§ The Principal Floquet Bundle

For $z \in \mathbb{R}_+$ and $c(x,t) \in C^{\beta, \beta/2}(\overline{\Omega} \times \mathbb{R})$, we say that the pair $(\varphi(x,t), H(t))$ is the principal Floquet bundle corresponding to (z, c) if they satisfy

$$
(1)\begin{cases}
\partial_t \varphi - z\Delta\varphi = c(x,t)\varphi + H(t)\varphi & \text{in } \Omega\times\mathbb{R} \\
n\cdot\nabla\varphi = 0 & \text{on } \partial\Omega\times\mathbb{R} \\
\int_\Omega \varphi(x,t)\,dx = 1 \text{ for } t\in\mathbb{R}, \quad \varphi > 0 & \text{in } \Omega\times\mathbb{R}.
\end{cases}
$$

and that $\psi(x,t)$ is the adjoint bundle if it satisfies

$$
(2)\begin{cases}
-\partial_t \psi - z\Delta\psi = c(x,t)\psi + H(t)\psi. & \text{in } \Omega\times\mathbb{R} \\
n\cdot\nabla\psi = 0 \text{ on } \partial\Omega\times\mathbb{R}, \quad \psi > 0 & \text{in } \Omega\times\mathbb{R}, \\
\int_\Omega \varphi(x,t)\psi(x,t)\,dx = 1 & \text{for } t > 0.
\end{cases}
$$

Thm. The mapping

$$\mathbb{R}_+ \times C^{\beta,\beta/2}(\bar{\Omega}\times\mathbb{R}) \longrightarrow \left[C^{2+\beta,1+\beta/2}(\bar{\Omega}\times\mathbb{R})\right]^2 \times C^{\beta/2}(\mathbb{R})$$

$$(z\ ,\ c) \longmapsto (\varphi, \psi, H)$$

is well-defined and smooth.

Ref. [Mierczynski JMAA (1997)]
[J. Húska, JDE (2006)] existence + cont. dependence.
[Cantrell-L., SIMA (2021)] smooth dependence.

Proof of existence (sketch).

For $k \in \mathbb{N}$, let (λ_k, φ_k) be the p.e.v. + p.e.f of

$$\begin{cases} \partial_t \varphi - z\Delta\varphi = c(x,t)\varphi + \lambda_k \varphi & \text{in } \Omega\times[-k,k] \\ n\cdot\nabla\varphi = 0 & \partial\Omega\times[-k,k] \\ \varphi(x,-k) = \varphi(x,k). \end{cases}$$

extend φ_k periodically in t, then

$$\hat{\varphi}_k(x,t) = e^{\lambda_k t}\varphi_k(x,t) \Big/ \|\varphi_k(\cdot,0)\|_{C(\bar{\Omega})}$$

satisfies the equation

$$(3)\begin{cases} \partial_t\hat{\varphi} - z\Delta\hat{\varphi} = c\hat{\varphi} & \text{in } \Omega\times[-k,k] \\ n\cdot\nabla\hat{\varphi} & \text{on } \partial\Omega\times[-k,k] \\ \hat{\varphi} > 0 & \text{in } \Omega\times[-k,k] \end{cases}$$

$$\frac{1}{C}e^{-\gamma|t|} \leq \frac{\hat{\varphi}_k(x,t)}{\|\hat{\varphi}_k(\cdot,0)\|_{C(\bar{\Omega})}} \leq Ce^{\gamma|t|} \quad \text{in } \Omega\times[-k+1, k-1]$$

This holds for any positive sol. of (3). (Exercise)

Here, we may pass to the limit to obtain the existence of $\tilde{\varphi}_k \longrightarrow \tilde{\varphi}$ which is a positive sol. of

$$\begin{cases} \partial_t \tilde{\varphi} - z\,\Delta \tilde{\varphi} = c\,\tilde{\varphi} & \text{in } \Omega \times \mathbb{R} \\ n \cdot \nabla \tilde{\varphi} = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ \tilde{\varphi} > 0 & \text{in } \Omega \times \mathbb{R} \end{cases}$$

Now, letting $H(t) = - \dfrac{\frac{d}{dt}\int_\Omega \tilde{\varphi}}{\int_\Omega \tilde{\varphi}} = - \dfrac{d}{dt} \log \int_\Omega \tilde{\varphi}$

and $\varphi(x,t) = \tilde{\varphi}(x,t) \exp\left(\int_0^t H(s)\,ds \right)$

we have the existence of <u>PFB</u>.

Check. $\displaystyle \int_\Omega \varphi = \int_\Omega \tilde{\varphi} \exp\left(\int_0^t H\right)$

$$= \left(\int_\Omega \tilde{\varphi}(x,t)\,dx \right) \exp\left(\log \int_\Omega \tilde{\varphi}(x,0)\,dx - \log \int_\Omega \tilde{\varphi}(x,t)\,dx \right)$$

$$= \cancel{\int_\Omega \tilde{\varphi}(x,t)\,dx} \cdot \frac{\int_\Omega \tilde{\varphi}(x,0)\,dx}{\cancel{\int_\Omega \tilde{\varphi}(x,t)\,dx}} = 1$$

Exponential separation

Let $\mathcal{L} = -z\Delta - c$

Consider the nonautonomous problem

$$(4)\begin{cases} \partial_t u + \mathcal{L}u = H_1(t)u & x \in \Omega;\ t \geq s. \\ n\cdot\nabla u = 0 & x \in \partial\Omega,\ t \geq s \\ u(x,s) = u_0(x) & x \in \Omega. \end{cases}$$

Denote the evolution operator by $\mathcal{U}(t,s)$.

s.t. $\mathcal{U}(t,s)[u_0] = u(\cdot,t)$.

If (φ,ψ,H_1) is the PFB of $\mathcal{L}$, then

$$\mathcal{U}(t,s)[\varphi(\cdot,s)] = \varphi(\cdot,t). \quad \forall t \geq s.$$

. Define, for each $t \in \mathbb{R}$,

$$X^1(t) = \operatorname{span}\{\varphi(\cdot,t)\}$$

$$X^2(t) = \{u_0 \in L^2(\Omega) : \int_\Omega u_0(x)\psi(x,t)\,dx = 0\}$$

Then $X^1(t) \oplus X^2(t) = L^2(\Omega)$ and they are forward invariant under $\mathcal{U}(t,s)$

Check. Let $u(\cdot,t) = \mathcal{U}(t,s)[u_0]$, $u_0 \in X^2(s)$.

$$\frac{d}{dt}\int_\Omega u(x,t)\psi(x,t)\,dx = \int_\Omega \psi u_t + u\psi_t$$

$$= \int_\Omega \psi(z\Delta u + cu + H_1 u) - u(z\Delta\psi + c\psi + H_1\psi)$$

$$= 0. \quad \text{for } t > s.$$

Denote $P^i(t) : L^2(\Omega) \to X^i(t)$ to be the projections.

Moreover, it follows from Thm 2.1 of [Hūska, JDE (2006)] that there is $C, \gamma > 0$ indep. of time such that

$$(5) \quad \| U(t,s)[v_0] \|_{L^2(\Omega)} \leq C e^{-\gamma(t-s)} \| v_0 \|_{L^2(\Omega)} \quad \text{for } t \geq s,\ v_0 \in X^2(s).$$

i.e. only the component in $X^1(t)$ persists under $U(t,s)$.
hence the name "principal" FB is used.

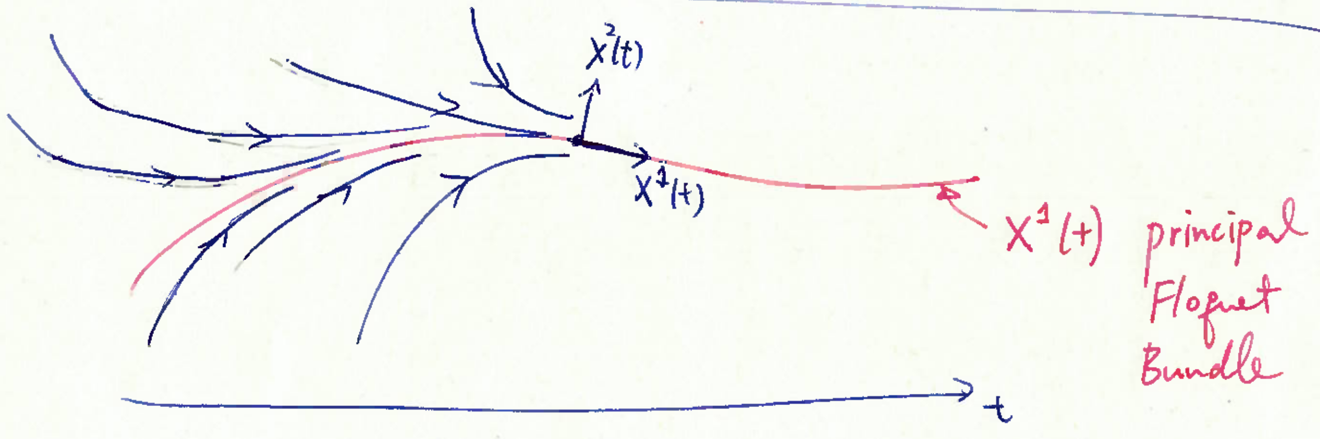
Sketch of the proof of smoothness:

Consider the mapping

$$\mathcal{F} : C_N^{2+\beta, 1+\beta/2}(\bar{\Omega} \times \mathbb{R}) \times C^{\beta/2}(\mathbb{R}) \times \left(\mathbb{R}_+ \times C^{\beta, \beta/2}(\bar{\Omega} \times \mathbb{R}) \right)$$
$$\longrightarrow C^{\beta, \beta/2}(\bar{\Omega} \times \mathbb{R}) \times C^{1+\beta/2}(\mathbb{R}).$$

$$\mathcal{F}(\varphi, H, z, c) = \begin{pmatrix} \partial_t \varphi - z \Delta \varphi - c\varphi - H(t)\varphi \\ \int_\Omega \varphi \, dx - 1 \end{pmatrix}.$$

To prove the smoothness of PFB on (z, c), it suffices to show that $D_{(\varphi, H)} \mathcal{F}$ is invertible when evaluated at a given PFB: (φ_1, H_1)



To this end, given $(f(x,t), G(t)) \in C^{\beta,\beta/2}(\overline{\Omega}\times\mathbb{R}) \times C^{1+\beta/2}(\mathbb{R})$

we show existence & uniqueness of $(w, Y) \in C_N^{2+\beta, 1+\beta/2}(\overline{\Omega}\times\mathbb{R})\times C^{\beta/2}(\mathbb{R})$

such that

$$\begin{cases} \partial_t w - z\Delta w - cw - H_1(t) w - Y(t)\varphi_1 = f(x,t) & \text{in } \Omega\times\mathbb{R} \\ n\cdot\nabla w = 0 & \text{on } \partial\Omega\times\mathbb{R} \\ \int_\Omega w(x,t)\,dx = G(t) & \text{for } t\in\mathbb{R}. \end{cases}$$

where (φ_1, H_1) is the PFB of $\mathcal{L} = -z\Delta - c(x,t)$.

① Set $w^\perp(x,t) = \int_{-\infty}^t U(t,\tau)[P^2(s) f(\cdot,\tau)]\,d\tau$

where, for each $t\in\mathbb{R}$, $P^2(t)$ is projection onto $X^2(t)$.

It is well-defined by (5).

Note that $\partial_t w^\perp + \mathcal{L}w^\perp - H_1 w^\perp = P^2(t)[f(\cdot,t)]$ in $\Omega\times\mathbb{R}$.

② Set $w(x,t) = w^\perp(x,t) + \left[-\int_\Omega w^\perp(y,t)\,dy + G(t)\right]\varphi_1(x,t)$.

Then $\int_\Omega w\,dx = G(t)$. and

$$\partial_t w + \mathcal{L}w - H_1(t) w = P^2(t)[f(\cdot,t)] + \left\{-\frac{d}{dt}\left[\int_\Omega w^\perp\right] + G'(t)\right\}\varphi_1(x,t).$$

$\longrightarrow$ Choose $Y(t)$ such that

$$Y(t)\varphi_1(x,t) + P^1(t)[f(\cdot,t)] = \left\{-\frac{d}{dt}\left[\int_\Omega w^\perp\right] + G'(t)\right\}\varphi_1(x,t).$$

This proves existence.

Uniqueness. Set $f=0$ and $G=0$. Then

$$\partial_t w + \mathcal{L}w - H_1(t)w = Y(t)\varphi_1(x,t).$$

Variation of constants formula implies

$$(6)\quad w(\cdot,t) - U(t,s)[w(\cdot,s)] = \int_s^t U(t,\tau)[Y(\tau)\varphi_1(\cdot,\tau)]\,d\tau$$
$$= \int_s^t Y(\tau)\{U(t,\tau)[\varphi_1(\cdot,\tau)]\}\,d\tau$$
$$= \int_s^t Y(\tau)\varphi_1(\cdot,t)\,d\tau$$
$$= \left(\int_s^t Y(\tau)\,d\tau\right)\varphi_1(\cdot,t) \quad \text{for } t>s.$$

Apply the projection onto $X^2(t)$.

$$P^2(t)[w(\cdot,t)] = P^2(t)\Big[U(t,s)[w(\cdot,s)]\Big] = U(t,s)\Big[P^2(s)[w(\cdot,s)]\Big]$$

$$\Rightarrow \quad \|P^2(t)[w(\cdot,t)]\|_{L^2(\Omega)} \le Ce^{-\gamma(t-s)}\|P^2(s)[w(\cdot,s)]\|$$
$$\le Ce^{-\gamma(t-s)}\|w(\cdot,s)\|_{L^\infty(\Omega)}$$
$$\le C'e^{-\gamma(t-s)}$$

Sending $s\to-\infty \Longrightarrow P^2(t)[w(\cdot,t)]=0$ for each $t\in\mathbb{R}$.

$\rightarrow\ w(\cdot,t)\in X^1(t)\ \forall t \iff w(\cdot,t)=\sigma(t)\varphi_1(\cdot,t)$.

Using $G(t)\equiv 0$, $0=\int_\Omega w(x,t)\,dt = \int_\Omega \sigma(t)\varphi_1(x,t)\,dx = \sigma(t)$

$\Rightarrow w\equiv 0$.

Substituting into (6), we have

$$\int_s^t Y(\tau)\,d\tau = 0 \quad \forall t>s \quad \Rightarrow \quad Y\equiv 0 \text{ as well}.$$

The smooth dependence now follows from the implicit function theorem. ※

A Generalized Reduction Principle

In static environment, mixing increases eigenvalue
decreases growth rate.

Let μ_1 be the p.e.v. of the elliptic problem

$$z\Delta\phi + c_0(x)\phi + \mu_1\phi = 0 \text{ in } \Omega, \quad n\cdot\nabla\phi = 0 \text{ on } \partial\Omega$$

Then $\frac{\partial}{\partial z}\mu_1 > 0$ provided $c_0(x) \not\equiv \text{const}$.

See also [L. Altenberg P.N.A.S. (2012)]

<u>Corollary</u>. Given $c_0(x) \not\equiv$ const and $\bar{I} \subseteq \mathbb{R}_+$ bounded interval.

There exists $\eta > 0$ s.t. for $z \in \bar{I}$ and $c \in C^{\beta, \beta/2}(\bar{\Omega}\times\mathbb{R})$

satisfying $\| c(x,t) - c_0(x) \|_{C^{\beta,\beta/2}} < \eta$,

the PFB $(\varphi_1(x,t), H_1(t))$ ~~(which depend smoothly on z)~~

~~satisfies~~ corresponding to $\mathcal{L} = -z\Delta - c(x,t)$,

(which depends smoothly on z), satisfies.

$$\inf_{t\in\mathbb{R}} \frac{\partial}{\partial z} H_1(t) > 0.$$

$$\Rightarrow \inf_{t\in\mathbb{R}} \left(H_1(t; z_2) - H_1(t; z_1) \right) \geq \hat{\eta}\,(z_2 - z_1).$$

Applications — Conjecture of Dockery et al.

Consider the N-species competition model.

$$(D)\begin{cases} \partial_t u_i - z_i \Delta u_i = u_i\left(m(x) - \sum_{j=1}^{N} u_j\right) & \text{in } \Omega\times\mathbb{R}_+ \\ n\cdot\nabla u_i = 0 & \text{on } \partial\Omega\times\mathbb{R}_+ \\ u_i(x,0) = u_{i,0}(x) & \text{in } \Omega \end{cases}$$

where $m(x) \not\equiv$ const.

<u>Conjecture</u>. For any $N \geq 2$, $0 < z_1 < z_2 < \dots < z_N$.

Let $u = (u_1, \dots, u_N)$ be a nonneg. nontrivial sol of (D).

then $u(\cdot,t) \longrightarrow E_{i_0}$ uniformly in Ω as $t \to \infty$,

where $i_0 \in \{1, \dots, N\}$ is the minimal index s.t. $u_i(x,0) \not\equiv 0$,

and $E_i = (0,\dots,0,\theta_{z_i},0,\dots,0)$ is the boundary equilibrium.

Define $\mathcal{D}$ to be the collect'n of all finite subsets of $\mathbb{R}_+$ such that the conj. holds,

$$\mathcal{D} = \bigcup_{N=1}^{\infty} \left\{ (z_i)_{i=1}^{N} : 0 < z_1 < \dots < z_N \text{ and conj. holds} \right\}.$$

Then by the result of Dockery et al.,

$\mathcal{D}$ contains all singleton and doubleton sets.

Can we say more?

Theorem For each $\hat{z} > 0$, there exists $\varepsilon > 0$ such that for any $N \geq 2$, $0 < z_1 < z_2 < \dots < z_N$
$\sup_i |z_i - \hat{z}| < \varepsilon$

we have $(z_i)_{i=1}^N \in \mathcal{D}$.

i.e. for N similar phenotypes, slower diffuser prevails.

Pf.
- Uniform in N estimates.
- $\limsup_{t\to\infty} \| \sum_{i=1}^N u_i(\cdot,t) - C_0(x) \| < \eta$.
- Use principal Floquet bundle.

- Apriori Bounds / Uniform in N estimates

① Let $\rho(t) = \sum_{i=1}^{N} \int_\Omega u_i(x,t)\,dx$.

$$\frac{d}{dt}\int_\Omega u_i = \int u_i \left(m - \sum_{j=1}^{N} u_j\right) dx.$$

$$\rho'(t) \leq \|m\|\,\rho(t) - \int_\Omega \left(\sum u\right)^2 dx$$

$$\leq a\,\rho(t) - b\,\rho(t)^2.$$

$$\Rightarrow \sup_{t>0} \sum_{i=1}^{N} \int_\Omega u_i\,dx \leq C. \quad \text{indep of } N.$$

② $\partial_t u_i - z_i \Delta u_i = u_i\left(m - \sum_j u_j\right) \leq m u_i$. $\boxed{\text{subsol}}$

Local max prin implies

$$\|u_i\|_{L^\infty} \leq C\,\|u_i\|_{L^1}$$

$$\Rightarrow \sum_{j=1}^{N} \|u_i\|_{L^\infty} \leq C \sum_{i=1}^{N} \|u_i\|_{L^1} = C\,\rho(t) \leq C.$$

indep. of N.

Parabolic regularity implies

$$\sum_{i=1}^{N} \|u_i\|_{C^{2+\beta,1+\beta/2}} \leq C,$$

since now the "coeffient" $m - \sum u_j$ is bounded, and u_i is <u>solution</u> to a linear parabolic eqn with bounded coeff.

Let $N \geq 2$ and $(z_i)_{i=1}^N$ be arbitrary

so that $\max_{1 \leq i \leq N} |z_i - \hat{z}| < \varepsilon$.

For each $\eta > 0$, we will show that the solution u to (D) satisfies

$$\limsup_{t \to \infty} \left\| \sum_{i=1}^N u_i(\cdot, t) - \theta_{\hat{z}}(\cdot) \right\|_{C(\bar{\Omega})} < \eta \quad \text{if } \varepsilon \text{ is small}.$$

where $\theta_{\hat{z}}$ is the unique positive solution to

$$\hat{z} \Delta \theta + \theta(m(x) - \theta) = 0 \text{ in } \Omega, \quad n \cdot \nabla \theta = 0 \text{ on } \partial\Omega.$$

Pf. Let $U(x,t) := \sum_{i=1}^N u_i(x,t)$.

then

$$\partial_t U - \hat{z} \Delta U - U(m - U) = \underbrace{\sum_{i=1}^N (z_i - z) \Delta u_i}_{f(x,t)}.$$

where

$$\|f(x,t)\|_\infty \leq \cancel{C\varepsilon \left\| \sum_{i=1}^N \Delta u_i \right\|} \; \varepsilon \sum_{i=1}^N \|u_i\|_{C^2(\bar{\Omega})} \leq C\varepsilon.$$

Choose ε small, indep of N

$$\Rightarrow \limsup_{t \to \infty} \| U(x,t) - \theta_{\hat{z}}(x) \|_{C(\bar{\Omega})} < \eta.$$

$$\Rightarrow \left\| \left(m(x) - \sum_{i=1}^N u_i(x,t) \right) - \left(m(x) - \theta_{\hat{z}}(x) \right) \right\| < \eta. \quad \text{for } t \gg 1$$

Use Principal Floquet Bundle

Take $c(x,t) = m(x) - \sum_{i=1}^{N} u_i(x,t)$. , then

$$\partial_t u_i - z_i \Delta u_i = c(x,t) u_i \qquad \underline{\underline{u_i > 0}}$$

The growth of u_i is controlled by

the PFB $\cancel{\varphi(x,t)}$ $\varphi = \varphi(x,t;z_i)$, $H = H(t;z_i)$.

$$\partial_t \varphi - z\Delta\varphi = c(x,t)\varphi + H(t;z)\varphi .$$

~~Set~~ ψ Since $\|c(x,t)\|_\infty \le C$ and.

$$0 = \int_\Omega c\varphi + H(t;z) \Rightarrow \|H(t);z)\|_\infty \le C .$$

Harnack ineq $\Rightarrow$ $\sup_{x\in\Omega} \varphi(x,t) \le C \inf_{x\in\Omega} \varphi(x,t)$.

$$\int_\Omega \varphi\, dx = 1 \quad \forall t \Rightarrow \frac{1}{C} \le \varphi(x,t) \le C .$$

By comparison,

$$\frac{1}{C}\varphi(x,t;z_i)\exp\left(-\int_0^t H(s;z_i)\,ds\right) \le u_i(x,t) \le C\varphi(x,t;z_i)\exp\left(-\int_0^t H(s;z_i)\,ds\right)$$

$$\frac{1}{C'}\exp\left(-\int_0^t H(s;z_i)ds\right) \le u_i(x,t) \le C'\exp\left(-\int_0^t H(s;z_i)ds\right)$$

$$\frac{u_i(x,t)}{u_1(x,t)} \le C''\exp\left(\int_0^t -H(s;z_i) + H(s;z_1)\,ds\right)$$

$$\le C''\exp\left(-\int_0^t \gamma(z_i - z)\,ds\right)$$

$$= C''\exp\left(-t\gamma(z_i - z)\right) \longrightarrow 0$$

This step uses the fact that $\left\|m(x) - \sum_{i=1}^{N} u_i(x,t) - C_0(x)\right\| < \eta$.

Thm [Cantrell-L. (2021)]

$\mathcal{D}$ is open in the Hausdorff topology.

The main proof ingredients are

- Uniform in $N \gg 1$ estimates.
- Morse decomposition + Lyapunov function due to Conley.
- Smoothness of PFB.

Recall $\mathrm{dist}_H(A,B) = \max\left\{ \sup_{x\in A} \inf_{y\in B} \mathrm{dist}(x,y), \sup_{y\in B} \inf_{x\in A} \mathrm{dist}(x,y) \right\}$

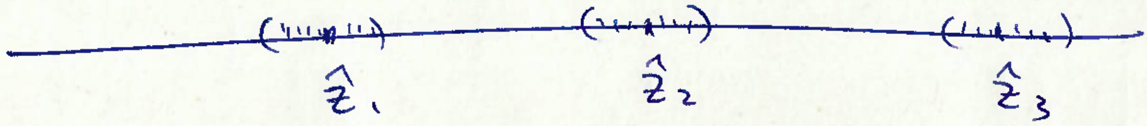
for $A, B \subseteq \mathbb{R}_+$.

Suppose the conjecture holds for some

$0 < \hat{z}_1 < \hat{z}_2 < \hat{z}_3$

Consider an arbitrary $N \geq 3$ with $0 < z_1 < \dots < z_N$

s.t. $|z_i - \hat{z}_k| < \varepsilon$ for $i \in I_k$, $k = 1, 2, 3$.



The three-species system with $\hat{z}_1 < \hat{z}_2 < \hat{z}_3$ admits a Morse-decomposition.

Let $U_k = \sum_{i \in I_k} u_i$, then

(U_1, U_2, U_3) approximately solves

$$\begin{cases} \partial_t \hat{u}_k = \hat{z}_k \Delta \hat{u}_k + \hat{u}_k (m - \hat{u}_1 - \hat{u}_2 - \hat{u}_3) & \text{in } \Omega \times (0, \infty), \ k = 1, 2, 3. \\ \text{Neumann b.c.} \end{cases}$$

$\xrightarrow[\text{decomposition}]{\text{Morse}}$ (U_1, U_2, U_3) eventually enters a neighborhood of $(\theta_{\hat{z}_1}, 0, 0)$.

$\rightarrow$ $m(x) - \sum_{k=1}^{3} U_k = m(x) - \sum_{i=1}^{N} u_i$ is close to $m(x) - \theta_{\hat{z}_1}$ eventually.

$\xrightarrow[\text{argument.}]{\text{PFB.}}$ $(u_1, u_2, \dots, u_N) \longrightarrow E_1$.

Thm [L.-Lou (2022)]

Given $m(x,t) \in C^{\beta,\beta/2}(\bar{\Omega} \times [0,\infty))$

such that $\liminf\limits_{T\to\infty} \frac{1}{T}\int_0^T \int_\Omega \left|m(x,t) - \fint_\Omega m\right|^2 dx\,dt > 0$,

there exists $B_0 > 0$ such that for any $N \geq 2$ and diffusion rates $B_0 < z_1 < z_2 < \cdots < z_N$, any positive solution u of (D) satisfies

$$\lim_{t\to\infty} \sup_{x\in\Omega} |u_i(x,t)| = 0 \quad \text{for } 2 \leq i \leq N.$$

Cor If $m = m(x)$ is nonconst., then

$$\mathcal{D} \supseteq \bigcup_{N=2}^{\infty} \left\{ (z_i)_{i=1}^N : B_0 < z_1 < \cdots < z_n \right\}$$

For $N=2$, $m(x,t)$ periodic in t, see

[Hutson et al. JMB (2001)] and

[He-Ni JEMS (2022) accepted].