u(a_1 - b_1 u) & \Omega \times (0,\infty) \\ n\cdot\nabla u = 0 & \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \end{cases}$$

since $a_1 > 0$ in $\bar{\Omega}$ $\begin{cases} (2) \text{ has a unique positive equilibrium } \tilde{u} \\ u(\cdot,t) \longrightarrow \tilde{u} \qquad \forall\, u_0 \geq 0 \end{cases}$

The linear stability of $\tilde{u}$ w.r.t. (2) :

$$(3)\begin{cases} d_1 \Delta \tilde{\phi} + \tilde{\phi}(a_1 - 2b_1\tilde{u}) + \tilde{\mu}\tilde{\phi} = 0 & \text{in } \Omega \\ n\cdot\nabla\tilde{\phi} = 0 & \text{on } \partial\Omega. \end{cases}$$

<u>Claim</u> $\tilde{\mu} > 0$ i.e. $\tilde{u}$ is stable w.r.t. (2).

<u>Pf.</u>

$$-\tilde{\mu}\int \tilde{u}\tilde{\phi} = \int \tilde{u}\left(d_1\Delta\tilde{\phi} + \tilde{\phi}(a_1 - 2b_1\tilde{u})\right)$$
$$= \int \tilde{\phi}\left(d_1 \Delta\tilde{u} + \tilde{u}(a_1 - 2b_1\tilde{u})\right)$$
$$= \int \tilde{\phi}\left(-\tilde{u}^2 b_1\right)$$

Similarly. denote $\tilde{v} \to \begin{cases} d_2\Delta\tilde{v} + \tilde{v}(a_2 - c_2\tilde{v}) = 0 & \Omega \\ n\cdot\nabla\tilde{v} = 0 & \partial\Omega \end{cases}$

1B

(1) generates a monotone dynamical system. S

$X = C(\bar{\Omega}, \mathbb{R}^2)$ $\quad K = K_1 \times (-K_1)$, $K_1 = C(\bar{\Omega}; \mathbb{R}_+)$.

Comparison principle.

If (u_0, v_0) and $(\hat{u}_0, \hat{v}_0)$ satisfies

$$u_0 \underset{\neq}{\leq} \hat{u}_0 , \quad v_0 \underset{\neq}{\geq} \hat{v}_0 \quad \text{in } \Omega$$

then the solutions (u,v) $(\hat{u}, \hat{v})$ satisfies

$$(*) \quad u(x,t) \leq \hat{u}(x,t) \quad v(x,t) \geq \hat{v}(x,t) \quad \text{in } \Omega \times [0,\infty).$$

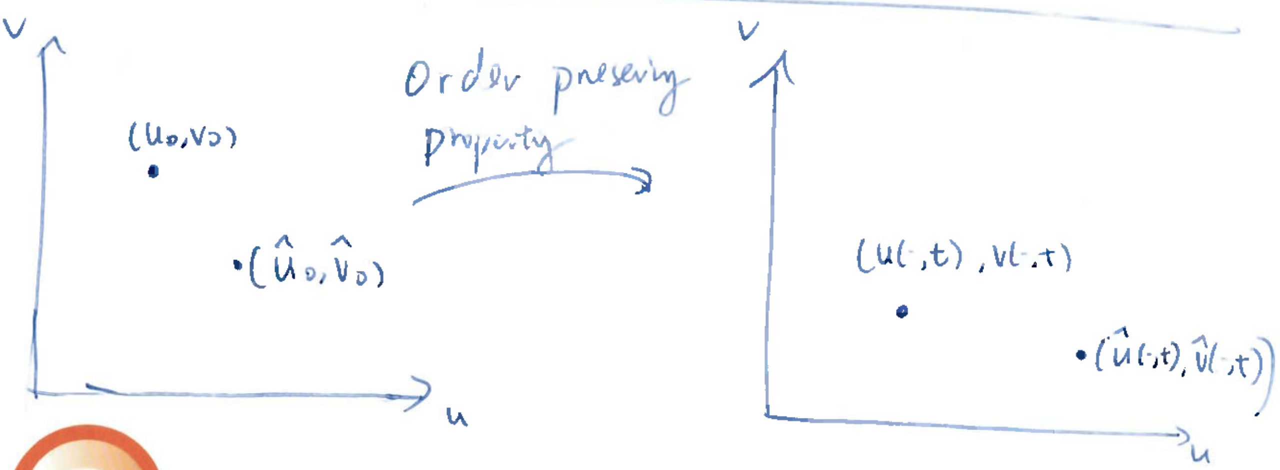
Moreover, if $u_0 > 0$, $v_0 > 0$ in $\bar{\Omega}$ (or $\hat{u}_0 > 0$, $\hat{v}_0 > 0$ in $\bar{\Omega}$)
then strict ineq holds in $(*)$.

- $(u_0, v_0) < (\hat{u}_0, \hat{v}_0) \Rightarrow S(u_0, v_0) < S(\hat{u}_0, \hat{v}_0)$

S is strictly monotone.

- If $(u_0, v_0) < (\hat{u}_0, \hat{v}_0)$ and one of them belongs to $\mathrm{Int}\, C(\bar{\Omega}; \mathbb{R}^2_+)$

then $S(u_0, v_0) \ll S(u_0, v_0)$.



Here System (1) has 3 equilibria

$E_0 = (0,0)$, $E_1 = (\hat{u}, 0)$, $E_2 = (0, \hat{v})$

Stability w.r.t. to the full system:

- E_0 is always linearly unstable.
- E_1. Consider the p.e.v. λ_1 of

$$(4)\begin{cases} d_1 \Delta \varphi + (a_1 - 2b_1\hat{u})\varphi - b_1 \hat{u} \psi + \lambda_1 \varphi = 0 & \text{in } \Omega \\ d_2 \Delta \psi + (a_2 - b_2 \hat{u})\psi \qquad + \lambda_1 \psi = 0 & \text{in } \Omega \\ \text{Neumann b.c.} \end{cases}$$

<u>Lemma</u> E_1 is lin. stable (resp. unstable) if and only if $\mu_{E_1} > 0$ (resp. $\mu_{E_1} < 0$),

where μ_{E_1} is the p.e.v. of $(5)\begin{cases} d_2 \Delta \phi + (a_2 - b_2 \hat{u})\phi + \mu \phi = 0 & \Omega \\ n \cdot \nabla \phi = 0 & \partial\Omega \end{cases}$

Pf. Rewrite (4):

$$(6)\begin{pmatrix} -d_1\Delta - (a_1 - 2b_1\hat{u}) & -b_1 \hat{u} \\ 0 & -d_2\Delta - (a_2 - b_2\hat{u}) \end{pmatrix}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \lambda_1 \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

<u>Case ①</u> $\mu_{E_1} > 0$. $\psi \neq 0 \Rightarrow \lambda_1$ is eigenvalue of (6) $\Rightarrow \lambda_1 \geq \mu_{E_1} > 0$

$\phi \neq 0, \psi = 0 \Rightarrow \lambda_1 = \tilde{\mu} > 0$

<u>Case ②</u> $\mu_{E_1} < 0$ $\rightarrow$ then μ_{E_1} is an eigenvalue of (4)

with $\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} [-d_1\Delta - (a_1 - 2b_1\hat{u}) - \mu_{E_1} I]^{-1}(-b_1 \hat{u} \tilde{\phi}) \\ \tilde{\phi} \end{pmatrix}$

<u>Thm</u> Suppose (1) has no positive equilibria

If E_1 is lin. unstable (ie. $\mu_{E_1} < 0$), then $(u,v) \longrightarrow E_2$.
whenever $v_0 \not\equiv 0$

<u>Pf</u>. (Due to [Hess-Lazer (1991)]). Works for noncompact flows.

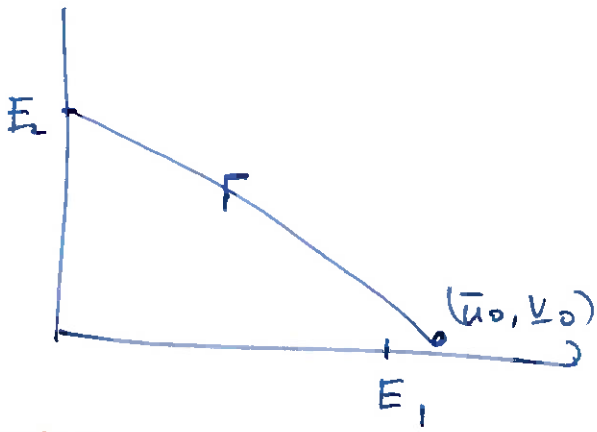
Define $(\bar{u}_0, \underline{v}_0) = ((1+\varepsilon)\tilde{u}, \delta\phi)$, $0 < \varepsilon, \delta << 1$.

<u>Step#1</u>. $(\bar{u}_0(x), \underline{v}_0(x))$ is a super-solution

$$d_1 \Delta \bar{u}_0 + \bar{u}_0 (a_1 - b_1 \bar{u}_0 - c_1 \underline{v}_0)$$
$$= (1+\varepsilon)\{ d_1 \Delta \tilde{u} + \tilde{u} [a_1 - b_1(1+\varepsilon)\tilde{u} - c_1 \delta \phi]\}$$
$$= (1+\varepsilon)\{ \tilde{u} [-b_1 \varepsilon \tilde{u} - c_1 \delta \phi]\} < 0.$$

$$d_2 \Delta \underline{v}_0 + \underline{v}_0 (a_2 - b_2 \bar{u}_0 - c_2 \underline{v}_0)$$
$$= \delta \{ d_2 \Delta \phi + \phi [a_2 - b_2(1+\varepsilon)\tilde{u} - c_2 \delta \phi + \mu_{E_1} - \mu_{E_1}]\}$$
$$= \delta \phi [-b_2 \varepsilon \tilde{u} - c_2 \delta \phi - \mu_{E_1}\}.$$
$$= \delta \phi [-\mu_{E_1} + o(1)] > 0.$$

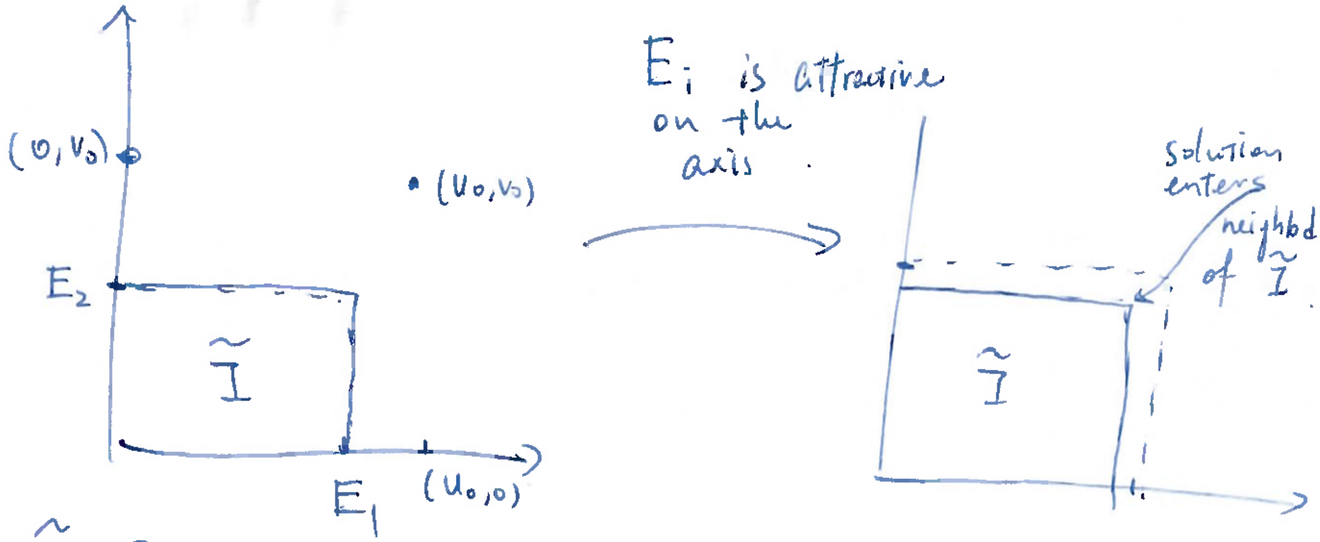
Since $\nexists$ positive equilibria, the sol. $(\bar{u}, \underline{v})$ initially at $(\bar{u}_0, \underline{v}_0)$ satisfies



Step #2

Given a solution (u,v) s.t. $u_0 \not\equiv 0$, $v_0 \not\equiv 0$.

$$(0, v_0) \le (u_0, v_0) \le (u_0, 0)$$



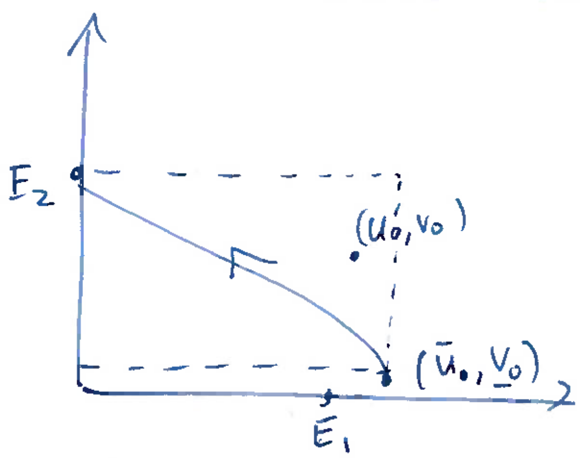
$$\tilde{I} = [E_2, E_1] = \{ (\hat{u}_0, \hat{v}_0) : (0, \tilde{v}) \le (\hat{u}_0, \hat{v}_0) \le (0, \tilde{v}) \}$$

By a translation in time, we may assume that

$$(0, v_0) \le (u_0, v_0) \le \big((1+\varepsilon)\hat{u}, \delta\phi \big)$$

$$\begin{array}{ccc} (0,v_0) & & \big((1+\varepsilon)\hat{u}, \delta\phi\big) \\ \Big\downarrow{\scriptstyle t\to\infty} & & \Big\downarrow{\scriptstyle t\to\infty} \\ E_2 & & E_2 \end{array}$$

$$\Longrightarrow \lim_{t\to\infty} (u(\cdot,t), v(\cdot,t)) = E_2.$$



Rmk E_1 lin. unstable $\Rightarrow$ E_1 repelling

Competition System with constant coeff.

Let a_i, b_i, c_i be positive const.

Thm Suppose $\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2}$, then any positive solution converges to the unique homogeneous equilibria (u^*, v^*)

Pf. By replacing (u,v) with $\left(\frac{b_1}{a_1}u, \frac{c_2}{a_2}v\right)$

$$u_t - d_1 \Delta u = a_1(1-u-cv)$$
$$v_t - d_2 \Delta v = a_2(1-bu-v)$$

for some $b, c \in (0,1)$.

and $(u^*, v^*) = \left(\frac{1-c}{1-bc}, \frac{1-b}{1-bc}\right)$.

Introduce the Lyapunov function

$$W(u,v) = \int_\Omega \frac{1}{a_1}\left[u-u^* - u^* \log\frac{u}{u^*}\right] + \frac{1}{a_2}\left[v-v^* - v^*\log\frac{v}{v^*}\right].$$

$$\frac{d}{dt}W(u,v) = \int_\Omega \frac{1}{a_1}\left[u_t - u^*\frac{u_t}{u}\right] + \int_\Omega \frac{1}{a_2}\left[v_t - v^*\frac{v_t}{v}\right] = I_1 + I_2.$$

$$I_1 = \int_\Omega \frac{1}{a_1}\left[\cancelto{0}{d_1 \Delta u} + a_1 u(1-u-cv) - u^*\left(d_1\frac{\Delta u}{u} + a_1(1-u-cv)\right)\right]$$

since $\int_\Omega \Delta u = \int_{\partial\Omega} n\cdot\nabla u = 0$, $\int_\Omega \frac{\Delta u}{u} = \int_\Omega \frac{|\nabla u|^2}{u}$, $1-u^*-cv^* = 0$

$$I_1 = \int_\Omega (u-u^*)(1-u-cv) = \int_\Omega (u-u^*)\left[-(u-u^*) - c(v-v^*)\right] - \int \frac{d_1 u^* |\nabla u|^2}{a_1 u^2}$$

Similarly,

$$I_2 = \int_\Omega (v-v^*)\left[-(v-v^*) - b(u-u^*)\right] - \int_\Omega \frac{d_2 v^*}{a_2}\frac{|\nabla v|^2}{v^2}$$

$$\Rightarrow \frac{d}{dt} W(u,v) = -\int |u-u^*|^2 - (b+c)\int (u-u^*)(v-v^*) - \int (v-v^*)^2 - \frac{d_1 u^*}{a_1}\int \frac{|Du|^2}{u^2} - \frac{d_2 v^*}{a_2}\int \frac{|Dv|^2}{v^2}$$

$$\leq -\delta \left[\int |u-u^*|^2 + \int |v-v^*|^2 \right] < 0.$$

Define $\dot{W}(u_0,v_0) = \frac{d}{dt} W(u,v) \Big|_{t=0}$.

LaSalle's invariance principle implies

$$\operatorname{dist}\left((u(\cdot,t), v(\cdot,t)), \mathcal{M} \right) \longrightarrow 0 \quad \text{as } t \to \infty$$

where $\mathcal{M}$ is the maximal invariant subset of

$$E = \{ (u_0,v_0) : \dot{W}(u_0,v_0) = 0 \}.$$

In this case, $E = \{ (u_0,v_0) \}$ is singleton set

$$\Rightarrow (u(\cdot,t), v(\cdot,t)) \longrightarrow (u^*,v^*) \quad \text{as } t \to \infty$$

#

Thm Let $a_i = b_i = c_i = 1$. Then for each non-negative, nontrivial solution (u,v),

$$(u(\cdot,t), v(\cdot,t)) \longrightarrow (s_0, 1-s_0)$$

for some $s_0 \in [0,1]$ depending on (u_0, v_0).

Pf. Consider

$$W(u,v) = \int_\Omega \left[u - \tfrac{1}{2} - \tfrac{1}{2}\log \tfrac{u}{1/2}\right] + \left[v - \tfrac{1}{2} - \tfrac{1}{2}\log \tfrac{v}{1/2}\right].$$

Then $$\frac{d}{dt} W(u,v) = -\int_\Omega |u+v-1|^2 - \frac{d_1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{d_2}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}.$$

In this case $E = \{(u_0,v_0) : |\nabla u_0| = |\nabla v_0| = 0,\ u_0 + v_0 \equiv 1\}$

$$\Rightarrow E = \{(s, 1-s) : s \in [0,1]\}.$$

and $M = E$.

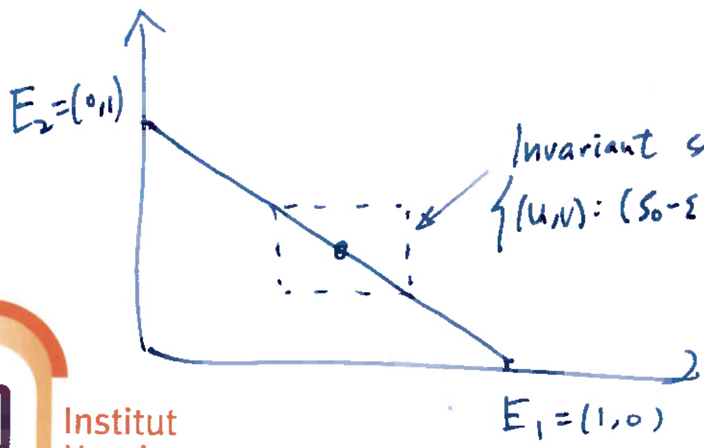
By LaSalle's Inv. prin., we have proved that

$$(u,v) \longrightarrow \{(s,(1-s)) : s \in [0,1]\}.$$

converges to the set of equilibria (quasiconvergence).

To conclude it suffices to note that omega limit set is non-ordered. Suppose $\exists\, s_0 \in [0,1]$ and $t_k \to \infty$ such that

$$(u(\cdot,t_k), v(\cdot,t_k)) \longrightarrow (s_0, 1-s_0)$$



Invariant set A_ε.
$\{(u,v) : (s_0-\varepsilon, 1-s_0+\varepsilon) \le (u,v) \le (s_0+\varepsilon, 1-s_0-\varepsilon)\}$

Slower diffuser prevails!

[Dockery-Hutson-Mischaikow-Pernarowski (1998)]

$$(7)\begin{cases} u_t - d_1\Delta u = u(m(x)-u-v) & \text{in } \Omega\times(0,\infty) \\ v_t - d_2\Delta v = v(m(x)-u-v) & \text{in } \Omega\times(0,\infty) \\ n\cdot\nabla u = 0 = n\cdot\nabla v & \text{on } \partial\Omega\times(0,\infty) \\ u(x,0)=u_0(x)\ ,\ v(x,0)=v_0(x) \end{cases}$$

Thm If $m(x) \neq$ const. and $0<d_1<d_2$

then $(u(\cdot,t), v(\cdot,t)) \longrightarrow E_1 = (\tilde{u}, 0)$ as $t\to\infty$

Pf It suffices to show two things.

① E_2 is linearly unstable

② There is no positive equilibria.

For $d>0$, and $h \in C(\bar{\Omega})$, define $\mu(d,h)$ to be the p.e.v. of

$$\begin{cases} d\Delta\phi + h\phi + \mu\phi = 0 & \text{in } \Omega \\ n\cdot\nabla\phi = 0 & \text{on } \Omega \end{cases}$$

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Step #1 Let $E_2 = (0, \tilde{v})$,

when $\tilde{v}$: $\begin{cases} d_2 \Delta \tilde{v} + \tilde{v}(m - \tilde{v}) = 0 & \Omega \\ \text{Neumann b.c.} \end{cases}$

Claim $m - \tilde{v} \not\equiv$ const.

Otherwise, $\Delta \tilde{v} = 0 \ \Omega \quad n \cdot \nabla \tilde{v} = 0 \Rightarrow \tilde{v} \equiv$ const.

$\Longrightarrow m = (m - \tilde{v}) + \tilde{v}$ constant as well!

Step #2. E_2 is unstable.

Consider μ_{E_2}: $\begin{cases} d_1 \Delta \phi + (m - \tilde{v})\phi + \mu_{E_2} \phi = 0 & \text{in } \Omega \\ \text{Neumann b.c.} \end{cases}$

Observe that $\mu(d_2, m - \tilde{v}) = 0$ (characterization of p.e.v.)

and $\mu_{E_2} = \mu(d_1, m - \hat{v}) \underset{\uparrow}{<} \mu(d_2, m - \hat{v}) = 0$

$m - \tilde{v} \not\equiv$ const.

μ monotone in d.

Step #3 There is no positive equilibria.

Suppose $(\hat{u}, \hat{v})$ is a positive equilibrium.

$\Longrightarrow \mu(d_i, m - \hat{u} - \hat{v}) = 0$ for $i = 1, 2$.

But this is impossible, since $m - \hat{u} - \hat{v} \not\equiv$ const (devoir)

and hence $d \longmapsto \mu(d, m - \hat{u} - \hat{v})$ is strictly increasing.

Compression Result [Hess-Lazer 1991]

Thm If E_1 and E_2 are linearly stable, then (1) has positive equilibria

$$E_* = (u_*, v_*) \quad \text{and} \quad E^* = (u^*, v^*)$$

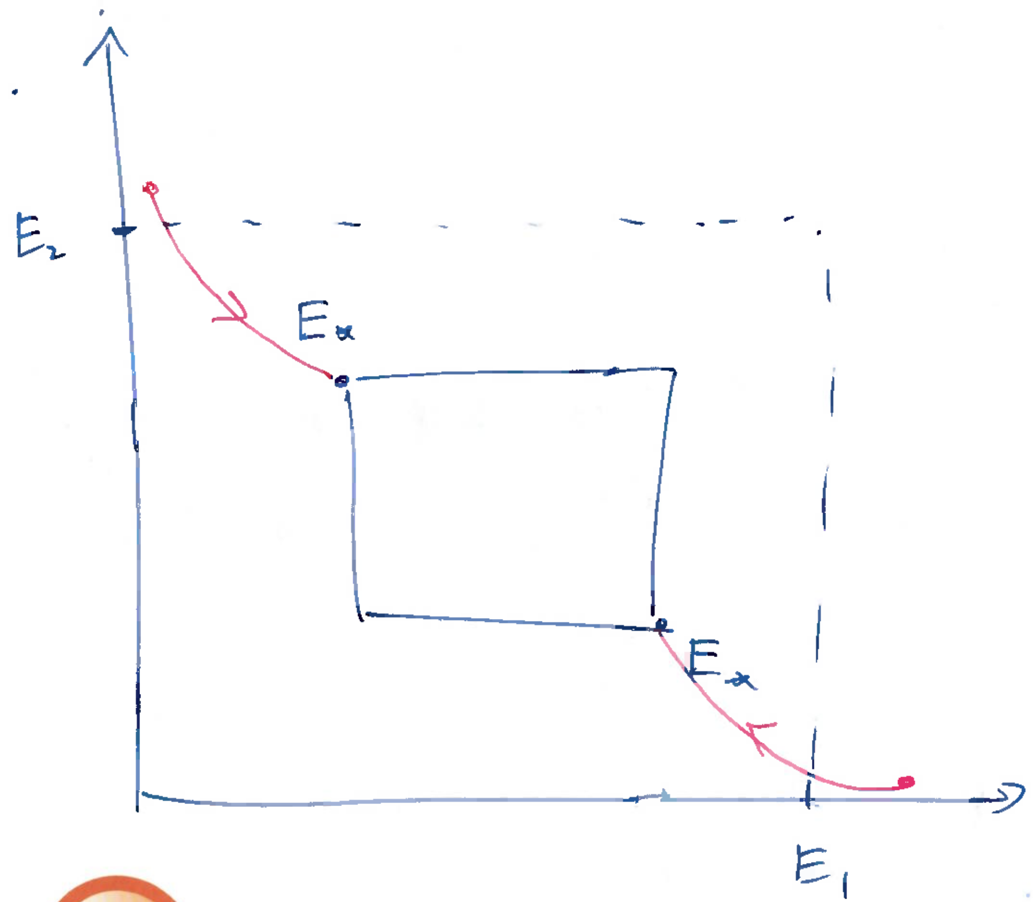
such that $E_2 \ll E_* \le E^* \ll E_1$, and

$$u_* \le \liminf_{t\to\infty} u(\cdot,t) \le \limsup_{t\to\infty} u(\cdot,t) \le u^*, \quad v_* \le \liminf_{t\to\infty} v(\cdot,t) \le \limsup_{t\to\infty} v(\cdot,t) \le v^*.$$

for all sol. (u,v) with $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$.

In particular, if $E_* = E^*$, then $(u,v) \longrightarrow E_*$.

pf.



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Uniqueness Criterion

Thm Assume

(i) E_1 and E_2 are linearly unstable

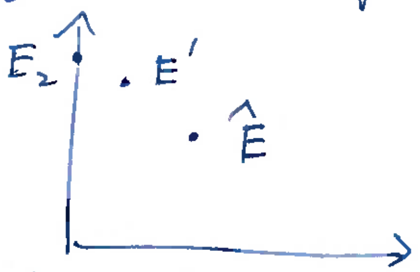
(ii) Every positive equilibrium is locally asymp. stable

Then (1) has a unique positive equilibrium $\hat{E}$.

Furthermore, $(u,v) \longrightarrow \hat{E}$ for all (u,v) with $u_0 \not\equiv 0, v_0 \not\equiv 0$

Pf By (i), $\exists$ at least one positive eq. $\hat{E}$, and $E_2 \ll \hat{E} \ll E_1$.

Let E' be a maximal equilibrium in $[E_2, \hat{E}] \setminus \{\hat{E}\}$.



This is possible since $\hat{E}$ is isolated, and the set of such equilibria is compact.

Then $E' \ll_K E^*$ and $[E', \hat{E}] \setminus \{E', \hat{E}\}$ contains no equilibrium.

It follows from the Dancer-Hess lemma (for semiflow) that there is a connecting orbit from E' to $\hat{E}$.

$\Rightarrow E'$ is unstable $\Rightarrow E' = E_2$.

Arguing similarly, $[\hat{E}, E_1] \setminus \{\hat{E}, E_1\}$ contains no equilibrium.

Hence, we may apply previous theorem with

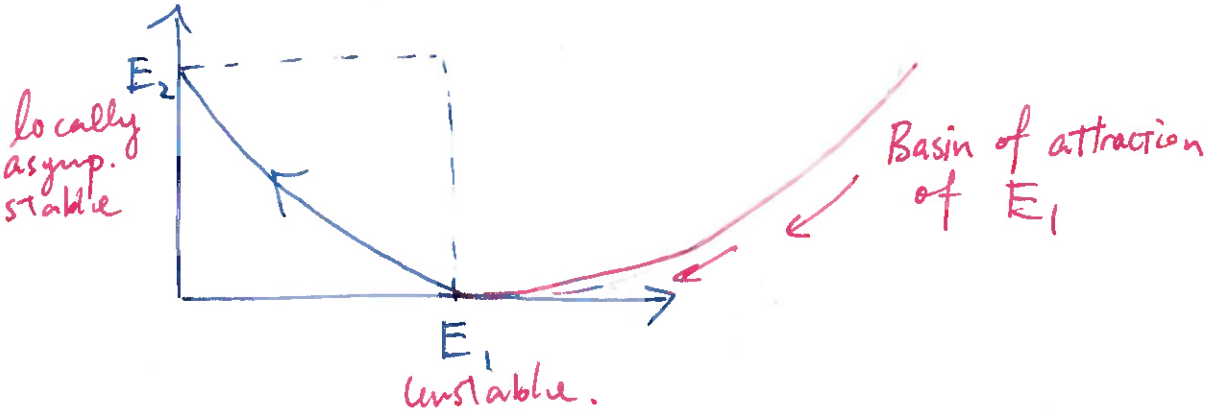
$$E^* = E_* = \hat{E}$$

to conclude. #

<u>Remark 1</u> The linear instability condition cannot be relaxed for general competition systems [Hsu-Smith-Waltman (1996)].

i.e. Nonexistence of positive eq $\not\Rightarrow$ one of E_i is globally attractive.

$$\begin{cases} \dfrac{d}{dt} x_1 = x_1(1-x_1-x_2) \\ \dfrac{d}{dt} x_2 = x_2(1-\mu x_1 - x_2)^3 \end{cases} \quad \text{where } \mu > 1.$$



The proof is left as an exercise.

However, we can show the following stronger trichotomy result for a class of models including the LV model of Reaction-diffusion type.

Theorem Consider (1), exactly one of the following hold.

(i) There exists at least one positive equilibrium.

(ii) E_1 attracts all solution (u,v) with $u_0 \neq 0$, $v_0 \neq 0$.

(iii) E_2 ——— " ———

Corollary If, in addition, every positive equilibrium is locally asymptotically stable, then (i) can be strengthened to

(i') There exists a unique positive equilibrium (u^*, v^*). Moreover, $(u,v) \to (u^*, v^*)$ if $u_0 \neq 0$ and $v_0 \neq 0$.

See Appendix E of Lecture Notes for a detailed presentation of [Hsu-Smith-Waltman (1996)], along with an improvement to prove the above results.

See also [L.-Munther (2016)]

Application. [He - Ni C.P.A.M. (1996)]

Consider

$$(8)\begin{cases} u_t - d_1\Delta u = u(m(x) - u - cv) & \Omega\times(0,\infty) \\ v_t - d_2\Delta v = u(m(x) - bu - v) & \Omega\times(0,\infty) \\ \text{Neumann b.c.} & \partial\Omega\times(0,\infty) \\ u(x,0)=u_0(x) \quad v(x,0)=v_0(x). \end{cases}$$

Thm Suppose $0<bc<1$, then exactly one of (i'), (ii), (iii) holds.

Pf. Suppose (8) has a positive equilibrium $(\tilde u, \tilde v)$.

It remains to show that $(\tilde u, \tilde v)$ is linearly stable.

Consider the eigenvalue problem

$$\begin{cases} -d_1\Delta\tilde\varphi = \tilde\varphi(m-2\tilde u - c\tilde v) - c\tilde u\tilde\psi + \mu_1\tilde\varphi \\ -d_2\Delta\tilde\psi = -b\tilde v\tilde\varphi + \tilde\psi(m - b\tilde u - 2\tilde v) + \mu_1\tilde\psi \\ \text{Neumann b.c.} \end{cases}$$

Set $\tilde\varphi = \tilde u\varphi$ and $\tilde\psi = -\tilde v\psi$

then
$$-d_1\Delta\tilde\varphi = -d_1\varphi\Delta\tilde u - 2d_1\nabla\varphi\nabla\tilde u - d_1\tilde u\Delta\varphi$$
$$= \varphi\tilde u(m-\tilde u - c\tilde v) - \frac{d_1}{\tilde u}\nabla\cdot[\tilde u^2\nabla\varphi]$$

$$-\frac{d_1}{\tilde u}\nabla\cdot[\tilde u^2\nabla\varphi] = -\tilde u^2\varphi + c\tilde u\tilde v\psi + \mu_1\tilde u\varphi$$

Similarly, we obtain

$$\frac{d_2}{\tilde v}\nabla\cdot[\tilde v^2\nabla\psi] = -b\tilde v\tilde u\varphi + \tilde v^2\psi - \mu_1\tilde v\psi.$$

$$\begin{cases} -d_1 \nabla\cdot[\tilde{u}^2 \nabla\varphi] = -\tilde{u}^3\varphi + c\tilde{u}^2\tilde{v}\psi + \mu_1 \tilde{u}^2\varphi \\ -d_2 \nabla\cdot[\tilde{v}^2\nabla\psi] = b\tilde{v}\tilde{u}\varphi - \tilde{v}^3\psi + \mu_1\tilde{v}^2\psi. \\ \text{Neuman b.c.} \end{cases}$$

This is an irreducible, cooperative system, and it has a principle eigenvalue $\mu_1 \in \mathbb{R}$ s.t. $\mu_1 < \mathrm{Re}\,\mu$ $\forall \mu \in \mathbb{C}$.
Moreover, one can choose $\varphi > 0$, $\psi > 0$ in $\overline{\Omega}$.

Claim $\mu_1 > 0$

Suppose $\mu_1 \le 0$. Mult. the 1st eq by $\tilde{\varphi}^2$ and integrate

$$2d_1 \int \tilde{u}^2 \varphi |\nabla\varphi|^2 = -\int(\tilde{u}\varphi)^3 + c\int(\tilde{u}\varphi)^2(\tilde{v}\psi) + \mu_1\int \tilde{u}^2\varphi^3.$$

Since $\mu_1 \le 0$, we have

$$\int(\tilde{u}\varphi)^3 \le c\int(\tilde{u}\varphi)^2(\tilde{v}\psi) \le c\left[\int_\Omega (\tilde{u}\tilde{\varphi})^3\right]^{2/3}\left[\int(\tilde{v}\psi)^3\right]^{\frac{1}{3}}$$

$$\Rightarrow \int(\hat{u}\varphi)^3 \le c^3\int(\tilde{v}\psi)^3.$$

Similarly, $\displaystyle\int(\tilde{v}\psi)^3 \le b^3\int(\tilde{u}\varphi)^3$.

$$\Rightarrow \quad 1 \le b^3c^3 \quad \text{contradictions}.$$

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Lecture 6

Competitive Systems in Ordered Banach Space

Let X_i be an ordered Banach space with a solid cone X_i^+
$\mathrm{Int}\, X_i^+ \neq 0$

$X^+ = X_1^+ \times X_2^+$, $K = X_1^+ \times (-X_2^+)$.

(e.g. $X = C(\bar\Omega) \times C(\bar\Omega)$, $X_i^+ = C(\bar\Omega; [0,\infty))$).

Consider a continuous map $S: X^+ \to X^+$ with properties...

(H1) S is compact and strictly monotone w.r.t. K.

(H2) $E_0 = (0,0)$ is a fixed pt. It is ejective,
i.e. $\exists\, U$ s.t. for each $x \in U \setminus \{E_0\}$, $S^n(x) \in X \setminus U$ for some $n(x)$.

(H3) $S(X_1^+ \times \{0\}) \subset X_1^+ \times \{0\}$.
$\exists\, \hat{x}_1 \in \mathrm{Int}\, X_1^+$ s.t. $E_1 = (\hat{x}_1, 0)$ attracts nonzero sol. in $X_1^+ \times \{0\}$.
i.e. $S^n(x_1, 0) \longrightarrow E_1$ provided $x_1 \neq 0$.
Symmetric condition hold for $E_2 = (0, \hat{x}_2)$.

(H4) If $x, y \in X^+$, $x < y$ and ($x \in \mathrm{Int}\, X^+$ or $y \in \mathrm{Int}\, X^+$)
Then $S(x) \ll S(y)$.

E.g. 1. Lotka-Volterra competition system.

$$\begin{cases} u_t + \mathcal{L}u = u(a_1 - b_1 u - c_1 v) \\ v_t + \mathcal{L}v = v(a_2 - b_2 u - c_2 v). \end{cases}$$

a_i, b_i, c_i autonomous or periodic in time.

Theorem [Hsu-Smith-Waltman (1996)]

Let (H1)-(H4) hold,

Let $I = [E_2, E_1] = \{(x_1, x_2) : 0 \le x_1 \le \hat{x}_1,\ 0 \le x_2 \le \hat{x}_2\}$.

Then • $\omega(x) \subseteq I$ for each $x \in X^+$

• If S has no fixed pts in $I \setminus \{E_0, E_1, E_2\}$, then for each $x \neq (0,0)$,

$$S^n(x) \longrightarrow E_1 \quad \text{or} \quad S^n(x) \longrightarrow E_2.$$

Moreover, one of E_1, E_2 attracts all of $\{(x_1, x_2) \in I : x_i \neq 0\}$.

Pf. Given $x = (x_1, x_2) \in X^+$, such that $x_1 \neq 0$, $x_2 \neq 0$.

$$\begin{array}{ccccc} (0, x_2) & \le & (x_1, x_2) & \le & (x_1, 0) \\ \downarrow S^n & & & & \downarrow S^n \\ E_2 & \le & \omega(x) & \le & E_1 \end{array} \Rightarrow \omega(x) \subseteq I.$$

If there is no other fixed pts: Apply Dancer-Hess Lemma

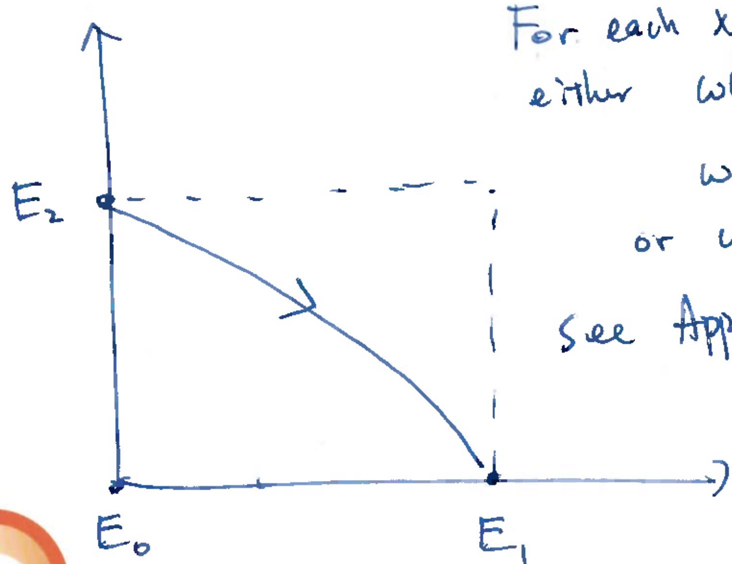
For each $x = (x_1, x_2)$, $x_1 \neq 0$, $x_2 \neq 0$.

either $\omega(x) \subseteq X_1^+ \times \{0\}$

$\omega(x) \subseteq \{0\} \times X_2^+$

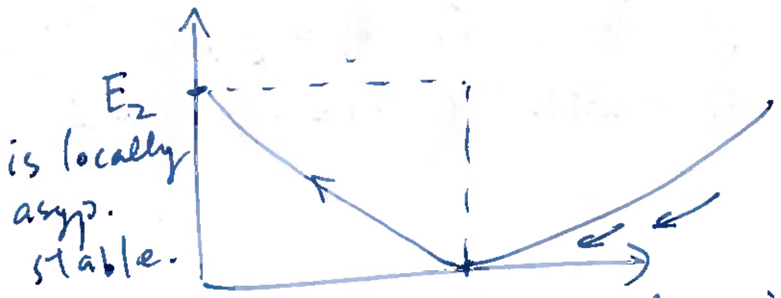
or $\omega(x) \cap \operatorname{Int} I \neq \emptyset$.

See Appendix E for details.



Remark Nonexistence of positive equilibria does not imply E_1 or E_2 is globally attractive in $\{(x_1, x_2) : x_1 \neq 0,\ x_2 \neq 0\}$.

$$\begin{cases} x_1' = x_1(1 - x_1 - x_2) \\ x_2' = x_2(1 - \mu x_1 - x_2)^3 \end{cases} \quad \text{when } \mu > 1.$$



E_1 unstable. with non-trivial basin of attraction (Devoir).

Lem If E_1 is linearly unstable, then it is repelling.
i.e. $S^n(x) \not\to E_1$ if $x = (x_1, x_2)$, $x_1 \neq 0$.

Pf. See prop. E.1 in the book.