# The principal eigenvalue of periodic-parabolic problems.

Consider

$$(1)\quad \begin{cases} \partial_t \varphi - d\Delta\varphi = c(x,t)\varphi + \lambda\varphi & \text{in } \Omega\times\mathbb{R} \\ n\cdot\nabla\varphi = 0 & \text{on } \partial\Omega\times\mathbb{R} \\ \varphi(x,t) = \varphi(x,t+1) & \text{in } \Omega\times\mathbb{R}. \end{cases}$$

where $c \in C(\overline{\Omega}\times\mathbb{R})$ and $c(x,t) = c(x,t+1)$.

It follows from the Krein-Rutman Theorem that (1) has a p.e.v. $\lambda_1 = \lambda(d,c)$.

## Examples

- $c = c(t)$, for some 1-periodic $c(t)$, then $\varphi = \varphi(t)$

$$\Rightarrow \varphi'(t) = c(t)\varphi(t) + \lambda_1\varphi(t)$$

$$\Rightarrow \int_0^1 \frac{\varphi'}{\varphi} = \int_0^1 c + \lambda_1 \Rightarrow \lambda_1 = -\int_0^1 c(s)\,ds.$$

- $c = c(x)$, then $\varphi = \phi(x)$ and.

$$\begin{cases} -d\Delta\phi(x) = c(x)\phi(x) + \lambda_1\phi(x) & \text{in } \Omega \\ n\cdot\nabla\phi = 0 & \text{on } \partial\Omega \end{cases}$$

$\Rightarrow \lambda_1 = \mu(d, c(\cdot))$. the p.e.v. of elliptic problem.

- $c(x,t) = a(x) + b(t)$ $\quad$ WLOG $\int_0^1 b = 0$.

$\lambda_1 = \mu(d, a(x))$ and $\varphi(x,t) = \phi(x)\, e^{\int_0^t b(s)\,ds}$

Indeed $\hat{\varphi}(x,t) = \varphi(x,t)e^{-\int_0^t b}$ satisfies

$$\partial_t\hat{\varphi} - d\Delta\hat{\varphi} = e^{-\int_0^t b}\left(\partial_t\varphi - b(t)\varphi - d\Delta\varphi\right) = e^{-\int_0^t b}\left(a(x)\varphi + \lambda_1\varphi\right)$$

$$= a(x)\hat{\varphi} + \lambda_1\hat{\varphi}.$$

The following says that $\lambda_1$ is a weighted average of $c(x,t)$.

<u>Lemma</u> ($L^\infty$-bound).

$$-\int_0^1 \max_{x\in\bar\Omega} c(x,t)\,dt \le \lambda_1 \le -\frac{1}{|\Omega|}\int_0^1\int_\Omega c(x,t)\,dx\,dt$$

and both equality holds iff $c=c(t)$.

<u>Pf</u> Normalize $\iint \varphi = 1$. and integrate (1). in $x$

$$\frac{d}{dt}\int_\Omega \varphi = \int_\Omega c\varphi + \lambda_1 \int_\Omega \varphi .$$

$$\frac{d}{dt}\int_\Omega \varphi \le \max_{\bar\Omega} c(\cdot,t)\int_\Omega \varphi + \lambda_1 \int \varphi .$$

Integrate in $t$ $\Rightarrow$ $\lambda_1 \ge -\int_0^1 \max_{x\in\bar\Omega} c(x,t)\,dt$.

Note that equality holds

$$\Leftrightarrow \int_0^1\int_\Omega \left[\max_{\bar\Omega} c(\cdot,t) - c(x,t)\right]\varphi(x,t)\,dx\,dt = 0$$

$$\Leftrightarrow \quad c(x,t) \equiv \max_{\bar\Omega} c(\cdot,t) \quad \forall t$$

$$\Leftrightarrow \quad c \equiv c(t).$$

For the 2nd ineq., divide (1) by $\varphi$ and integrate

$$\int_0^1\int_\Omega \left(\partial_t \log\varphi - d\frac{\Delta\varphi}{\varphi}\right) = \int_0^1\int_\Omega (c(x,t)+\lambda_1).$$

$$0 \ge -d\int_0^1\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2} = \iint c\,dx\,dt + |\Omega|\lambda_1$$

Equality holds iff $\nabla_x\varphi \equiv 0$ iff $c=c(t)$.

# The Hutson-Shen-Vickers Lemma

Let $\hat{\mu}$ be the p.e.v. of the elliptic problem.

$$(2)\begin{cases} -d\Delta\phi = \hat{c}(x)\phi + \hat{\mu}\phi & \text{in } \Omega \\ n\cdot\nabla\phi = 0 & \text{on } \partial\Omega \end{cases}$$

where $\hat{c}(x) = \int_0^1 c(x,t)dt$.

<u>Lemma</u>. $\lambda_1 \le \hat{\mu}$ for all $d>0$.

The equality holds iff $c(x,t) = \hat{c}(x) + b(t)$.

<u>Pf</u>. For simplicity set $d=1$ and let

$$w(x) = e^{\int_0^1 \ln\varphi(x,t)dt}.$$

Then $w_{x_i} = w\int_0^1 \frac{\varphi_{x_i}}{\varphi}dt$.

$$w_{x_ix_i} = w\left(\int_0^1 \frac{\varphi_{x_i}}{\varphi}dt\right)^2 + w\int_0^1 \frac{\varphi_{x_ix_i}\varphi - (\varphi_{x_i})^2}{\varphi^2}$$

$$\le w\int_0^1 \frac{\varphi_{x_ix_i}}{\varphi} \quad \text{by Cauchy-Schwartz.}$$

Here, $n\cdot\nabla w = 0$ on $\partial\Omega$, and

$$\Delta w \le w\int_0^1 \frac{\Delta\varphi}{\varphi}$$

$$\le w\int_0^1 \frac{\varphi_t - (c+\lambda_1)\varphi}{\varphi}$$

$$= -w\int_0^1 (c+\lambda_1) = -w\left(\hat{c}(x) + \lambda_1\right)$$

Let $\phi(x)$ denote an eigenfcn of (2).

$$\int_\Omega (\hat{c}(x)+\hat{\mu})\phi w = -d\int_\Omega (\Delta\phi) w$$

$$= -d\int_\Omega \phi \Delta w$$

$$\geq \int_\Omega (\hat{c}(x)+\lambda_1)\phi w \Rightarrow \lambda_1 \leq \hat{\mu}.$$

Equality holds $\Leftrightarrow$ $\nabla \ln \varphi$ is indep. of $t$.

$\Leftrightarrow$ $\ln\varphi = a(x) + b(t)$

$\Leftrightarrow$ $\varphi = u(x)v(t)$

where $v'(t)u(x) - v(t)\Delta u(x) = c(x,t)u(x)v(t) + \lambda_1 u(x)v(t)$.

Divide by $v(t)>0$ and integrate,

$\Rightarrow -\Delta u(x) = \hat{c}(x)u(x) + \lambda_1 u(x)$

$\Rightarrow v'(t)\cancel{u(x)} = (c(x,t)-\hat{c}(x))\cancel{u(x)}v(t)$

$\Rightarrow c(x,t) - \hat{c}(x) = \dfrac{v'(t)}{v(t)}$.

On the other hand, if $c(x,t) = \hat{c}(x) + b(t)$

one can show that $\varphi(x,t) = u(x)v(t)$.

We omit the details.

w/ temporal variation $\hookrightarrow \lambda_1 \le \hat{\mu} \hookleftarrow$ w/o temporal variation

Biological interpretation

The addition of temporal variation tends to destabilize an equilibrium under all circumstance.

$\Longrightarrow$ Spatio-temporal heterogeneity always favor persistence.

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Frequency $\omega$. Let $\lambda(\omega)$ and $\varphi_\omega$ be the p.e.v, p.e.f of.

Consider
$$\begin{cases} \partial_t\varphi - d\Delta\varphi = c(x,\omega t)\varphi + \lambda(\omega)\varphi & \Omega\times\mathbb{R} \\ n\cdot\nabla\varphi = 0 & \partial\Omega\times\mathbb{R} \\ \varphi(x,0) = \varphi(x,\tfrac{1}{\omega}) = \varphi(x,\tfrac{2}{\omega}) = \dots & \Omega\times\mathbb{R}. \end{cases}$$

Normalize $\sup \varphi_\omega = 1$,

$L^p$ estimate $\Rightarrow$ $\|\varphi_\omega\|_{W^{2,p}(\Omega\times\mathbb{R})} \le C$ and $\|\varphi_\omega\|_{C^{\alpha,\alpha/2}(\bar{\Omega}\times\mathbb{R})} \le C$.

We may pass to the limit $\omega \to \infty$

$$\varphi_\omega(x,t) \longrightarrow \varphi_\infty(x), \quad \lambda(\omega) \longrightarrow \lambda_\infty$$

where $\lambda_\infty$, $\varphi_\infty$ are p.e.v./p.e.f of the elliptic problem

$$-d\Delta\int_0^{[\omega]/\omega}\varphi_\omega = \int_0^{[\omega]/\omega} c(x,t)\varphi_\omega(x,t)\,dt + \lambda(\omega)\int_0^{[\omega]/\omega}\varphi_\omega$$

$$\to \begin{cases} -d\Delta\varphi_\infty(x) = \hat{c}(x)\varphi_\infty(x) + \lambda_\infty\varphi_\infty(x) & \text{in } \Omega \\ n\cdot\nabla\varphi_\infty & \text{on } \partial\Omega \end{cases}$$

$$\Longrightarrow \lambda_\infty = \hat{\mu}$$

The previous lemma says that $\lambda(\omega) < \lambda(+\infty) \quad \forall \omega > 0$.

Q [Hutson et al] Is $\omega \mapsto \lambda(\omega)$ monotone increasing in $\omega$?

A [Liu-Lou-Peng-Zhou (2019)] Yes.

Thm [L-L-P-Z] $\lambda(\omega)$ is non-decreasing in $\omega > 0$.
Furthermore, (i) $c = \int_0^1 c(x,t)\,dt + g(t) \Rightarrow \lambda(\omega)$ is constant in $\omega$.
(ii) otherwise, $\frac{\partial \lambda}{\partial \omega}(\omega) > 0 \quad \forall \omega > 0$.

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→ Increasing frequency of temporal fluctuation harms persistence.

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See, Ch. 7 in Lecture notes for the proof.

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Go back to the case $\omega = 1$.

$$(1) \quad \begin{cases} \partial_t \varphi - d\Delta\varphi = c(x,t)\varphi + \lambda_1 \varphi & \Omega \times \mathbb{R} \\ n \cdot \nabla \varphi = 0 & \partial\Omega \times \mathbb{R} \\ \varphi(x,t) = \varphi(x,t+1) & \Omega \times \mathbb{R} \end{cases}$$

Lemma (Small diffusion limit)

$$\lim_{d\to 0} \lambda_1 = -\max_{x\in\bar\Omega} \int_0^1 c(x,t)\,dt = -\max_{\bar\Omega} \hat{c}(x).$$

Pf. By Hutson-Shen-Vickers' lemma,

$$\lim_{d\to 0} \lambda_1 \le \lim_{d\to 0} \mu(d, \hat{c}(x)) = -\max_{\bar\Omega} \hat{c}.$$

Need to show $\liminf_{d\to 0} \lambda_1 \geq -\max_{\bar\Omega} \hat{c}$.

Suppose not, then pass to a seq $d_k \to 0$.

$$\lambda_1 \to \lambda_0 = \liminf_{d\to 0} \lambda_1.$$

Normalize $\sup_{\bar\Omega\times\mathbb{R}} \varphi_d = 1$ and $\varphi_d(x_d, t_d) = 1$ $\exists x_d \in \bar\Omega, 0 \le t_d \le 1$.

Set $x = x_d + \sqrt{d}\, y$, and define

$$\psi_d(y,t) = \varphi_d(x,t).$$

Then $0 \le \psi_d \le 1$, $\psi_d(0, t_d) = 1$, and

$$\begin{cases} \partial_t \psi_d - \Delta \psi_d = c(x_d + \sqrt{d}\, y, t)\psi_d + \lambda_1 \psi_d & \Omega_d \times [0,1] \\ n\cdot\nabla\psi_d = 0 & \partial\Omega_d \times [0,1] \\ \psi_d(y,0) = \psi_d(y,1) & \text{in } \Omega_d. \end{cases}$$

where $\Omega_d = \{y : x_d + \sqrt{d}\, y \in \Omega\}$.

Pass to a seq s.t. $x_d \to x_0 \in \overline{\Omega}$ $\lambda_1 \to \lambda_0$

Case ① $x_0 \in \Omega$.

then $\psi_d \to \psi_0$ in $C^{2,1}_{loc}(\mathbb{R}^n \times [0,1])$.

$$\begin{cases} \partial_t \psi_0 - \Delta \psi_0 = c(x_0,t)\psi_0 + \lambda_0 \psi_0 & \text{in } \mathbb{R}^n \times [0,1] \\ \psi(x,0) = \psi(x,1) & \text{in } \mathbb{R}^n \\ \psi(0,0) = \psi(0,1) = 1 & 0 \le \psi(x,t) \le 1 \end{cases}$$

let $p(t) = \exp\left( \int_0^t (c(x_0,s) + \lambda_0)\, ds \right)$.

then $p'(t) = (c(x_0,t) + \lambda_0)\, p(t)$, $p(0) = 1 = \sup \psi(\cdot,0)$

By comparison principle,

$p(1) \geq \sup \psi(\cdot,1) = 1$

$\Rightarrow \int_0^1 c(x_0,t)\,dt + \lambda_0 \geq 0$.

$\Rightarrow \liminf_{d\to 0} \lambda_1 = \lambda_0 \geq -\int_0^1 c(x_0,t)\,dt \geq -\max_{\Omega} \hat{c}(x)$.

## Lemma (The large diffusion limit).

$$\lim_{d\to\infty} \lambda_1 = \frac{1}{|\Omega|}\int_\Omega \int_0^1 c(x,t)\,dt\,dx.$$

Pf. Set $\int_\Omega\int_0^1 \varphi^2 = 1$.

Mult. by $\varphi$ and integrate

$$\frac{1}{2}(\varphi^2)_t - d\,\varphi\Delta\varphi = (c+\lambda_1)\varphi^2$$

$$d\iint |\nabla_x \varphi|^2 = \iint (c+\lambda_1)\varphi_1^2$$

Set $\Phi(x,t) = \varphi(x,t) - \frac{1}{|\Omega|}\int_\Omega \varphi(y,t)\,dy$

By Poincaré's Ineq.

$$c_0 \iint |\Phi|^2 \leq \iint |\nabla_x \Phi|^2 = \iint |\nabla_x\varphi|^2 \leq C_1/d.$$

Integrate (1) in $\Omega$

$$\partial_t \int_\Omega \varphi = \int_\Omega (c+\lambda_1)\varphi \qquad \forall t>0$$

Set $\bar\varphi(t) = \int_\Omega \varphi(x,t)\,dx$ and write $\varphi(x,t) = \bar\varphi(t) + \Phi(x,t)$,

$$\frac{d}{dt}\bar\varphi = \int_\Omega (c+\lambda_1)\bar\varphi(t) + \int_\Omega (c+\lambda_1)\Phi.$$

$|\lambda_1| \leq C$

$$\Rightarrow \iint |(c+\lambda_1)\Phi| \leq C_2 \iint |\Phi| \leq C_3\left(\iint |\Phi|^2\right)^{1/2} = o(1)$$

$$\exp\left(-\int_0^t\int_\Omega (c+\lambda_1)\right)\bar\varphi(t) = \bar\varphi(0) + o(1)$$

Since $\bar{\varphi}(0) = \bar{\varphi}(1)$,

$$\exp\left( - \int_0^1 \int_\Omega (c+\lambda_1) \right) \bar{\varphi}(0) = \bar{\varphi}(0) + o(1).$$

$\Rightarrow$ either $\bar{\varphi}(0) = \bar{\varphi}(1) \to 0$ or $\lambda_1 \to \frac{1}{|\Omega|} \iint_{\Omega}{}_0^1 c$.

<u>Claim</u> $\bar{\varphi}(0) \not\to 0$

Suppose $\bar{\varphi}(0) \to 0$, then $\sup\limits_{[0,1]} \bar{\varphi}(t) \to 0$.

Since $\iint |\varphi(x,t) - \bar{\varphi}(t)|^2 \to 0$,

$\Rightarrow \iint |\varphi(x,t)|^2 \to 0$ which is impossible in view of the normalization $\iint |\varphi|^2 = 1$.

This proves the Lemma.

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See [Bai-He-Ni, J.E.M.S., accepted] for higher order expansions of $\lambda_1$ in $d \gg 1$.

[L.-Lou] for results related to principal Floquet bundle.

**Cor** [Thm 2.2 (a) [Hutson et al. JMB (2001)]].

Suppose $c \not\equiv c(t)$ and $\hat{c}(x) = \text{const}$.

then (i) $\exists\, d > 0$ s.t. $\partial_d \lambda(d, c) < 0$ $\exists\, d$.

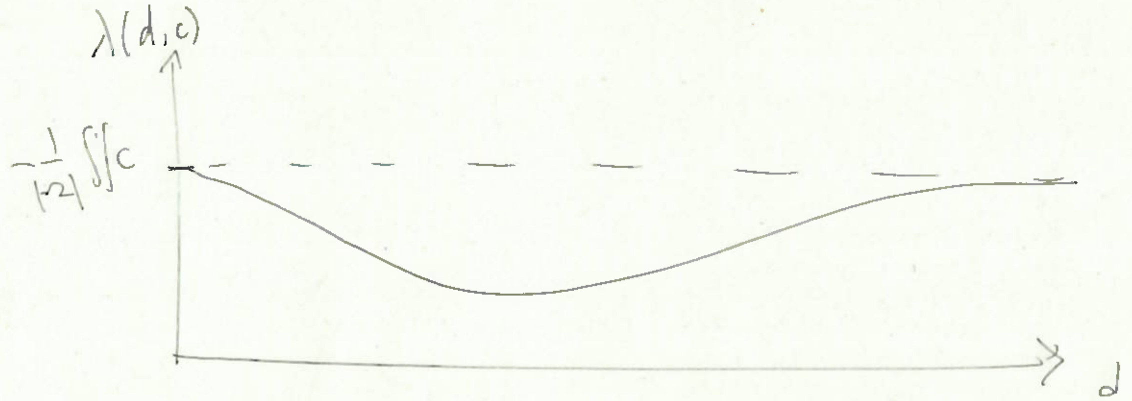
(ii) $\exists\, 0 < d_1 < d_2$ s.t. $\lambda(d_1, c) = \lambda(d_2, c)$.

Pf. • $\lambda(d, c) < \frac{1}{|\Omega|} \int_\Omega \int_0^1 c \, dt \, dx$ $\boxed{c \not\equiv c(t)}$

• $\lim_{d \to 0} \lambda(d, c) = - \max_{\Omega} \int_0^1 c(x,t) \, dt = \frac{-1}{|\Omega|} \iint c \, dx \, dt$.

$\boxed{\hat{c}(x) \equiv \text{const}}$

• $\lim_{d \to \infty} \lambda(d, c) = - \frac{1}{|\Omega|} \iint c \, dx \, dt$.



## Competition in Time-periodic Environment.

$$(*)\quad \begin{cases} u_t - d_1 \Delta u = u\,(m(x,t) - u - v) & \Omega \times (0,\infty) \\ v_t - d_2 \Delta v = v\,(m(x,t) - u - v) & \Omega \times (0,\infty) \\ n\cdot \nabla_x u = n \cdot \nabla_x v = 0 & \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \Omega \end{cases}$$

where $m(x,t)$ , 1-periodic in $t$.

Theorem [Hutson-Mischaikow-Polacik (2001)].

There are $m(x,t)$, and $0 < d_1 < d_2$ such that.

- $E_1 = (\tilde{u}, 0)$, $E_2 = (0, \tilde{v})$ exists.
- $E_2$ is globally asymptotically stable among solutions with initial data $u_0 \neq 0, v_0 \neq 0$.

## Step 1

Fix a $c_1(x,t)$ and $d_1 > 0$ s.t. $\partial_d \lambda(d_1, c_1) < 0$.

and let $\varphi_1(x,t)$ be the corresponding eigenfcn.

For $\varepsilon$ small, set

$$\hat{u}(x,t) = \varepsilon \varphi_1(x,t)$$

$$m(x,t) = c_1(x,t) + \lambda(d_1, c_1) + \varepsilon \varphi_1(x,t),$$

then $\hat{u}(x,t)$ is the unique positive periodic sol. of the single species problem.

$$\begin{cases} u_t - d_1 \Delta u = u(m(x,t) - u) & \text{in } \Omega \times (0,\infty) \\ n \cdot \nabla u = 0 & \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x). & \Omega. \end{cases}$$

and $E_1 = (\hat{u}, 0)$ is boundary attractor of the system (*).

## Step 2. For $d_2 = d_1 + \varepsilon$,

$E_1 = (\hat{u}, 0)$ is linearly unstable.

Pf. To show the p.e.v. $\lambda_{E_1}$ of

$$\begin{cases} \phi_t - d_2 \Delta \phi = (m(x,t) - \hat{u})\phi + \lambda_{E_1} \phi & \text{in } \Omega \times [0,1] \\ n \cdot \nabla \phi = 0 \quad \partial\Omega \times [0,1], & \phi(x,0) = \phi(x,1) \text{ in } \Omega \end{cases}$$

satisfies $\lambda_{E_1} < 0$.

Indeed,
$$\begin{aligned} \lambda_{E_1} &= \lambda(d_2, m - \hat{u}) \\ &= \lambda(d_1 + \varepsilon, c_1 + \lambda(d_1, c_1)) \\ &= \lambda(d_1 + \varepsilon, c_1) - \lambda(d_1, c_1). \\ &< 0. \end{aligned}$$

<u>Step 3</u> For $d_2 = d_1 + \varepsilon$, $0 < \varepsilon \ll 1$,

(*) has no positive periodic sol.

Pf. Suppose to the contrary that

$\exists\ \varepsilon_k \to 0+$ s.t. (*) with $(d_1, d_2) = (d_1, d_1 + \varepsilon_k)$

has positive periodic sol. $(u_k, v_k)$.

$$(\star_k)\quad \begin{cases} \partial_t u_k + d_1 \Delta u_k = u_k (m - u_k - v_k) \\ \partial_t v_k + (d_1 + \varepsilon) \Delta v_k = v_k (m - u_k - v_k). \end{cases}$$

Since $\|u_k\|_\infty + \|v_k\|_\infty \le 2 \|m\|_\infty$ (exercise)

we may use parabolic regularity to pass to

the limit $(u_k, v_k) \longrightarrow (u_\infty, v_\infty)$ s.t.

$$\begin{cases} \partial_t u_\infty + d_1 \Delta u_\infty = u_\infty (m - u_\infty - v_\infty) \\ \partial_t v_\infty + d_2 \Delta v_\infty = v_\infty (m - u_\infty - v_\infty). \end{cases}$$

Since $u_k + v_k \not\to 0$ (exercise),

$\Rightarrow$ $u_\infty + v_\infty$ is a positive periodic sol of the single species problem

$\Rightarrow$ $u_\infty + v_\infty = \tilde{u}$

$\Rightarrow$ $m - u_k - v_k = m - \tilde{u} + o(1) = c_1(x,t) + \lambda(d_1, c_1) + o(1)$

By $(\star)_k$ $\Rightarrow$ $\lambda(d_i, m - u_k - v_k) = 0$ $\forall\, i = 1, 2$, and $k \ge 1$.

$\Rightarrow$ $\partial_d \lambda(\hat{d}_k, m - u_k - v_k) = 0$ $\exists\ \hat{d}_k \in (d_1, d_1 + \varepsilon_k)$.

Let $k \to \infty$, $\Rightarrow 0 = \partial_d \lambda(d_1, m - \hat{u}) = \partial_d (d_1, c_1 + \lambda(d_1, c_1))$

$\Rightarrow$ $0 = \partial_d \lambda(d_1, c_1)$. Contradictions.

By Steps 2 and 3, we can apply the theory of monotone dynamical systems in [Hsu-Smith-Waltman (1996)] to conclude that $E_2 = (0, \tilde{v})$ is globally asymptotically stable.

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<u>Interpretation</u>. In periodic habitat, the slower diffuser could be selected against.

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# Dockery's Problem in Time-periodic Environment

Let $0<d_1<d_2$ and consider

$$(1)\begin{cases} u_t - d_1\Delta u = u(m(x,\tau t) - u - v) & \Omega\times(0,\infty) \\ v_t - d_2\Delta v = v(m(x,\tau t) - u - v) & \Omega\times(0,\infty) \\ n\cdot\nabla u = 0 = n\cdot\nabla v & \partial\Omega\times(0,\infty) \\ u(x,0) = u_0(x),\quad v(x,0) = v_0(x) & \Omega \end{cases}$$

[Hutson-Mischaikow-Polacik (2001)].

$\boxed{\tau : \text{frequency}}$. Let $d_1<d_2$.

- Slower diffuser prevails if $d_2 \gg 1$ or $\omega$ is small or large.
- For intermediate $\omega$, the two phenotypes may coexist or the fast diffuser may win.
- For $d_2\gg 1$, see [L.-Lou (2021) preprint] for n-species competition model in generic environment.

Thm [Hutson et al. (2001) Thm 5.3].

Assume $\iint_{\Omega_T} m \, dx dt > 0$ and $\hat{m}(x) \neq \text{const}$.

Fix $0 < d_1 < d_2$.

For $\tau$ sufficiently large, $(\tilde{u}, 0)$ is globally asymptotically stable among nonneg, nontrivial solutions of (1).

Pf. Step 1. Let $E_2 = (0, \tilde{v}_\tau)$,

where $\tilde{v}_\tau$ is the unique positive $\frac{1}{\tau}$-periodic sol. of

$$(2) \quad \begin{cases} \partial_t \hat{v} - d_2 \Delta \hat{v} = \hat{v}(m(x, \tau t) - \hat{v}) & \text{in } \Omega \times \mathbb{R} \\ n \cdot \nabla \hat{v} = 0 \text{ on } \partial\Omega \times \mathbb{R}, & \hat{v}(x,t) = \hat{v}(x, t + \frac{1}{\tau}). \end{cases}$$

Claim. $\lim_{\tau \to \infty} \| \tilde{v}_\tau(x,t) - \tilde{v}(x) \|_{C^0(\bar{\Omega} \times [0, \frac{1}{\tau}])} \to 0$

where $\tilde{v}(x)$ is the unique positive equilibrium of

$$(3) \quad d_2 \Delta \tilde{v} + \tilde{v}(\hat{m}(x) - \tilde{v}) = 0 \quad \Omega \quad n \cdot \nabla \tilde{v} = 0 \text{ on } \partial\Omega.$$

Indeed, by $L^p$-estimate for parabolic eq,

$$\| \tilde{v}_\tau \|_{C^{\alpha, \alpha/2}(\bar{\Omega} \times \mathbb{R})} \le C \quad \text{uniformly in } \tau.$$

and $\hat{v}_\tau(x, \frac{n}{\tau}) = \hat{v}_\tau(x, \frac{m}{\tau}) \quad \forall n, m \in \mathbb{Z}$.

$\Rightarrow$ passing to a seq $\tau_k \to \infty$, $\tilde{v}_\tau \to \tilde{v}(x)$ in $C(\bar{\Omega} \times \mathbb{R})$.

Finally, $\tilde{v}(x)$ satisfies, in distribution sense, (3).

This proves the Claim.

Step 2. $E_2 = (0, \tilde{V}_\tau)$ is linearly unstable for $\tau \gg 1$.

Let $\lambda_\tau$, $\phi_\tau(x,t)$ be the p.e.v. / p.e.f of

$$\begin{cases} \partial_t \phi_\tau - d_1 \Delta \phi_\tau = (m(x, \tau t) - \tilde{V}_\tau(x,t))\phi_\tau + \lambda_\tau \phi_\tau & \text{in } \Omega \times \mathbb{R} \\ n \cdot \nabla \phi_\tau = 0 \quad \text{on } \partial\Omega \times \mathbb{R}, \\ \phi_\tau(x, t + \frac{1}{\tau}) = \phi(x,t) & \text{in } \Omega \times \mathbb{R}. \end{cases}$$

Then similarly, we can pass to a subsequential limit $\tau_k \to \infty$ such that

$$\lambda_\tau \longrightarrow \lambda_\infty \text{ and } \phi_\tau(x,t) \longrightarrow \phi_\infty(x) \text{ in } C(\bar{\Omega} \times \mathbb{R})$$

where $(\lambda_\infty, \phi_\infty)$ is an eigenpair of

$$\begin{cases} d_1 \Delta \phi_\infty + (\hat{m}(x) - \tilde{V}(x))\phi_\infty + \lambda_\infty \phi_\infty & \text{in } \Omega \\ n \cdot \nabla \phi_\infty = 0. & \text{on } \partial\Omega. \end{cases}$$

$\phi_\infty \underset{\neq}{\geq} 0 \Rightarrow \lambda_\infty = \mu(d_1, \hat{m}(x) - \tilde{v}(x)) < \mu(d_2, \hat{m} - \tilde{v}) = 0$

$\hat{m}(x) - \tilde{v}(x) \neq \text{const}$

$\Rightarrow \lambda_\tau < 0$ for $\tau \gg 1$.

Step 3. (1) has no positive periodic sol. if $\tau$ is large

Suppose to the contrary that

$\exists \tau_k \to \infty$, $\exists (u_k, v_k)$ $\frac{1}{\tau_k}$ periodic positive sol. of (1),

then $(u_k(x,t), v_k(x,t)) \longrightarrow (u_\infty(x), v_\infty(x))$,

where $(u_\infty, v_\infty)$ is an equilibrium of

the average problem:

$$(4)\ \begin{cases} d_1 \Delta u_\infty + u_\infty(\hat{m} - u_\infty - v_\infty) = 0 \\ d_2 \Delta v_\infty + v_\infty(\hat{m} - u_\infty - v_\infty) = 0 \\ n \cdot \nabla u_\infty = n \cdot \nabla v_\infty = 0 \end{cases}$$

But (4) has no positive equilibria.

$\Rightarrow (u_\infty, v_\infty) = (\tilde{u}(x), 0)$ or $(0, \tilde{v}(x))$.

and. $\hat{m}(x) - u_\infty - v_\infty$ is nonconstant.

$\Rightarrow \mu(d_1, \hat{m} - u_\infty - v_\infty) < \mu(d_2, \hat{m} - u_\infty - v_\infty)$.

Next, consider

$$(\varphi_k, \psi_k) = \left(\frac{u_k}{\|u_k\|_\infty}, \frac{v_k}{\|v_k\|_\infty}\right) \longrightarrow (\varphi_\infty(x), \psi_\infty(x))$$

$$\begin{cases} d_1 \Delta \varphi_\infty + \varphi_\infty(\hat{m}(x) - u_\infty - v_\infty) = 0 & \Omega \\ d_2 \Delta \psi_\infty + \psi_\infty(\hat{m}(x) - u_\infty - v_\infty) = 0 & \Omega \\ \text{Neumann b.c.} & \partial\Omega \end{cases}$$

$\rightarrow \mu(d_1, \hat{m} - u_\infty - v_\infty) = \mu(d_2, \hat{m} - u_\infty - v_\infty) = 0$.