EPIDEMIC SPREADING IN COMPLEX NETWORKS AS FRONT PROPAGATION INTO UNSTABLE STATES

WORKSHOP ON COMPETITION DYNAMICS IN BIOLOGY (OHIO STATE 12/15)

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Two themes

1. Arrival times: if a disease originates in one city, how long will it take to appear in some other city?

    - Meta-population SIR model
    - Airline transportation network

2. Traveling fronts on networks

    - Nonlocal connections
    - Linear determinacy: linear prediction for nonlinear solution *far from equilibrium*
    - Use insights gleaned from PDEs to make predictions for network ODE

Mathematical Model

Meta-population model (Brockmann and Helbing 2013)

$$\begin{aligned}
\dot{s}_n &= -\alpha s_n j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(s_m - s_n) \\
\dot{j}_n &= \alpha s_n j_n - \beta j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(j_m - j_n) \\
\dot{r}_n &= \beta j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(r_m - r_n)
\end{aligned}$$

Variables: s_n (susceptible proportion), j_n (infected), r_n (recovered) at city n

Parameters: α (infection rate), β (recovery rate),γ (average mobility rate, typically small), Row stochastic matrix P describes passenger flux between cities

Global airline network a) as of 2001 (Guimera *et. al* 2005) with 3618 cities (nodes) with 14,142 connections (edges) b) as of 2014 (Openflights.org) 3304 cities with 19,082 connections

Prior Work

Defn: The *arrival time* at node m of an epidemic initated at node n with $j_n(0) = j_0$ is

$$T_{nm}(\alpha, \beta, \gamma, \mathrm{P}, \kappa, j_0) = \inf\{t > 0 \mid j_m(t) = \kappa\}$$

Prior Work

Brockmann and Helbing (2013) – front propagation with respect to some effective distance

$$T_{nm}(\alpha, \beta, \gamma, \mathrm{P}, \kappa, j_0) = \frac{D_{\mathrm{eff}}(\mathrm{P})}{v_{\mathrm{eff}}(\alpha,\beta,\gamma,\kappa,j_0)}$$

Colizza *et al.* 2006, Balcan *et al.* 2010, Pastor-Satorras *et al.* 2015, disease spread in stochastic models

Gautreau, Barrat, Barthelemy 2007/2008, metapopulation arrival time estimates

Fu, Guo, Wu (2016), Wu (2017), lattice SIR wavespeed selection

Outlook

Goal: Use PDE theory to make estimates/ qualitative predictions for network dynamical system

Fact: Some systems are *linearly determined* and their invasion speeds equal the invasion speeds for the system linearized near the unstable state

Prediction: In some examples $T_{nm}(\alpha,\beta,\mathrm{P}) \approx T_{nm}^{\mathrm{lin}}(\alpha,\beta,\mathrm{P})$

Fact: Some systems are *nonlinearly determined* and their invasion speeds are typically faster than the linearized equation

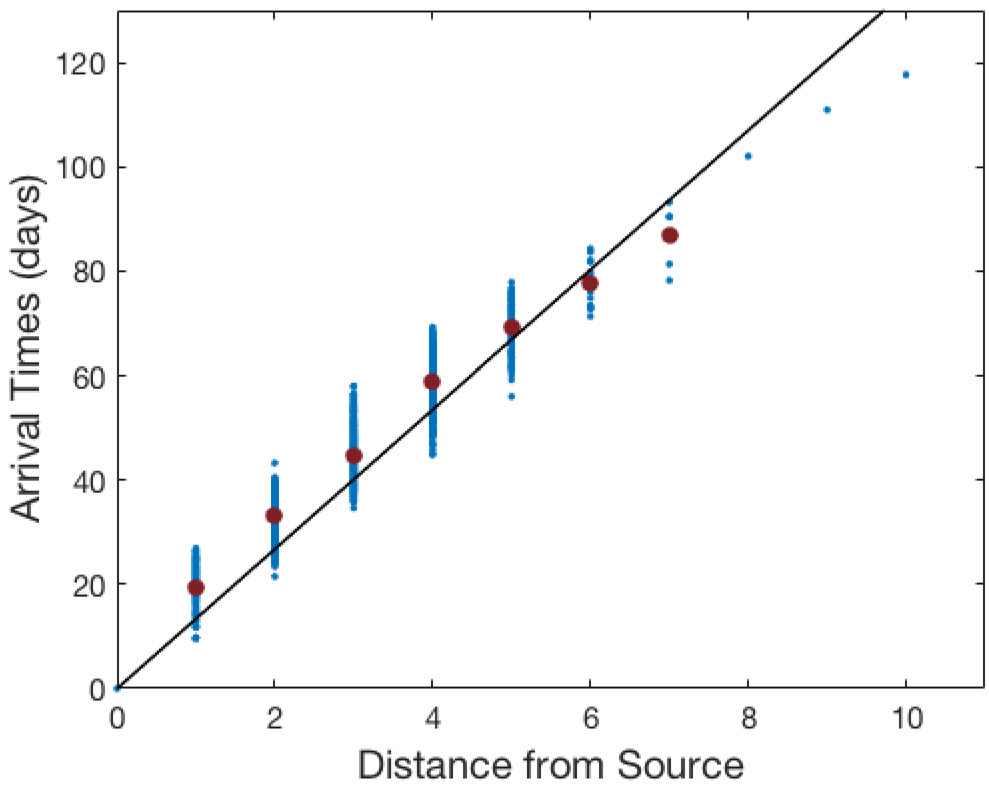
Prediction: In some examples $T_{nm}(\alpha,\beta,\mathrm{P}) < T_{nm}^{\mathrm{lin}}(\alpha,\beta,\mathrm{P})$

Fact: Heterogeneities can increase the invasion velocity

Prediction: Heterogeneities may lead to faster (on average) arrival times

Arrival times vs Distance

Simulation: $\alpha = 0.5$, $\beta = 0.25$, $\gamma = 0.01$, epidemic originating in Paris, France



Linear Determinacy (super-solution)

Write system in vector form

$$\begin{aligned} \mathbf{s}_t &= -\alpha\mathbf{s}\circ\mathbf{j} + \gamma(\mathrm{P}-\mathrm{I})\mathbf{s} \\ \mathbf{j}_t &= \alpha\mathbf{s}\circ\mathbf{j} - \beta\mathbf{j} + \gamma(\mathrm{P}-\mathrm{I})\mathbf{j} \end{aligned}$$

Theorem(informal) $T_{nm}^{\mathrm{lin}} \leq T_{nm}$

Let $\bar{\mathbf{s}}(t) = \mathbf{1}$ and $\bar{\mathbf{j}}_t = (\alpha-\beta)\bar{\mathbf{j}} + \gamma(\mathrm{P}-\mathrm{I})\bar{\mathbf{j}}$

Define

$$\begin{aligned} N_s(\mathbf{s},\mathbf{j}) &= \mathbf{s}_t + \alpha\mathbf{s}\circ\mathbf{j} - \gamma(\mathrm{P}-\mathrm{I})\mathbf{s} \\ N_j(\mathbf{s},\mathbf{j}) &= \mathbf{j}_t - \alpha\mathbf{s}\circ\mathbf{j} + \beta\mathbf{j} - \gamma(\mathrm{P}-\mathrm{I})\mathbf{j} \end{aligned}$$

Compute

$$\begin{aligned} N_s(\mathbf{1},\overline{\mathbf{j}(t)}) &= \alpha\bar{\mathbf{j}}(t) \geq 0 \\ N_j(\mathbf{1},\bar{\mathbf{j}}(t)) &= \mathbf{j}_t - (\alpha-\beta)\bar{\mathbf{j}}(t) - \gamma(\mathrm{P}-\mathrm{I})\bar{\mathbf{j}}(t) = 0 \end{aligned}$$

Arrival time estimates (small diffusion limit)

Linearize about the unstable disease free steady state $(\mathbf{s},\mathbf{j}) = (\mathbf{1},\mathbf{0})$ and obtain

$$\begin{aligned} \partial_t \mathbf{s}_l &= -\alpha \mathbf{j}_l + \gamma(\mathrm{P} - \mathrm{I})\mathbf{s}_l \\ \partial_t \mathbf{j}_l &= (\alpha - \beta)\mathbf{j}_l + \gamma(\mathrm{P} - \mathrm{I})\mathbf{j}_l. \end{aligned}$$

Equation for infected population decouples with solution,

$$j_m(t) = j_n(0) v_m^T e^{(\alpha-\beta-\gamma)t} e^{\gamma \mathrm{P} t} v_n,$$

$$v_m^T e^{\gamma \mathrm{P} t} v_n = \sum_{k=0}^{\infty} \frac{\gamma^k t^k}{k!} v_m^T \mathrm{P}^k v_n.$$

Let $\rho_k = v_m^T \mathrm{P}^k v_n$ and let d be the minimal number of flights between the two cities.

$$j_m(t) = j_0 e^{(\alpha-\beta-\gamma)t} \left[\frac{\gamma^d \rho_d t^d}{d!} + \frac{\gamma^{d+1} \rho_{d+1} t^{d+1}}{(d+1)!} + \dots \right],$$

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Let $\rho_k = v_m^T \mathrm{P}^k v_n$ and let d be the minimal number of flights between the two cities. For γ sufficiently small, we anticipate

$$T_{nm} \approx \frac{d}{\alpha - \beta - \gamma} W\left(\frac{(d!)^{1/d}}{d} \frac{\alpha - \beta - \gamma}{\gamma (\rho_d)^{1/d}} \left(\frac{\kappa}{j_0}\right)^{1/d} \right),$$

Asymptotic Expansions

Asymptotic expansions are computed as follows

$$T_{nm}^{\text{lin}} = \frac{-d}{\alpha-\beta}\log\gamma - \frac{d}{\alpha-\beta}\log(-\log\gamma) - \frac{d}{\alpha-\beta}\log\frac{d}{\alpha-\beta} - \frac{1}{\alpha-\beta}\log\frac{\rho_d}{d!} + o(1). \quad (1)$$

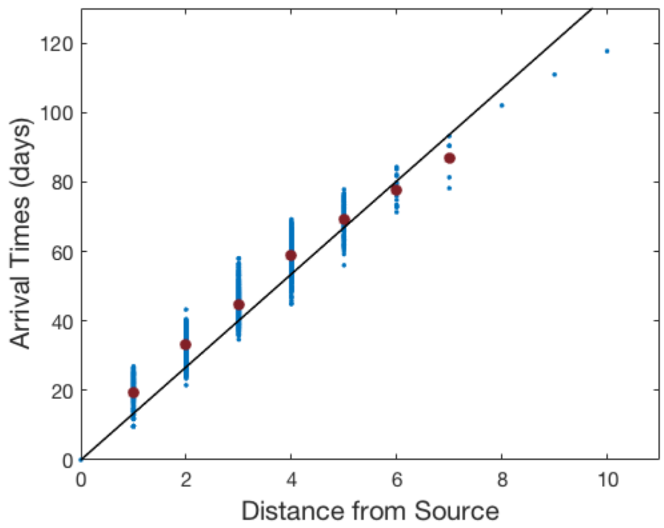
Observations

- To leading order, the effective distance is the graph distance
- To leading order, the effective velocity is that of a front on a 1-dimensional lattice
- Network properties enter at $\mathcal{O}(1)$ where the relevant paramter is the random walk probability between node n and m

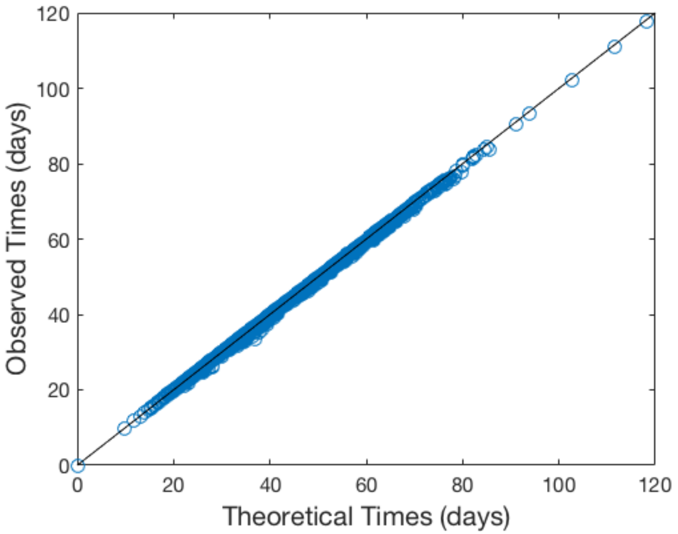
Prediction versus Observation

Simulation: $\alpha = 0.5$, $\beta = 0.25$, $\gamma = 0.01$, epidemic originating in Paris, France

Distance versus observed



Heat kernel prediction versus observed



Alternate Derivation

$$j_1(t) \approx j_0 e^{(\alpha-\beta)t}$$

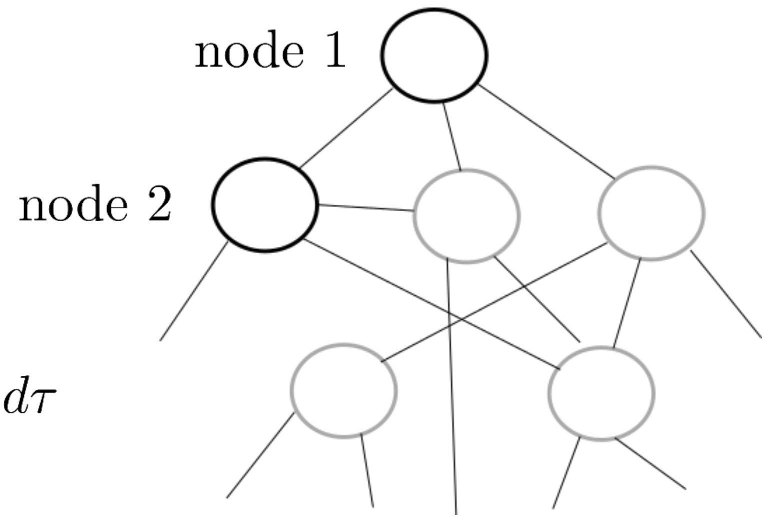
$$\frac{dj_2}{dt} \approx (\alpha - \beta) j_2 + \gamma \mathrm{P}_{21} j_1(t)$$

$$j_2(t) \approx \gamma \mathrm{P}_{21} e^{(\alpha-\beta)t} \int_0^t e^{-(\alpha-\beta)\tau} j_1(\tau) d\tau$$

$$j_2(t) \approx \gamma j_0 \mathrm{P}_{21} e^{(\alpha-\beta)t} \int_0^t d\tau$$

$$j_2(t) \approx \gamma j_0 \mathrm{P}_{21} t e^{(\alpha-\beta)t}$$

$$t_2 \approx \frac{1}{\alpha-\beta} W\left(\frac{\kappa(\alpha-\beta)}{\gamma j_0 \mathrm{P}_{21}}\right)$$



Alternate Derivation

$$\frac{dj_4}{dt} \approx (\alpha - \beta) j_4 + \gamma \mathrm{P}_{42} j_2(t) + \gamma \mathrm{P}_{43} j_3(t)$$

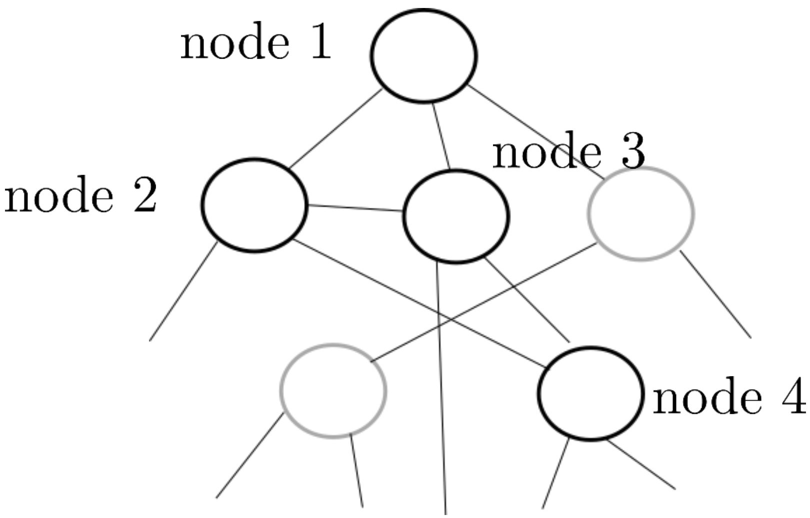
$$j_2(t) \approx \kappa e^{(\alpha-\beta)(t-t_2)} \quad t > t_2$$

$$j_3(t) \approx \kappa e^{(\alpha-\beta)(t-t_3)} \quad t > t_3$$

$$\begin{aligned} j_4(t) \quad &\approx \quad \gamma\kappa \mathrm{P}_{42} e^{(\alpha-\beta)t} \int_{t_2}^{t} e^{-(\alpha-\beta)t_2} d\tau \\ &+ \quad \gamma\kappa \mathrm{P}_{43} e^{(\alpha-\beta)t} \int_{t_3}^{t} e^{-(\alpha-\beta)t_3} d\tau \end{aligned}$$

$$j_4(t) \approx \gamma \left(\mathrm{P}_{42} e^{-(\alpha-\beta)t_2} + \mathrm{P}_{43} e^{-(\alpha-\beta)t_3} \right) t e^{(\alpha-\beta)t}$$

$$j_4(t) \approx \frac{\gamma^2 j_0}{\kappa(\alpha - \beta)} \left(\mathrm{P}_{42}\mathrm{P}_{21} t_2 + \mathrm{P}_{43}\mathrm{P}_{31} t_3 \right) t e^{(\alpha-\beta)t}$$



Pushed front: model of social epidemics

Social epidemics involving higher order interactions via simplicial complexes (Iacopini *et al.* 2019))

$$\dot{s}_n = -\alpha s_n j_n - \rho s_n j_n^2 + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(s_m - s_n)$$

$$\dot{j}_n = \alpha s_n j_n + \rho s_n j_n^2 - \beta j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(j_m - j_n)$$

- Linear arrival times are equal to SIR
- $(\mathbf{1}, \bar{\mathbf{j}}(t))$ no longer super-solution
- Enhanced growth for $\rho > 0$ leads to faster arrival times
- Pushed front

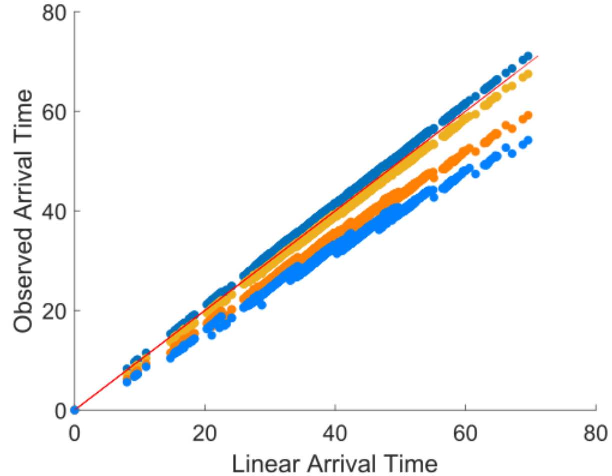
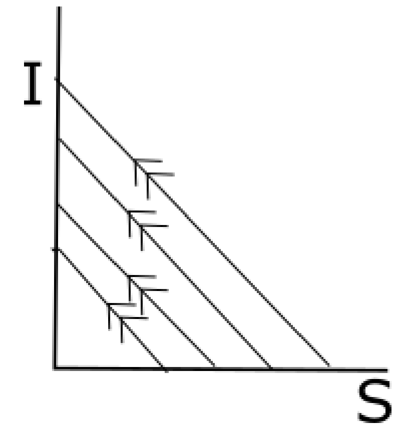


Figure: arrival times for $\rho = 0, 10, 30, 50$

Pushed front: local dynamics

At each node the local equations are

$$\begin{aligned} S' &= -\alpha S I - \rho S I^2 \\ I' &= \alpha S I + \rho S I^2 - \beta I \end{aligned}$$



Let $\rho = \frac{1}{\epsilon}$ and rescale $t = \epsilon\tau$

$$\begin{aligned} \frac{dS}{d\tau} &= -SI^2 - \epsilon\alpha S I \\ \frac{dI}{d\tau} &= SI^2 + \epsilon\alpha S I - \epsilon\beta I \end{aligned}$$

Let $W = S + I$ then formally

$$\begin{aligned} \frac{dW}{d\tau} &= \mathcal{O}(\epsilon) \\ \frac{dI}{d\tau} &= (W - I)I^2 + \mathcal{O}(\epsilon) \end{aligned}$$

Approximate

$$I(t) = \begin{cases} \kappa e^{(\alpha-\beta)t} & t \leq \Omega \\ e^{-\beta(t-\Omega)} & t > \Omega \end{cases}$$

$$\Omega = \frac{1}{\alpha - \beta} \log\left(\frac{\epsilon}{\kappa}\right)$$

Pushed front: arrival time estimates

Suppose $j_1(t) = e^{-\beta(t-\Omega)}$ for $t > \Omega$

$$\frac{dj_2}{dt} \approx (\alpha - \beta)j_2 + \gamma P_{21} j_1(t)$$

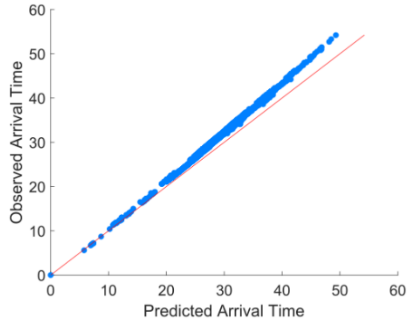
$$j_2(t) \approx \gamma P_{21} e^{(\alpha-\beta)t} \int_{\Omega}^{t} e^{-(\alpha-\beta)\tau} e^{-\beta(\tau-\Omega)} d\tau$$

$$j_2(t) \approx \gamma P_{21} e^{(\alpha-\beta)(t-\Omega)}$$

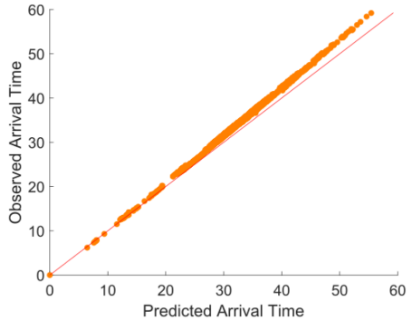
$$t_2 = \frac{1}{\alpha-\beta} \log\left(\frac{\kappa\alpha}{\gamma P_{21}}\right) + \Omega$$

$$\Delta(\epsilon) = t_2(\epsilon) - t_2(\infty) = \frac{-1}{\alpha-\beta}\left(\log(\epsilon) + \log(\alpha) - \log(\alpha - \beta) + \log(-\log(\gamma))\right)$$

$\rho = 50$



$\rho = 30$



Inhomogeneous Reaction Terms

Consider original model with inhomogeneous infection rate $\alpha + \omega_n$, $\sum \omega_n = 0$

$$\begin{aligned}\dot{s}_n &= -\alpha s_n j_n - \omega_n s_n j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(s_m - s_n) \\ \dot{j}_n &= \alpha s_n j_n + \omega_n s_n j_n - \beta j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(j_m - j_n)\end{aligned}$$

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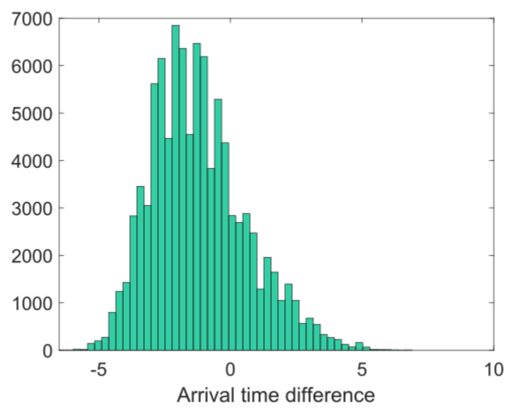
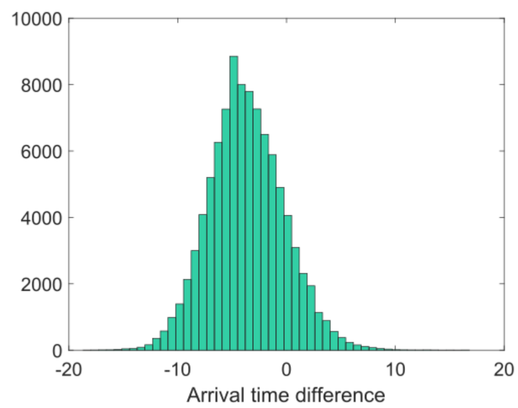
$$\begin{aligned} \dot{s}_n &= -\alpha s_n j_n - \omega_n s_n j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(s_m - s_n) \\ \dot{j}_n &= \alpha s_n j_n + \omega_n s_n j_n - \beta j_n + \gamma \sum_{m \neq n} \mathrm{P}_{nm}(j_m - j_n) \end{aligned}$$



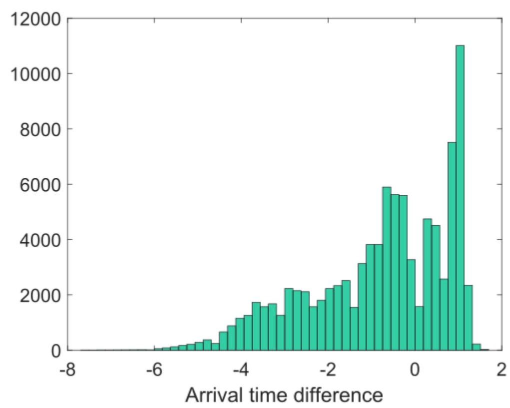
$50\%\ \omega_n = 0.2$

$50\%\ \omega_n = -0.2$



$\omega_n \sim \mathcal{N}(0, 0.2^2)$

normalized to mean zero



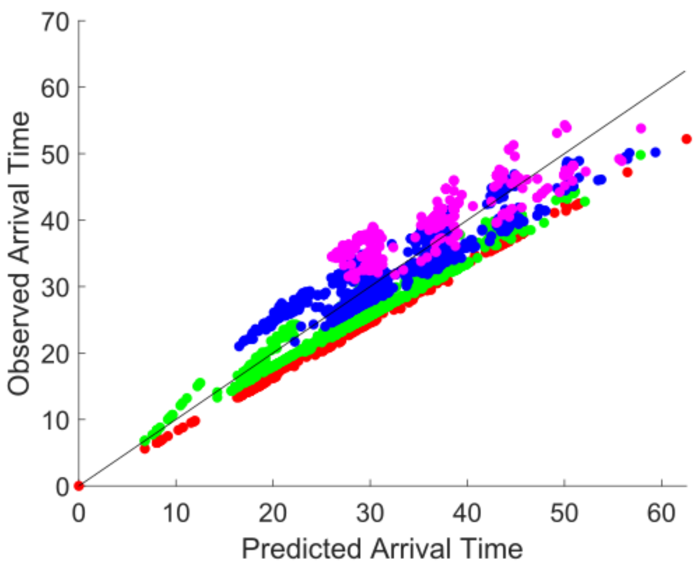
$90\%\ \omega_n = -0.02$

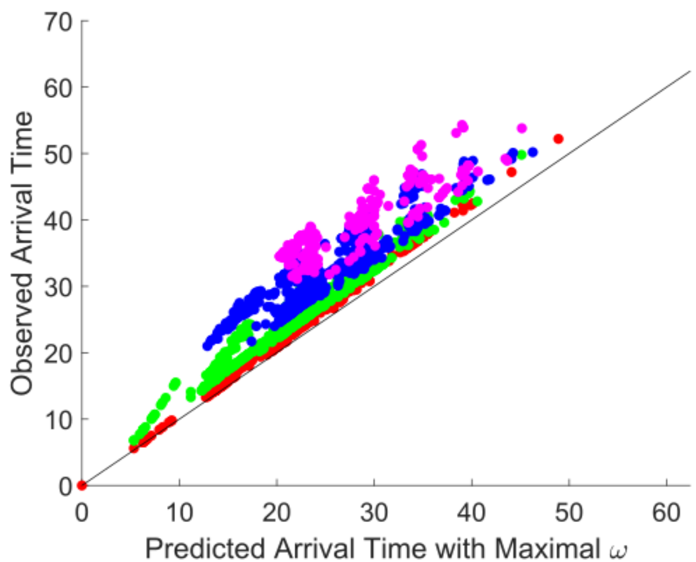
$10\%\ \omega_n = 0.18$

Inhomogeneous Reaction Terms

- Arrival times are faster on average since there typically exist multiple paths of minimum distance connecting two nodes
- Recall $T_{nm} \approx \frac{-d}{\alpha-\beta}\log(\gamma)$

Ex: $50\%\ \omega_n = 0.2 \qquad 50\%\ \omega_n = -0.2$





Part II: invasion fronts in trees

Consider the Fisher-KPP equation on a homogeneous tree of degree $k+1$

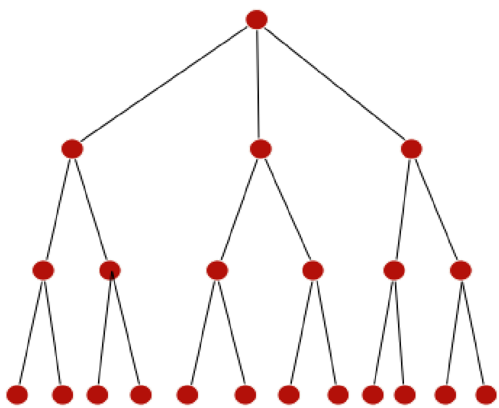
$$\frac{du_n}{dt} = \frac{\gamma}{k+1}(u_{n-1} - 2u_n + u_{n+1}) + \frac{\gamma}{k+1}(k-1)(u_{n+1} - u_n) + u_n(1-u_n).$$

Two Observations

- If $\gamma \ll 1$, then reaction dominates and we expect traveling fronts
- If $\gamma \gg 1$, $\gamma = \frac{1}{(\Delta x)^2}$ the system can be viewed as a discretization of the PDE

$$u_t = u_{xx} + \frac{k-1}{\Delta x} u_x + u,$$

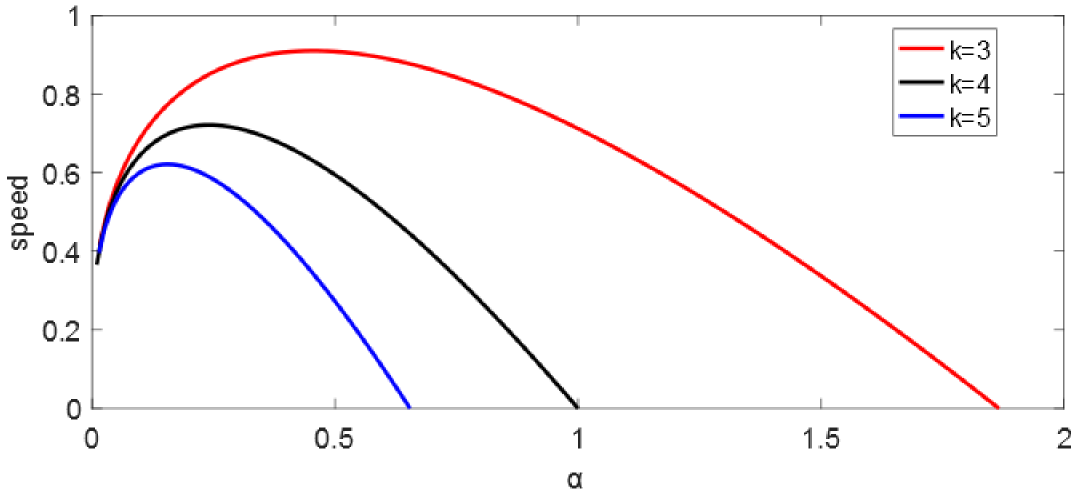
advection dominates and solution propagates up the tree



Linear invasion speed

Recall: The *linear spreading speed* can be computed from roots of

$$F(s,\gamma) = \begin{pmatrix} \frac{\gamma}{k+1}\left(e^{\nu} - k - 1 + ke^{-\nu}\right) - s\nu + 1 \\ \frac{\gamma}{k+1}\left(e^{\nu} - ke^{-\nu}\right) - s \end{pmatrix} = 0$$



Observation: Non-monotone invasion speeds – front slows down as population leaves front interface

Critical diffusion coefficients

For $s_{lin} = 0$ we solve

$$\begin{pmatrix} \frac{\gamma}{k+1}\left(e^{\nu} - k - 1 + ke^{-\nu}\right) + 1 \\ \frac{\gamma}{k+1}\left(e^{\nu} - ke^{-\nu}\right) \end{pmatrix} = 0$$

We find solution for

$$\begin{aligned} \nu_2 &= \frac{\log(k)}{2} \\ \gamma_2 &= \frac{1(k+1)}{k+1-2\sqrt{k}} \end{aligned}$$

Note that ν_2 is l^2-critcal

$$||u||_2 = \left(\sum_{n\in\mathbb{N}} k^{n-1} u_n^2(t)\right)^{\frac{1}{2}}$$

For s_{lin} maximal we observe

$$\begin{aligned} \nu_1 &= \log(k) \\ \gamma_1 &= \frac{k+1}{(k-1)\log(k)} \\ s_1 &= \frac{1}{\log(k)} \end{aligned}$$

solves $F(s_1, \gamma_1, \alpha_1) = 0$

Note that ν_1 is l^1-critcal

$$||u||_1 = \sum_{n\in\mathbb{N}} k^{n-1} u_n(t)$$

Conclusions and References

References

- Hoffman and Holzer, *Invasion fronts on graphs: the Fisher-KPP equation on homogeneous trees and Erdos-Reyni graphs*, DCDS-B, 2019.

- Chen, Holzer and Shapiro, *Estimating epidemic arrival times using linear spreading theory*, Chaos, 2018.

- Armbruster, Holzer, Roselli, Underwood, preprint, 2021.

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