Global dynamics of a Lotka-Volterra competition patch model

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Outline

1. Introduction
2. Global dynamics of Lotka-Volterra competition patch model
   - Global dynamics
   - Two main techniques
   - Two special cases
3. Conclusion

Single species patch model

Consider a single species $u = (u_1, ..., u_n)$ living in n patches:

$$u_i' = \mu \sum_{j=1}^{n} l_{ij} u_j + u_i(r_i - u_i) \qquad i = 1, \dots, n. \tag{1}$$

- $r_i > 0$ is the growth rate for patch i. $\mu > 0$ is the dispersal rate.
- Dispersal matrix $L = (l_{ij})$ satisfies $l_{ij} \geq 0$ for $i \neq j$ and $l_{jj} = -\sum_{i \neq j} l_{ij}$ ($-L$ is (column) Laplacian).
- Suppose L is irreducible. L has a left eigenvector (1, ..., 1) corresponding with eigenvalue 0. By Perron-Frobenius Theorem, spectral bound $s(L) = 0$ is the unique eigenvalue corresponding with a positive eigenvector $(\alpha_1, \alpha_2, \dots, \alpha_n)^T > 0$.

$$L = \begin{pmatrix} -\sum\limits_{k\neq 1} l_{k1} & l_{12} & \cdots & l_{1n} \\ l_{21} & -\sum\limits_{k\neq 2} l_{k2} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & -\sum\limits_{k\neq n} l_{kn} \end{pmatrix}$$

Single species patch model

The dynamics of the single species n-patch model is similar to the logistic model:

Theorem (Lu-Takeuchi 1993; Cosner 1996; Li-Shuai 2010)

The single species n patch model has a unique positive equilibrium, which is globally asymptotically stable in $\mathbb{R}^n_+ \backslash \{\mathbf{0}\}$.

Proof.

Since the Jacobian matrix of the right hand side of the model is cooperative (off-diagonal entries are nonnegative) and irreducible, the solutions have strong monotonicity, i.e. for any two solutions $u^1(t)$ and $u^2(t)$ of the model, if $u^1(0) > u^2(0) \geq \mathbf{0}$ then $u^1(t) \gg u^2(t)$ for all $t > 0$. (In particular, if $u^1(0) > \mathbf{0}$ then $u^1(t) \gg \mathbf{0}$ for all $t > 0$.) □

Two species competition patch model

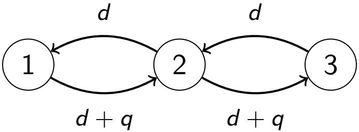
1. Let $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ = two competing species in n patches.
2. The classic Lokta-Volterra competition patch model:

$$\begin{cases} u_i' = \mu_u \sum_{j=1}^n l_{ij} u_i + u_i(p_i - u_i - cv_i), & i = 1, \dots, n, t > 0, \\ v_i' = \mu_v \sum_{j=1}^n l_{ij} v_j + v_i(q_i - bu_i - v_i), & i = 1, \dots, n, t > 0. \end{cases}$$

3. The model has one trivial equilibria $E_0 = (\mathbf{0}, \mathbf{0})$ and two semi-trivial equilibria: $E_1 = (w^*(\mu_u, p), \mathbf{0})$ and $E_2 = (\mathbf{0}, w^*(\mu_v, q))$, where $w^*(\mu, r)$ is the unique positive equilibrium of

$$w_i' = \mu \sum_{j=1}^n l_{ij} w_j + w_i(r_i - w_i) \qquad i = 1, \dots, n.$$

The following model was considered in [Jiang-Lam-Lou BMB 2020&2021]:



$$\begin{cases} u_1' = d(u_2 - u_1) - qu_1 + u_1(1 - \frac{u_1+v_1}{k_1}) \\ u_2' = d(u_1 + u_3 - 2u_2) + qu_1 - qu_2 + u_2(1 - \frac{u_2+v_2}{k_2}) \\ u_3' = d(u_2 - u_3) + qu_2 + u_3(1 - \frac{u_3+v_3}{k_3}) \\ v_1' = D(v_2 - v_1) - qv_1 + v_1(1 - \frac{u_1+v_1}{k_1}) \\ v_2' = D(v_1 + v_3 - 2v_2) + qv_1 - qv_2 + u_2(1 - \frac{u_2+v_2}{k_2}) \\ v_3' = D(v_2 - v_3) + qv_2 + v_3(1 - \frac{u_3+v_3}{k_3}). \end{cases}$$

The resident species u and invading species v are only different by the diffusion rate D. Under what conditions, can v invade u?

Related works: special cases

- There is a recent paper [Slavik SIADS 2020] considers the case when L is symmetric.
- When $n = 2$, [Lin-Lou-Shih-Tsai MBE 2014; Cheng-Lin-Shih MBE 2019] studied a conjecture by [Gourley-Kuang MBE 2005] for

$$\begin{cases} u_1' = d(u_2 - u_1) + u_1(\alpha_1 - u_1 - v_1) \\ u_2' = d(u_1 - u_2) + u_2(\alpha_2 - u_1 - v_1) \\ v_1' = d(v_2 - v_1) + v_1(\beta_1 - u_1 - v_1) \\ v_2' = d(v_1 - v_2) + v_2(\beta_2 - u_2 - v_2). \end{cases}$$

They proved:

Theorem (Lin-Lou-Shih-Tsai MBE 2014; Cheng-Lin-Shih MBE 2019)

Suppose $0 < \alpha_1 = \beta_1 - \sigma < \beta_1 < \beta_2 < \alpha_2 = \beta_2 + \sigma$ *with* $\sigma > 0$. *Then there exists* $d^* > 0$ *such that if* $d > d^*$ *the model has a globally stable positive equilibrium; if* $d < d^*$ *the* $u-$*only semitrivial equilibrium is globally stable.*

Related works: reaction-diffusion models

- The reaction-diffusion Lotka-Volterra competition model has been studied by many people...
  (1) R. S. Cantrell and C. Cosner. Spatial Ecology via Reaction-Diffusion Equations, 2004.
  (2) W.-M. Ni. The mathematics of diffusion, 2011.
  (3) P. Hess. Periodic-Parabolic Boundary Value Problems and Positivity, 1991.
- See [Lou JDE 2006, Lam-Ni SIAM 2012, He-Ni CPAM 2016] for global dynamics when $bc < 1$ of

$$\begin{cases} \partial_t u = \mu_u \Delta u + u(p(x) - u - cv), \\ \partial_t v = \mu_v \Delta v + v(q(x) - bu - v). \end{cases}$$

- Similar result was obtained in [Zhou-Xiao JFA 2018] for

$$\begin{cases} \partial_t u = \mu_u (\Delta u - \nabla \cdot u \nabla m) + u(p(x) - u - cv), \\ \partial_t v = \mu_v (\Delta v - \nabla \cdot v \nabla m) + v(q(x) - bu - v). \end{cases}$$

Outline

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One patch

If $n = 1$, then

$$u' = u(p - u - cv),$$
$$v' = v(q - bu - v).$$

- One trivial equilibrium: $E_0 = (0, 0)$. Two semi-trivial equilibria: $E_1 = (p, 0)$, $E_2 = (0, q)$.
- Case $bc < 1$: weak competition. If $b < q/p$ and $c < p/q$, coexistence equilibrium $E = (\frac{p-cq}{1-bc}, \frac{q-bp}{1-bc})$ is globally stable; if $b \geq q/p$ and $c < p/q$, E_1 is globally stable; if $b < q/p$ and $c \geq p/q$, E_2 is globally stable; if $b = q/p$ and $b = m/n$, there are infinitely many positive equilibria.
- Case $bc > 1$: strong competition.

Lotka-Volterra competition patch model

We consider the classic Lokta-Volterra competition patch model:

$$\begin{cases} u_i' = \mu_u \sum_{j=1}^n l_{ij} u_i + u_i(p_i - u_i - cv_i), & i = 1, \dots, n, t > 0, \\ v_i' = \mu_v \sum_{j=1}^n l_{ij} v_j + v_i(q_i - bu_i - v_i), & i = 1, \dots, n, t > 0. \end{cases}$$

We make the following assumptions:

(A1) $b > 0$, $c > 0$, and $0 < bc \le 1$; $p_i, q_i > 0$ for all $i = 1, 2, ..., n$.

(A2) The matrix $-L$ is irreducible and Laplacian.

(A3) The weighted digraph $\mathcal{G}$ associated with L is cycle-balanced.

Each of the following graph is cycle-balanced:

(1) L is symmetric;

(2) $n = 2$;

(3) Every cycle of the weighted digraph $\mathcal{G}$ has two vertices.

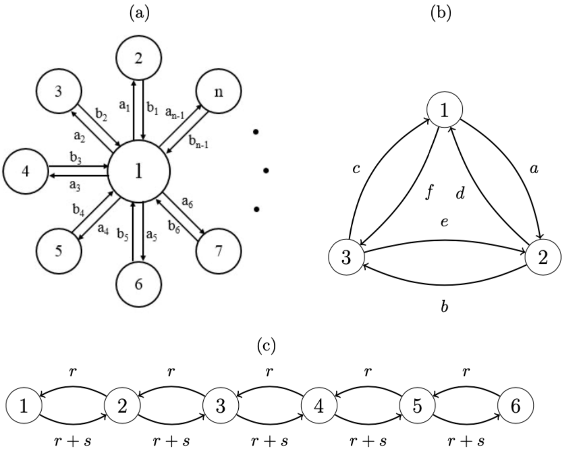

Figure 1: (a) A star migration graph. (b) A digraph that is cycle-balanced if $abc = def$. (c) A stepping stone graph.

Notation

- The competition model has two semi-trivial equilibria, $E_1 = (w^*(\mu_u, p), \mathbf{0})$ and $E_2 = (\mathbf{0}, w^*(\mu_v, q))$, and one trivial equilibria $E_0 = (\mathbf{0}, \mathbf{0})$.
- Denote by $\lambda_1(\mu, h)$ the principal eigenvalue of

$$\mu \sum_{j=1}^{n} l_{ij}\psi_j + h_i\psi_i = \lambda\psi_i, \quad i = 1, \ldots, n.$$

-

$$\begin{aligned}
S_u &= \{(\mu_u, \mu_v) : E_1 \text{ is linearly stable}\} = \{(\mu_u, \mu_v) : \lambda_1(\mu_v, q - bw^*(\mu_u, p)) < 0\}, \\
S_v &= \{(\mu_u, \mu_v) : E_2 \text{ is linearly stable}\} = \{(\mu_u, \mu_v) : \lambda_1(\mu_u, p - cw^*(\mu_v, q)) < 0\}, \\
S_- &= \{(\mu_u, \mu_v) : \lambda_1(\mu_v, q - bw^*(\mu_u, p)) > 0,\ \lambda_1(\mu_u, p - cw^*(\mu_v, q)) > 0\}, \\
S_{u,0} &= \{(\mu_u, \mu_v) : \lambda_1(\mu_v, q - bw^*(\mu_u, p)) = 0\}, \\
S_{v,0} &= \{(\mu_u, \mu_v) : \lambda_1(\mu_u, p - cw^*(\mu_v, q)) = 0\}, \\
S_{0,0} &= \{(\mu_u, \mu_v) : \lambda_1(\mu_v, q - bw^*(\mu_u, p)) = \lambda_1(\mu_u, p - cw^*(\mu_v, q)) = 0\}.
\end{aligned}$$

Complete dynamics for the patch model

Global dynamics is completely determined by local dynamics.

Theorem (Chen-Shi-Shuai-W Nonlinearity 2022)

Suppose that (A1)-(A3) hold. Then we have the following mutually disjoint decomposition of $Q := \{(\mu, \nu) : \mu, \nu > 0\}$*:*

$$Q = (S_u \cup S_{u,0} \setminus S_{0,0}) \bigcup (S_v \cup S_{v,0} \setminus S_{0,0}) \bigcup S_- \bigcup S_{0,0}. \quad (2)$$

Moreover, the following statements hold:

(i) *For any* $(\mu_u, \mu_v) \in S_u \cup S_{u,0} \setminus S_{0,0}$*,* $E_1 = (w^*(\mu_u, p), \mathbf{0})$ *is globally asymptotically stable.*

(ii) *For any* $(\mu_u, \mu_v) \in S_v \cup S_{v,0} \setminus S_{0,0}$*,* $E_2 = (\mathbf{0}, w^*(\mu_v, q))$ *is globally asymptotically stable.*

(iii) *For any* $(\mu_u, \mu_v) \in S_-$*, there exists a unique positive equilibrium* (u, v)*, which is globally asymptotically stable.*

(iv) *For any* $(\mu_u, \mu_v) \in S_{0,0}$*, we have* $bc = 1$*,* $w^*(\mu_u, p) \equiv cw^*(\mu_v, q)$ *and the model has a compact global attractor consisting of a continuum of equilibria*

$$\{(\rho w^*(\mu_u, p), (1 - \rho) w^*(\mu_u, p)/c) : \rho \in [0, 1]\}. \quad (3)$$

Outline

First tool: monotone dynamical system theory

The main tool to investigate the competition model is monotone dynamical system theory:

1. H. L. Smith. Monotone Dynamical Systems. 1995.
2. P. Hess. Periodic-Parabolic Boundary Value Problems and Positivity, volume 247, 1991.
3. M. W. Hirsch, H. L. Smith. Monotone dynamical systems, 2006.
4. S. B. Hsu, H. L. Smith, and P. Waltman. Competitive exclusion and coexistencefor competitive systems on ordered Banach spaces. Trans. Amer. Math. Soc., 348 (10): 4083–4094, 1996.
5. ...

First tool: monotone dynamical system theory

The global dynamics of the competition model is determined by the local dynamics of the equilibria:

1. if E_2 is unstable and the model has no positive equilibrium, then E_1 is globally asymptotically stable; if E_1 is unstable and the model has no positive equilibrium, then E_2 is globally asymptotically stable;
2. if E_1 and E_2 are both unstable, then the model has at least one stable positive equilibrium, which is globally asymptotically stable if it is unique;
3. if E_1 and E_2 are both locally asymptotically stable, then the model has at least one unstable positive equilibrium.

Key step: a positive (or coexistence) equilibrium, if exists, is stable.

Second tool: graph theory

Let $L = (l_{ij})_{n\times n}$ be an $n \times n$ matrix with nonnegative off-diagonals.

- A *weighted digraph* $\mathcal{G} = (V, E)$ associated with the matrix L consists of a set $V = \{1, 2, \ldots, n\}$ of vertices and a set E of arcs (i, j) (i.e., directed edges from i to j) with weight l_{ji}, where $(i, j) \in E$ if and only if $l_{ji} > 0$, $i \neq j$.
- A weighted digraph $\mathcal{G}$ is strongly connected if and only if the weight matrix L is irreducible.
- A list of distinct vertices $i_1, i_2, ..., i_k$ with $k \geq 2$ form a *directed cycle* if $(i_m, i_{m+1}) \in E$ for all $m = 1, 2, ..., k-1$ and $(i_k, i_1) \in E$.
- A subdigraph $\mathcal{H}$ of $\mathcal{G}$ is *spanning* if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set.
- The *weight* of a subdigraph $\mathcal{H}$ is the product of the weights of all its arcs.

Second tool: graph theory

- A connected subdigraph $\mathcal{T}$ of $\mathcal{G}$ is a *rooted out-tree* if it contains no directed cycle, and there is one vertex, called the root, that is not an terminal vertex of any arcs while each of the remaining vertices is a terminal vertex of exactly one arc.
- A subdigraph $\mathcal{Q}$ of $\mathcal{G}$ is *unicyclic* if it is a disjoint union of two or more rooted out-trees whose roots are connected to form a directed cycle. Every vertex of unicyclic digraph $\mathcal{Q}$ is a terminal vertex of exactly one arc.
- For $\mathcal{G}$, we associate it with a row Laplacian matrix:

$$\mathcal{L} = \begin{pmatrix} \sum_{k\neq 1} l_{1k} & -l_{12} & \cdots & -l_{1n} \\ -l_{21} & \sum_{k\neq 2} l_{2k} & \cdots & -l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{n1} & -l_{n2} & \cdots & \sum_{k\neq n} l_{nk} \end{pmatrix}. \tag{4}$$

Tree-cycle identity

The following result is proved in [Li-Shuai JDE 2010]:

Theorem (Tree-Cycle Identity)

Let $\mathcal{G}$ be a strongly connected weighted digraph and let $\mathcal{L}$ be the Laplacian matrix of $\mathcal{G}$ as defined in (4). Let α_i denote the cofactor of the i-th diagonal element of $\mathcal{L}$. Then the following identity holds for $x_i, x_j \in D \subset \mathbb{R}^N, 1 \le i, j \le n$ and any family of functions $\{F_{ij} : D \times D \to \mathbb{R}\}_{1 \le i,j \le n}$

$$\sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} \alpha_i l_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_{\mathcal{Q}})} F_{sr}(x_s, x_r), \tag{5}$$

where $\mathbb{Q}$ is the set of all spanning unicyclic digraphs of $(\mathcal{G}, L)$, $w(\mathcal{Q}) > 0$ is the weight of $\mathcal{Q}$ (the product of weights of all directed edges on $\mathcal{Q}$), and $\mathcal{C}_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$ with arc set $E(\mathcal{C}_{\mathcal{Q}})$.

Cycle balanced graph

A weighted digraph $\mathcal{G}$ is said to be *cycle-balanced* if for any cycle $\mathcal{C}$ in $\mathcal{G}$ it has a corresponding reversed cycle $-\mathcal{C}$ and $w(\mathcal{C}) = w(-\mathcal{C})$. Here $-\mathcal{C}$, the reverse of $\mathcal{C}$, have the same vertices but edges with reserved direction as $\mathcal{C}$.

We note that each of the following graph is cycle-balanced:

(1) $\mathcal{L}$ is symmetric;

(2) $n = 2$;

(3) Every cycle of the weighted digraph $\mathcal{G}$ has two vertices.

Theorem (Chen-Shi-Shuai-W Nonlinearity 2022)

Let $\mathcal{G}$ be a strongly connected weighted digraph that is cycle-balanced, and let $\mathcal{L}$ be the Laplacian matrix of $\mathcal{G}$ as defined in (4). Let α_i denote the cofactor of the i-th diagonal element of $\mathcal{L}$. Assume that $x_i, x_j \in D \subset \mathbb{R}^N$ for all $1 \leq i, j \leq n$ and $\{F_{ij} : D \times D \to \mathbb{R}\}_{1 \leq i,j \leq n}$ be a family of functions satisfying

$$F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i) \geq 0, \quad 1 \leq i, j \leq n, \ j \neq i. \tag{6}$$

Then the following holds

$$\sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} \alpha_i l_{ij} F_{ij}(x_i, x_j) = \sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} \alpha_i l_{ij} F_{ji}(x_j, x_i) \geq 0. \tag{7}$$

In addition, the double sum in (7) equals 0 if and only if $F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i) = 0$ for all distinct i, j.

Outline

Special case 1

Two species compete for the common resource:

$$\begin{cases} u_i' = \mu_u \sum_{j=1}^n l_{ij} u_j + u_i(r_i - u_i - cv_i), & i = 1, \ldots, n, t > 0, \\ v_i' = \mu_v \sum_{j=1}^n l_{ij} v_j + v_i(r_i - bu_i - v_i), & i = 1, \ldots, n, t > 0. \end{cases}$$

Theorem (Chen-Shi-Shuai-W Nonlinearity 2022)

Suppose that $(A1) - (A3)$ *holds. Let* $\theta = (\theta_1, \theta_2, ..., \theta_n)$ *be a positive right eigenvector of* L *corresponding to* $s(L) = 0$ *with* $\sum_{i=1}^n \theta_i = 1$. *Then,*

(i) *If* $r = \delta\theta$ *for some* $\delta > 0$, *then we have:*

- (i_1) *If* $(b, c) = (1, 1)$, *there exists a compact global attractor consisting of a continuum of equilibria* $\{(\rho r, (1-\rho)r) : \rho \in [0, 1]\}$;
- (i_2) *If* $b \geq 1$ *and* $c < 1$, E_1 *is globally asymptotically stable;*
- (i_3) *If* $b < 1$ *and* $c \geq 1$, E_2 *is globally asymptotically stable;*
- (i_4) *If* $b < 1$ *and* $c < 1$, *there exists a unique globally stable positive equilibrium*

$$E = \left(\frac{1-c}{1-bc} r, \frac{1-b}{1-bc} r \right).$$

Theorem

(ii) *If $r \neq \delta\theta$ for any $\delta > 0$, then we have:*

- (ii$_1$) *Suppose $\mu_u < \mu_v$. Then there exist $b^* < 1$ and $c^* > 1$ with $b^* c^* > 1$ such that if $b < b^*$ and $c < c^*$ the model has a unique positive equilibrium which is globally asymptotically stable; if $b \geq b^*$, E_1 is globally asymptotically stable; if $c \geq c^*$, E_2 is globally asymptotically stable;*
- (ii$_2$) *Suppose $\mu_u > \mu_v$. Then there exist $b^* > 1$ and $c^* < 1$ with $b^* c^* > 1$ such that if $b < b^*$ and $c < c^*$ the model has a unique positive equilibrium which is globally asymptotically stable; if $b \geq b^*$, E_1 is globally asymptotically stable; if $c \geq c^*$, E_2 is globally asymptotically stable;*
- (ii$_3$) *Suppose $\mu_u = \mu_v$. Then if $b < 1$ and $c < 1$, the model has a unique positive equilibrium which is globally asymptotically stable; if $b \geq 1$ and $c < 1$, E_1 is globally asymptotically stable; if $c \geq 1$ and $b < 1$, E_2 is globally asymptotically stable; if $(b, c) = (1, 1)$, there exists a compact global attractor consisting of a continuum of steady states $\{(\rho w^*(\mu_u, r), (1-\rho) w^*(\mu_v, r)) : \rho \in [0, 1]\}$.*

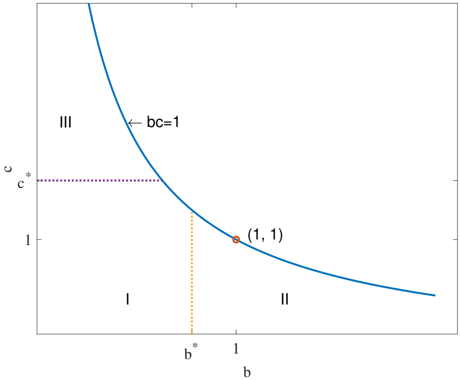

Figure: Illustration of Theorem (ii$_1$) ($r \neq \delta\theta$ and $\mu_u < \mu_v$). Here, $b^* < 1$, $c^* > 1$, and $b^* c^* > 1$. In I, there exists a globally asymptotically stable positive equilibrium; in II, E_1 is globally asymptotically stable; in III, E_2 is globally asymptotically stable.

Special case 2

Two species have the same intraspecific competition coefficients and diffusion rates, but different resources availability:

$$\begin{cases} u_i' = \mu \sum_{j=1}^{n} l_{ij} u_j + u_i(p_i - u_i - v_i), & i = 1, \ldots, n, t > 0, \\ v_i' = \mu \sum_{j=1}^{n} l_{ij} v_j + v_i(q_i - u_i - v_i), & i = 1, \ldots, n, t > 0, \\ u(0) = u_0 \geq (\not\equiv) \mathbf{0}, \; v(0) = v_0 \geq (\not\equiv) \mathbf{0}. \end{cases} \tag{8}$$

Theorem (Chen-Shi-Shuai-Wu Nonlinearity 2022)

Suppose that (A1)-(A3) hold. If $p > q$, then $E_1 = (w^*(\mu, p), \mathbf{0})$ is globally asymptotically stable for (8); if $p < q$, then $E_2 = (\mathbf{0}, w^*(\mu, q))$ is globally asymptotically stable for (8).

Theorem (Chen-Shi-Shuai-W Nonlinearity 2022)

Suppose that (A1)-(A3) hold, and $p \not\geq q$ *and* $q \not\geq p$. *Let* $\theta = (\theta_1, \theta_2, ..., \theta_n)$ *be a positive right eigenvector of L corresponding to* $s(L) = 0$ *satisfying* $\sum_{i=1}^{n} \theta_i = 1$. *Then the following statements hold for* (8)*:*

1. *There exists* $\mu_1 > 0$ *such that the model has a unique positive equilibrium which is globally asymptotically stable for* $0 < \mu < \mu_1$.
2. *If* $\sum_{j=1}^{n} \theta_j(p_j - q_j) > 0$, *then there exists* $\mu_2 > \mu_1$ *such that* E_1 *is globally asymptotically stable for* $\mu > \mu_2$*; on the other hand if* $\sum_{j=1}^{n} \theta_j(p_j - q_j) < 0$, *then there exists* $\mu_3 > \mu_1$ *such that* E_2 *is globally asymptotically stable for* $\mu > \mu_3$.

Theorem (Chen-Shi-Shuai-W Nonlinearity 2022)

Suppose that (A1)-(A3) hold. Let $\Omega_u = \{i : 1 \le i \le n, p_i > q_i\}$ *and* $\Omega_v = \{i : 1 \le i \le n, p_i < q_i\}$. *Suppose that* Ω_u *and* Ω_v *are not empty with* $\Omega_u \cup \Omega_v = \{1, 2, ..., n\}$. *Let* $U_0 = (U_{01}, U_{02}, ..., U_{0n})$ *and* $V_0 = (V_{01}, V_{02}, ..., V_{0n})$, *where*

$$U_{0i} = \begin{cases} p_i, & \text{if } i \in \Omega_u, \\ 0, & \text{if } i \in \Omega_v, \end{cases} \quad \text{and} \quad V_{0i} = \begin{cases} 0 & \text{if } i \in \Omega_u, \\ q_i & \text{if } i \in \Omega_v. \end{cases}$$

Let (u, v) *be the unique positive equilibrium of the model when* μ *is small, then* $\lim_{\mu \to 0}(u, v) = (U_0, V_0)$.

Outline

Future works

- Can we remove cycle-balanced condition?
- Strong competition? $bc > 1$.
- More general:

$$\begin{cases} u_i' = \mu_u \sum_{j=1}^n l_{ij} u_i + u_i(p_i - a_i u_i - c_i v_i), & i = 1, \dots, n, t > 0, \\ v_i' = \mu_v \sum_{j=1}^n l_{ij} v_j + v_i(q_i - b_i u_i - d_i v_i), & i = 1, \dots, n, t > 0. \end{cases}$$

- Radom and directed movement rates are not equal?

Competition in stream

Two-species competition model in spatially homogeneous stream environment:

$$
\begin{cases}
\dfrac{du_i}{dt} = \displaystyle\sum_{j=1}^{n} (d_1 D_{ij} + q_1 Q_{ij}) u_j + u_i(r - u_i - v_i), \quad i = 1, \dots, n, \ \ t > 0, \\
\dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{n} (d_2 D_{ij} + q_2 Q_{ij}) v_j + v_i(r - u_i - v_i), \quad i = 1, \dots, n, \ \ t > 0, \\
u(0) = u_0 \geq (\not\equiv) \mathbf{0}, \ v(0) = v_0 \geq (\not\equiv) \mathbf{0}.
\end{cases}
\tag{9}
$$

Competition in stream

Three cases:

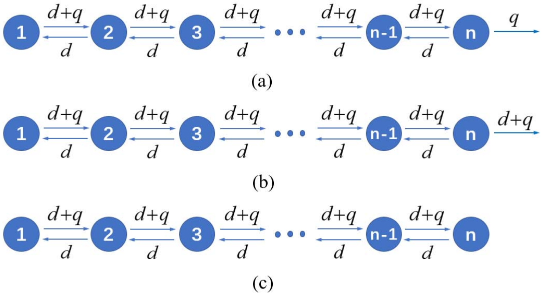

Figure: Movements of the species among patches. Here, (a): stream to lake; (b): stream to ocean; and (c): inland stream.

We have the following result for Case a.

Theorem (Chen-Shi-Shuai-W, Submitted)

Then the following statements hold:

(*i*) *Fix* $q_2 = q_1$. *If* $d_2 < d_1$, $(u^*, \mathbf{0})$ *is globally asymptotically stable; and if* $d_2 > d_1$, $(\mathbf{0}, v^*)$ *is globally asymptotically stable;*

(*ii*) *Fix* $d_2 = d_1$. *If* $q_2 > q_1$, $(u^*, \mathbf{0})$ *is globally asymptotically stable ; and if* $q_2 < q_1$, $(\mathbf{0}, v^*)$ *is globally asymptotically stable.*

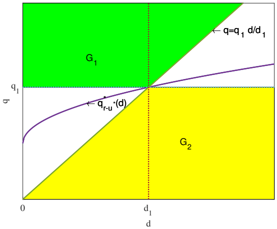

Figure: If $(d_2, q_2) \in G_1$, $(u^*, \mathbf{0})$ is globally asymptotically stable; and if $(d_2, q_2) \in G_2$, $(\mathbf{0}, v^*)$ is globally asymptotically stable.

Thank you!