Asymptotic spreading of KPP reactive fronts in heterogeneous shifting environments II: Flux-limited solutions

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Abstract

We consider the spreading dynamics of the Fisher-KPP equation in a shifting environment, by characterizing the limit of the rate function of the solution. For the environment with a weak monotone condition, it was demonstrated in a previous paper that the rate function converges to the unique viscosity solution of the underlying Hamilton-Jacobi equations. In case the environment does not satisfy the weak monotone condition, we show that the rate function is then characterized by the Hamilton-Jacobi equation with a dynamic junction condition, which depends additionally on the generalized principal eigenvalue derived from the environmental function. This approach applies to the case when the environment has multiple shifting speeds, and clarifies the transition between nonlocally pulled fronts and forced traveling waves.

Contents

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1 Introduction

Consider the following KPP equation with heterogeneous coefficients

$$\begin{cases} u_t - u_{xx} = u(g(x - c_1 t) - u) & \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $c_1 \in \mathbb{R}$, g is a continuous and positive function on $\mathbb{R}$ with $g(\pm\infty) > 0$, and $u_0 \in L^\infty(\mathbb{R})$ is nonnegative, nontrivial and compactly supported. This type of equations models the growth and dispersal of a population with density u in a shifting habitat, which arises from the ecological question of whether the species can survive in the midst of climate change [52, 7, 45]. In a previous work [44], we considered (1.1) as a special case of a class of integro-differential equations with a distributed time-delay in heterogeneous shifting environments. Under the assumption $\sup g \leq \max\{g(\pm\infty)\}$, we utilized the theory of viscosity solutions of Hamilton-Jacobi equations, specifically the uniqueness of viscosity solutions in the sense of Ishii [42], to obtain the complete explicit formulas of rightward spreading speeds for (1.1) in terms of c_1 as well as leftward spreading speeds. However, the assumption $\sup g \leq \max\{g(\pm\infty)\}$ was crucial in [44], as the uniqueness of viscosity solutions in the sense of Ishii can no longer be expected if such a condition is relaxed [3, 31].

To recover uniqueness and further develop the Hamilton-Jacobi approach, we will utilize the notion of flux-limited solution recently introduced by Imbert and Monneau [40, 41], which were motivated by the study of Hamilton-Jacobi equations on networks. See also [37] for a recent application of this approach to study the shape of expansion in the road-field propagation model introduced by Berestycki et al. [12]. Throughout our paper, we are mainly working on (1.1) which has one shifting speed c_1, and we are able to determine the spreading properties of (1.1) for any g for which $g(\pm\infty)$ exist. Furthermore, our treatment naturally extends to the case of multiple shifting speeds; see Section 2.4 for the precise statements. Before stating our main results, we provide a brief account of several related works.

The asymptotic speed of spread, or spreading speed in short, is a crucial quantity in spatial ecology that determines the expansion boundary of a population under the joint influence of the diffusion rate and environmental conditions. For simplicity, the diffusion rate has been normalized to 1 in (1.1). In a homogeneous environment, i.e. $g(\cdot) \equiv g_0$ for a positive constant g_0, model (1.1) reduces to the classical Fisher-KPP equation. A well known result of Kolmogorov et al. [43] states that there is a number $c_* = 2\sqrt{g_0} > 0$ such that

$$\lim_{t\to\infty} \sup_{x \geq ct} u(t,x) = 0 \quad \text{for } c \in (c_*, +\infty), \quad \text{and} \quad \liminf_{t\to\infty} \inf_{0\leq x \leq ct} u(t,x) > 0 \quad \text{for } c \in (0, c_*).$$

Moreover, the same value also coincides with the minimal wave speed of traveling wave solutions $U(x - ct)$ of (1.1). This result was later extended to more general nonlinearity and in higher dimensions in [1]. It is also remarkable that in homogeneous media, the spreading speed can be obtained via local information, where $c_* = 2\sqrt{g_0}$ is the smallest value of $c > 0$ such that

$$-c\phi_x + \phi_{xx} + g_0\phi = \Lambda\phi$$

admits a zero generalized principal eigenvalue.

Since then, spreading speeds for various reaction-diffusion models including Fisher-KPP equations have been intensively investigated [55, 46, 9, 53, 10, 25, 11]. Among those, an elaborate method was proposed by Weinberger [55] to establish the existence of spreading speeds for discrete-time order-preserving recursions with a monostable structure and its characterization as the minimal speed of traveling wave solutions. These results were subsequently generalized to monotone semiflows [46, 25]. By combining the Hamilton-Jacobi approach [23] and homogenization ideas [47, 22], Berestycki et al. [10, 11] showed the existence of spreading speed for spatially almost periodic, random stationary ergodic, and other general environments, whose speed was characterized as a min-max formula in terms of suitable notions of generalized principal eigenvalues in unbounded domains.

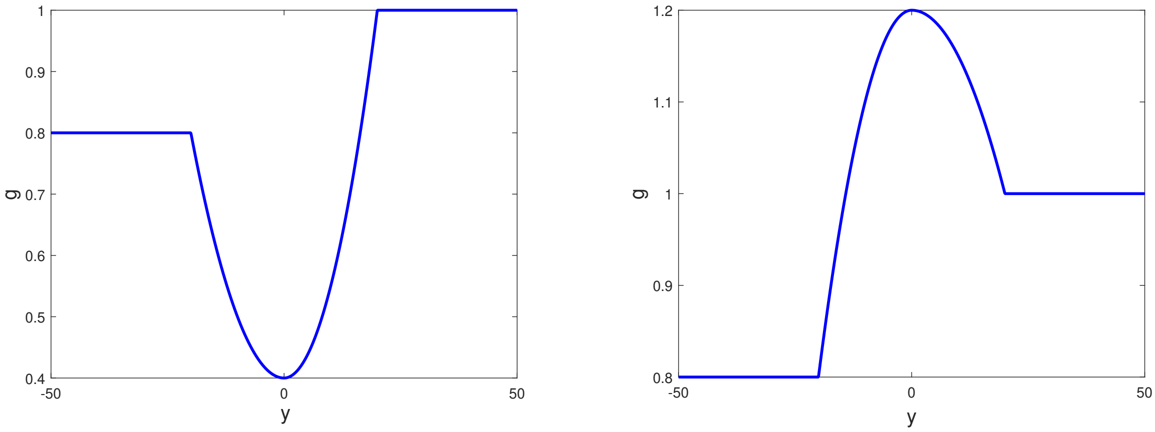
The heterogeneous shifting environment, which is the focus of this article, was introduced by Potapov and Lewis [52] and Berestycki et al. [7] to investigate the impact of shifting climate on the persistence of one or several focal species. As a simple formulation, the temporal-spatial heterogeneity $x-c_1t$ was incorporated into various diffusion models including (1.1) for the single species, where c_1 is regarded as the velocity of the shifting climate. For (1.1), the propagation dynamics have been rigorously explored in [52, 7, 13, 14] for the case of a moving patch of a finite length, and in [45, 39] for a retreating semi-infinite patch. The latter problem is a special case of (1.1) in case g is increasing and $g(-\infty) \leq 0 < g(+\infty)$, where it is proved that the species persists if and only if it can spread faster than the environment with the spreading speed being given by the KPP formula $c_* = 2\sqrt{g(+\infty)}$. A shifting environment can also arise in other ways. Holzer and Scheel [38] considered a partially decoupled reaction-diffusion system of two equations, where a wave solution for the first equation induces a shifting environment for the second one. See also [21, 18, 33, 50] for further results on competition or prey-predator systems. A similar modeling idea was also adapted in Fang et al. [24], where (1.1) was also retrieved from an SIS disease model to study whether pathogen can keep pace with its host. If g is non-increasing, then (1.1) becomes a special case of the cylinder problem studied by Hamel [35]. Du and collaborators [20, 19] proposed a free boundary version of (1.1). Yi and Zhao [56, 57] established a general theory on the propagation dynamics without spatial translation invariance. See also [26] for a model with shifting diffusivity. Finally, we refer to Wang et al. [54] for a survey on reaction-diffusion models in shifting environments.

Indeed, the shifting habitat brings about new spreading phenomena in case that the intrinsic growth rate profile g is strictly positive everywhere. When $0 < \inf g < \sup g \leq \max\{g(\pm\infty)\} = g(+\infty)$, the results of [38, 44] clarified that, for a certain range of shift speed c_1, the initially compactly supported population selects a supercritical speed of spread $c_* > 2\sqrt{g(-\infty)}$ in a phenomenon called non-local pulling [38, 33]; see Figure 1(a). This falls into the biological scenario when the species fails to keep up with the climate shifting, but is still influenced by the presence of a favorable habitat which is located at a distance of order t ahead of the front. When $\sup g > \max\{g(\pm\infty)\}$, Holzer and Scheel [38] proved the existence of forced traveling wave solutions, which move at the same speed as the environment. Subsequently, Berestycki and Fang [8] classified such forced traveling wave solutions and proved global attractivity results.

Our main contribution, in the case of (1.1), where the environment has a single speed c_1, is to completely determine the existence of the rightward spreading speed c_* and its dependence on the environmental speed c_1, whenever $g(\pm\infty)$ exist and $\inf g > 0$; see Figure 1(b). Moreover, our framework provides the context in which the spreading results in [38, 44] (where $c_* < c_1$ with nonlocal pulling) connects with those in [8] (where $c_* = c_1$). Furthermore, our method readily generalizes to the case when the environment has more than one shifting speed (Subsection 2.4).

Organization of the paper

Our approach is to study the spreading speed via the asymptotic limit of the rate function, following [23] (see also [3, Chapter 29] and [44]). However, the consideration of a shifting habitat leads to a discontinuous Hamiltonian. Also, the rate function has unbounded and discontinuous



(a) $\sup g \leq \max\{g(\pm\infty)\}$ (b) $\sup g > \max\{g(\pm\infty)\}$

Figure 1: For panel (a), the spreading speed $c_* = s_{base}$, see Theorem 2.14 (due to [44]) for the explicit formula. For panel (b), the exact formula of c_* is given by Theorem 3.2 which is not covered by [44].

initial data since the initial population was compactly supported. In our previous work [44], the solution concept of Ishii was used and the corresponding comparison principle was established. Nevertheless, the previous results in [44] are not applicable in case $\sup g > \max\{g(\pm\infty)\}$, because then the invasion is enhanced by the specific profile of g, and the solutions in the sense of Ishii are generally non-unique. To overcome the non-uniqueness of viscosity solution and connect with the results regarding forced waves, we need to incorporate further information of (1.1) in deriving the limiting Hamilton-Jacobi equation. To this end, we recall some results of an eigenvalue problem with weight coefficient $g(\cdot)$. Then in Subsection 2.1, we introduce the concept of a flux-limited solution and prove the comparison principle needed in our context. In Subsection 2.2, we state our main theorems and extensions of our results. We also discuss the viscosity solutions in sense of Ishii and recall some earlier results from [44] in Subsection 2.6.

Section 3 presents the application of Theorem 2.9, and we place it immediately after stating our main theorem. In this section, we take Theorem 2.9 for granted and apply it to obtain several explicit formulas for the spreading speed in terms of $g(+\infty), g(-\infty)$ and Λ_1, where Λ_1 is the principal eigenvalue given in (2.5). This provides a general context connecting previous results of [8, 38] concerning forced wave (where $c_* = c_1$) and of [44] concerning nonlocal pulling (where $c_* < c_1$ but is influenced by the presence of the shifting environment).

In Section 4, we present preliminary results. In particular, we recall the properties of Λ_1 in Proposition 4.2 (Subsection 4.1), as well as a few technical results for Hamilton-Jacobi equations.

Section 5 is devoted to the proof of the main results, namely, Proposition 2.6, Corollary 2.7, and Theorem 2.9. This section only logically depends on Proposition 4.2 (proved in Appendix A), Lemma 4.4 concerning continuity of subsolutions (proved in Subsection 4.2), the critical slope lemmas inspired by Imbert and Monneau [40] (Lemmas 4.5 and 4.7, proved in Appendix B), as well as the comparison principle (proved in Appendix C).

Finally, the appendices present the proofs of the technical results mentioned above.

2 Preliminaries and Statements of Results

The concepts of maximal and minimal spreading speeds are introduced in [36, Definition 1.2] for a single species; see also [30, 50]. In our setting, we define

$$\begin{cases} \overline{c}_* = \inf\{c > 0 \mid \limsup_{t\to\infty} \sup_{x>ct} u(t,x) = 0\}, \\ \underline{c}_* = \sup\{c > 0 \mid \liminf_{t\to\infty} \inf_{0<x<ct} u(t,x) > 0\}, \end{cases} \tag{2.1}$$

where $\overline{c}_*$ and $\underline{c}_*$ are the maximal and minimal (rightward) spreading speeds of species u, respectively. If $\overline{c}_* = \underline{c}_* > 0$, we say that the population has the (rightward) spreading speed given by the common value c_*.

Motivated by the large deviations technique [23, 27, 28], we introduce, for fixed solution u of (1.1), the scaling $u^\epsilon(t,x) = u(\frac{t}{\epsilon}, \frac{x}{\epsilon})$ with $\epsilon > 0$. The resulting function $u^\epsilon(t,x)$ satisfies the following equation:

$$\begin{cases} u^\epsilon_t - \epsilon u^\epsilon_{xx} = \frac{1}{\epsilon} u^\epsilon \left(g\left(\frac{x-c_1 t}{\epsilon}\right) - u^\epsilon\right) & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}, \\ u^\epsilon(0,x) = u_0(x/\epsilon) & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Observe that the spreading speed of the population is given by $c_* > 0$ if and only if

$$\lim_{\epsilon\to 0} u^\epsilon(t,x) \to 0 \quad \text{in } C_{loc}(\{x > c_* t\}), \quad \liminf_{\epsilon\to 0} u^\epsilon(t,x) > 0 \quad \text{in } C_{loc}(\{0 \le x < c_* t\}). \tag{2.3}$$

To fully characterize these limits, we introduce the following eigenvalue problem:

$$\Phi'' + g(y)\Phi = \Lambda\Phi \quad \text{for } y \in \mathbb{R}. \tag{2.4}$$

In this paper, we define the principal eigenvalue Λ_1 of (2.4) as follows:

$$\Lambda_1 := \Lambda_1(g) = \inf\left\{\Lambda \in \mathbb{R} : \ \exists \phi \in C^2_{loc}(\mathbb{R}),\ \phi > 0,\ \phi'' + g(y)\phi \le \Lambda\phi \text{ in } \mathbb{R}\right\}. \tag{2.5}$$

This and several other notions of principal eigenvalues are analyzed in [15]. We will recall some basic properties of Λ_1 and the associated positive eigenfunction in Proposition 4.2.

As we shall see in Section 3, the four quantities c_1, Λ_1, $g(+\infty)$ and $g(-\infty)$ completely determine the spreading speed c_*.

2.1 Flux-limited solution due to Imbert and Monneau

To determine the exact spreading speed in Theorem 2.9, we will study the rate function $w^\epsilon(t,x) := -\epsilon \log u^\epsilon(t,x)$. More precisely, we will show that $w^\epsilon(t,x) \to t\hat{\rho}(x/t)$ in C_{loc}, where the limit $\hat{\rho}(s)$ is to be interpreted using the notion of *flux-limited solution* introduced by Imbert and Monneau [40]. This notion is well-adapted to catch the influence of the coefficients along a discontinuity at $s = c_1$.

We begin with a few notations regarding the effective Hamiltonian and effective junction condition.

Definition 2.1. Define the Hamiltonian $H(s,p)$ by

$$H(s,p) = -sp + p^2 + \chi_{\{s>c_1\}} g(+\infty) + \chi_{\{s\le c_1\}} g(-\infty),$$

and its increasing (resp. decreasing) part by

$$H^+(s,p) = \inf_{p'>p} H(s,p') \qquad \left(\text{resp.} \quad H^-(s,p) = \inf_{p'<p} H(s,p')\ \right).$$

Definition 2.2. For each $A \in \mathbb{R}$, define the flux-limited junction condition to be

$$F_A(\tilde{p}_+, \tilde{p}_-) = \max\{A, H^-(c_1+, \tilde{p}_+), H^+(c_1-, \tilde{p}_-)\}, \tag{2.6}$$

where $H^+(c_1\pm, p) = \lim_{s\to c_1\pm} H^+(s,p)$ and $H^-(c_1\pm, p) = \lim_{s\to c_1\pm} H^-(s,p)$, as defined above, can be expressed as follows:

$$H^+(c_1\pm, p) = \begin{cases} -\frac{|c_1|^2}{4} + g(\pm\infty), & \text{for } p \le c_1/2, \\ -c_1 p + p^2 + g(\pm\infty) & \text{for } p > c_1/2, \end{cases} \tag{2.7}$$

$$H^-(c_1\pm, p) = \begin{cases} -c_1 p + p^2 + g(\pm\infty) & \text{for } p \le c_1/2, \\ -\frac{|c_1|^2}{4} + g(\pm\infty) & \text{for } p > c_1/2. \end{cases} \tag{2.8}$$

Remark 2.3. The above definitions are adapted from [41, Section 2] with $s/2$ being the minimum point of $p \to H(s,p)$.

The information of the profile of g can be incorporated into the Hamilton-Jacobi equation by an additional *junction condition* as follows:

$$\begin{cases} \min\{\rho, \rho + H(s, \rho')\} = 0 & \text{for } s > 0, s \ne c_1, \\ \min\{\rho(c_1), \rho(c_1) + F_A(\rho'(c_1+), \rho'(c_1-))\} = 0, \end{cases} \tag{2.9}$$

where the flux-limiter $A \in \mathbb{R}$ will be specified in (2.13), and $F_A(\tilde{p}_+, \tilde{p}_-)$ are given in (2.13) and (2.6) respectively. The above equations are to be considered using continuous and *piecewise* C^1 test functions whose left and right derivatives at c_1 are well defined but maybe unequal:

$$C^1_{pw} = \{\psi \in C((0,\infty)) : \ C^1((0, c_1]) \cap C^1([c_1, \infty)). \tag{2.10}$$

In the following, we provide the definitions of FL-super/subsolutions introduced in [40]. Note that the boundary conditions are satisfied in a strong sense, in contrast to the usual relaxed sense in classical viscosity solutions [2].

Definition 2.4. Let $A \in \mathbb{R}$ be given.

(a) We say that $\hat{\rho} : (0,\infty) \to \mathbb{R}$ is a FL-subsolution of (2.9) provided (i) $\hat{\rho}$ is upper semicontinuous, and (ii) if $\hat{\rho} - \psi$ (with $\psi \in C^1_{pw}$) attains a local maximum point at some $s_0 > 0$ such that $\hat{\rho}(s_0) > 0$, then

$$\hat{\rho}(s_0) + H(s_0, \psi'(s_0)) \le 0 \quad \text{in case } s_0 \ne c_1,$$

$$\hat{\rho}(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \le 0 \quad \text{in case } s_0 = c_1.$$

(b) We say that $\hat{\rho} : (0,\infty) \to \mathbb{R}$ is a FL-supersolution of (2.9) provided (i) $\hat{\rho}$ is lower semicontinuous, (ii) $\hat{\rho} \ge 0$ for all $s > 0$, and (iii) if $\hat{\rho} - \psi$ (with $\psi \in C^1_{pw}$) attains a local minimum point at some $s_0 > 0$, then

$$\hat{\rho}(s_0) + H(s_0, \psi'(s_0)) \ge 0 \quad \text{in case } s_0 \ne c_1,$$

$$\hat{\rho}(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \ge 0 \quad \text{in case } s_0 = c_1.$$

(c) We say that $\hat{\rho}$ is a FL-solution of (2.9) if it is both FL-subsolution and FL-supersolution of (2.9).

Remark 2.5. In practice, $\hat{\rho} - \psi$ having a local maximum point at s_0 is equivalent to $\tilde{\psi}(s) := \psi(s) - \psi(s_0) + \hat{\rho}(s_0)$ touching $\hat{\rho}$ at the point s_0 from above. In addition, one can also assume without loss of generality that $\hat{\rho} - \psi$ has a *strict* local maximum [2, Proposition 3.1]. Hence we will sometimes reduce our consideration to this smaller class of test functions, without loss of generality, for the verification of subsolution property. Analogous statements hold when considering supersolutions.

Next, we discuss the uniqueness of FL-solution of (2.9) by first showing the following comparison principle.

Proposition 2.6. *Let $A \in \mathbb{R}$ be given. If $\underline{\rho}$ and $\overline{\rho}$ are, respectively, a FL-subsolution and a FL-supersolution of* (2.9)*, and such that*

$$\underline{\rho}(0) \leq \overline{\rho}(0) \quad \text{and} \quad \lim_{s \to +\infty} \frac{\overline{\rho}(s)}{s} = +\infty, \tag{2.11}$$

then $\underline{\rho}(s) \leq \overline{\rho}(s)$ in $[0, +\infty)$.

Corollary 2.7. *For each $A \in \mathbb{R}$,* (2.9) *has a unique FL-solution $\hat{\rho}_A$ which satisfies the following boundary conditions (in a strong sense)*

$$\rho(0) = 0 \quad \text{and} \quad \lim_{s \to +\infty} \frac{\rho(s)}{s} = +\infty. \tag{2.12}$$

These two results will be proved in Subsection 5.4.

We apply the half-relaxed limit method, due to Barles and Perthame [4, 5], to pass to the (upper and lower) limits of $w^\epsilon(t, x)$. Moreover, we can show that $w^\epsilon(t, x) \to t\hat{\rho}_A(x/t)$ in C_{loc}, where the *flux limiter* A is identified by

$$A = \Lambda_1 - \frac{c_1^2}{4}. \tag{2.13}$$

The spreading speed c_* will then be fully characterized by $\hat{\rho}_A$ with this specific flux-limiter A.

Remark 2.8. Note that $A = \Lambda_1 - \frac{c_1^2}{4}$ could be regarded as the principal eigenvalue of of $\Psi'' - c_1\Psi' + g(y)\Psi = A\Psi$ in the sense that:

$$A = \inf\left\{\Lambda \in \mathbb{R} : \ \exists \phi \in C^2_{loc}(\mathbb{R}), \ \phi > 0, \ \phi'' - c_1\phi' + g(y)\phi \leq \Lambda\phi \text{ in } \mathbb{R}\right\}, \tag{2.14}$$

which quantifies the influence of the coefficient $g(x - c_1t)$ in the moving coordinate $y = x - c_1t$.

2.2 Main results

We are now in position to state our main result.

Theorem 2.9. *Let u be a solution of* (1.1)*. Then the following statements hold.*

(a) *The spreading speed c_* of u exists, and is given by*

$$c_* = \hat{s}_A = \sup\{s \in [0, \infty) : \hat{\rho}_A(s) = 0\}, \tag{2.15}$$

where $\hat{\rho}_A$ is the unique FL-solution of (2.9)*with $A = \Lambda_1 - \frac{c_1^2}{4}$ that also satisfies the boundary conditions* (2.12)*.*

(b) *Furthermore, if $\Lambda_1 = \max\{g(\pm\infty)\}$, then $c_* = s_{base} = \sup\{s : \hat{\rho}_{base}(s) = 0\}$, where $\hat{\rho}_{base}$ is the unique viscosity solution of* (2.26)*–*(2.12) *in the sense of Ishii (see Definition D.1).*

Remark 2.10. In Section 3, we will give explicit formulas of c_* in terms of of $g(\pm\infty)$, c_1 and Λ_1.

Remark 2.11. After the research of this work has finished, the preprint of Giletti et al. [32] was brought to our attention, where the authors treated the case when g is piecewise constant:

$$g(t,x) = r_1\chi_{\{x<A(t)\}} + r_2\chi_{\{A(t)\le x<A(t)+L\}} + r_3\chi_{\{x\ge A(t)+L\}}. \tag{2.16}$$

Here $t \mapsto A(t)$ is either linear or slowly oscillating between two shifting speeds. Interestingly, they obtained the formula of Theorem 3.2 assuming that g is given by (2.16) with $A(t) = c_1 t$.

Furthermore, it was remarked that their construction can be generalized to treat (1.1) provided that $g(y)$ is constant near $y = \pm\infty$. They also conjectured that the last condition may not be necessary. Their proof is based on the direct construction of super/subsolution for the parabolic problem.

Incidentally, our main result can be considered as an affirmative answer of their conjecture, by passing to the limiting Hamilton-Jacobi problem with junction condition. It is worth mentioning that (i) we need only $A(t) = c_1 t + o(t)$ and (ii) we merely require $g(\pm\infty)$ exist (but not necessarily constant for $|x| \gg 1$). In particular, the spreading speed can be determined by the value of the eigenvalue Λ_1 and the exact shape of g is not important.

Here we mention several outstanding open questions.

1. The effect of more general initial data that are not compactly supported [51].
2. The consideration of models with nonlocal coupling in space (nonlocal diffusion) or in time (time-delay) [44].
3. More precise estimate of the level set such as the analysis of logarithmic correction term, and the convergence of the solution profile to the traveling wave profile [16].
4. If g is bounded and $g(\pm\infty)$ does not exist, and/or when $A(t)/t$ is bounded but does not tend to a limit, then it is not clear whether the spreading speed exists, or that there exist distinct maximal and minimal speeds, as in [36, Definition 1.2].

2.3 Monotonicity of the flux-limited solutions

Since $F_A(\tilde{p}_+, \tilde{p}_-)$ is monotone increasing in the variable A, the effect of the flux limiter A is as follows.

Corollary 2.12. *Let $A \in \mathbb{R}$ and $\hat{\rho}_A$ be the unique FL-solution of* (2.9)-(2.12)*.*

(a) *If $A \ge A'$, then $\hat{\rho}_A(s) \le \hat{\rho}_{A'}(s)$ for all $s \ge 0$. In particular, the free boundary point $\hat{s}_A$ is monotone increasing with respect to A, i.e. $\hat{s}_A \ge \hat{s}_{A'}$, where*

$$\hat{s}_A := \sup\{s : \hat{\rho}_A(s) = 0\}. \tag{2.17}$$

(b) *If $A_0 := \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$, then for any $A \le A_0$, $\hat{\rho}_A$ coincides with the unique viscosity solution of* (2.26)-(2.12) *in the sense of Ishii, and is independent of $A \le A_0$. In case $A > A_0$, $\hat{\rho}_A$ is a viscosity subsolution (but not necessarily a supersolution) in the sense of Ishii.*

Proof. Assertion (a) is a direct consequence of Proposition 2.6, since $\hat{\rho}_A(0) = \hat{\rho}_{A'}(0) = 0$, $\hat{\rho}_{A'}$ is a FL-supersolution of (2.9) and satisfies (2.12). Assertion (b) is proved in Subsection 5.2. □

2.4 Generalizations

Our arguments can also be applied in the following setting where $f = f(t,x,u)$ possesses multiple junction points $\mathcal{P} = \{c_i\}_{i=1}^n$ for some $0 < c_1 < c_2 < ... < c_n$.

$$\begin{cases} u_t - u_{xx} = uf(t,x,u) & \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{2.18}$$

(H1′) $f(t,x,u) \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$.

(H2′) There exists $g_i \in L^\infty(\mathbb{R})$ with $g_i(\pm\infty) > 0$, $i = 1, ..., n$, and for some $\delta_1 > 0$,

$$\lim_{t\to\infty} \text{ess sup}_{|x-c_it|<\delta_1 t} |f(t,x,0) - g_i(x - c_i t)| = 0.$$

(H3′) There exist $R, \underline{R} : [0,\infty) \to [0,\infty)$ such that $\inf \underline{R} > 0$, $R(s) = \underline{R}(s)$ a.e. and

$$R(s) = \limsup_{\substack{\epsilon\to 0^+ \\ (t,x)\to(1,s)}} f\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, 0\right) \quad \text{and} \quad \underline{R}(s) = \liminf_{\substack{\epsilon\to 0^+ \\ (t,x)\to(1,s)}} f\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, 0\right).$$

Moreover, R is locally monotone in $\mathbb{R} \setminus \mathcal{P}$, i.e. for each $s_0 \in \mathbb{R} \setminus \mathcal{P}$,

$$\text{either} \quad \liminf_{\delta\to 0} \inf_{(s,s')\in S(\delta)} [R(s) - R(s')] \geq 0 \quad \text{or} \quad \limsup_{\delta\to 0} \sup_{(s,s')\in S(\delta)} [R(s) - R(s')] \leq 0,$$

where $S(\delta) = \{(s,s') : \ s_0 - \delta < s < s' < s_0 + \delta\}$.

(H4′) There exist positive constants C', σ' such that

$$-C'|u|^{\sigma'} \leq f(t,x,u) - f(t,x,0) \leq 0 \quad \text{for all } (t,x), \text{ and } 0 \leq u < \sigma'.$$

(H5′) There exists $M > 0$ such that $f(t,x,u) < 0$ for $(t,x,u) \in \mathbb{R}_+ \times \mathbb{R} \times [M,\infty)$.

(H6′) For each $i = 1, ..., n$, let $\Lambda_1^{(i)}$ be the principal eigenvalue given by

$$\Lambda_1^{(i)} = \inf \left\{\Lambda \in \mathbb{R} : \ \exists \phi \in C^2_{loc}(\mathbb{R}), \ \phi > 0, \ \phi'' + g_i(y)\phi \leq \Lambda\phi\right\}. \tag{2.19}$$

Theorem 2.13. *Given $\mathcal{P} = \{c_i\}_{i=1}^n$, and f satisfying* (H1′) – (H6′). *If $u_0 \in L^\infty(\mathbb{R}, \mathbb{R}_+)$ is nontrivial and compactly supported, then the solution of* (2.18) *spreads to the right at speed c_*, where*

$$c_* = \sup\{s \geq 0 : \ \rho^\dagger(s) = 0\},$$

and $\rho^\dagger$ is the unique FL-solution of

$$\begin{cases} \min\{\rho, \rho + H(s,\rho')\} = 0 & \textit{for } s \in (0,\infty) \setminus \mathcal{P}, \\ \min\{\rho(c_i), \rho(c_i) + F^{(i)}(\rho'(c_i+), \rho'(c_i-))\} = 0 & \textit{for all } 1 \leq i \leq n, \\ \rho(0) = 0 \quad \textit{and} \quad \lim\limits_{s\to+\infty} \rho(s)/s = +\infty, \end{cases} \tag{2.20}$$

where

$$H(s,p) = -sp + p^2 + R(s),$$

$$F^{(i)}(\tilde{p}_+, \tilde{p}_-) = \max\left\{\Lambda_1^{(i)} - \frac{c_i^2}{4}, \ H^-(c_i+, \tilde{p}_+), \ H^+(c_i-, \tilde{p}_-)\right\},$$

such that $H^\mp(c_i\pm, \cdot)$ are the decreasing/increasing parts of $H(c_i\pm, \cdot)$ given by

$$H^-(c_i+, p) = H(c_i+, \min\{p, c_i/2\}), \quad H^+(c_i-, p) = H(c_i-, \max\{p, c_i/2\}),$$

and $\Lambda_1^{(i)}$ is the principal eigenvalue given by (2.19).

Proof. For $s \in (0,\infty) \setminus \mathcal{P}$, the limiting Hamilton-Jacobi equation (2.20) can be derived exactly as in [44, Section 2]. The derivation of the second equation in (2.20) for $s \in \mathcal{P}$, the uniqueness of $\rho^\dagger$, and the rest of the assertions can be developed in exactly the same procedure to establish Theorem 2.9 in Section 5. □

2.5 Related optimal control formulations

It is well known that the viscosity solutions of Hamilton-Jacobi equations correspond naturally to the value functions of certain optimal control problems, whereas the viscosity solutions of variational inequalities correspond to the value functions of two-player, zero-sum deterministic differential games [23]. In fact, following the control formulation for Hamilton-Jacobi equations stated in [40], we conjecture that the following formulation holds for the unique FL-solution of (2.20):

$$\rho(s) = \inf_{\substack{z\in X\\ z(0)=s}} \left[\sup_{T\in[0,\infty]} \int_0^T e^{-s'} \ell(z(s'), z(s') - \dot z(s'))\, ds' \right], \tag{2.21}$$

where $X = H^1([0,\infty);[0,\infty))$, $T \in [0,\infty]$ is any constant and the cost function ℓ is given by

$$\ell(s,\alpha) := \frac{\alpha^2}{4} - R(s) \text{ if } s \in \mathbb{R}\setminus\{c_i\}_{i=1}^n, \quad \ell(s,\alpha) := \frac{\alpha^2}{4} - \Lambda_1^{(i)} \text{ if } s = c_i.$$

By a change of variables $\tau = t(1-e^{-s'})$, one can show that $w(t,x) = t\rho(x/t)$ satisfies

$$w(t,x) = \inf_{\substack{\gamma(0)=x\\ \gamma(t)=0}} \left[\sup_{\theta\in[0,\infty]} \int_0^{\min\{t,\theta\}} \ell\left(\frac{\gamma(\tau)}{t-\tau}, -\dot\gamma(\tau)\right) d\tau \right]. \tag{2.22}$$

Applying the arguments in [29, Lemma 2.4], one can check that the above is consistent with the known max-min formulas involving stopping times when the running cost ℓ is a continuous function [23, 11]. When the minimum with ρ is not taken in the Hamilton-Jacobi equation (2.20), then the unique viscosity solution can be characterized via the optimal control formulation (with T and θ taken to be $+\infty$ in (2.21) and (2.22) respectively); see [40, 3].

2.6 An earlier result: Viscosity solution in the sense of Ishii

First, let $\tilde H$ be the truncated version of the Hamiltonian H, given by

$$\tilde H(s,p) = -sp + p^2 + g(-\infty) \quad \text{for } s < c_1, \quad \tilde H(s,p) = -sp + p^2 + g(+\infty) \quad \text{for } s > c_1 \tag{2.23}$$

and then set $\tilde H(c_1,p) = -c_1p + p^2 + g(-\infty) \vee g(+\infty)$. Note that $\tilde H$ uses only the information $g(\pm\infty)$ but does not depend on the specific form of the profile of g nor does it depend on the eigenvalue Λ_1. The following left and right limits of $H(s,p)$ at $s = c_1$, as functions of p, will be used later.

$$\tilde H(c_1-,p) = -c_1p + p^2 + g(-\infty) \quad \text{and} \quad \tilde H(c_1+,p) = -c_1p + p^2 + g(+\infty). \tag{2.24}$$

In a previous paper [44], we studied equation (1.1) in the case

$$\sup_{y\in\mathbb{R}} g(y) \le \max\{g(+\infty), g(-\infty)\}, \tag{2.25}$$

which is connected to the unique $\hat\rho_{base}$ such that (2.12) holds and satisfies, in viscosity sense,

$$\min\{\rho, \rho + \tilde H(s,\rho')\} = 0 \quad \text{for } s > 0. \tag{2.26}$$

Since the Hamiltonian function $\tilde H$ is discontinuous at c_1, the viscosity solution needs to be interpreted in a relaxed sense introduced by Ishii [42]. As this definition is well known, we skip it here and refer the reader to Definition D.1 in the appendix.

We will show in Section 5.2 that ρ is an Ishii solution to the equation (2.26) if and only if it is a FL-solution of (2.9) with $A \in (-\infty, A_0]$, where $A_0 := \max\{\min_{\mathbb{R}} \tilde H(c_1+,\cdot), \min_{\mathbb{R}} \tilde H(c_1-,\cdot)\}$. See Proposition 5.8.

In [44], we showed that the rate function of problem (1.1) selects $\hat\rho_{base}$ provided that (2.25) holds. This means that the spreading speed of (1.1) is as predicted by the equations (2.26)-(2.12) (with the solution in the sense of Ishii).

Theorem 2.14. *For each* $c_1 > 0$*, there exists a unique* $\hat{\rho}_{base}$ *which satisfies* (2.26) *in the sense of Ishii and the boundary conditions* (2.12)*. Furthermore, if* (2.25) *is valid, then*

$$-\epsilon \log u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right) \to t\hat{\rho}_{base}(\frac{x}{t}) \quad \text{as } \epsilon \to 0 \text{ which implies} \quad \bar{c}_* = \underline{c}_* = s_{base}.$$

Recall that $s_{base} = \sup\{s \geq 0 : \hat{\rho}_{base}(s) = 0\}$*. With the notation* $r_\pm = g(\pm\infty)$*, the following formulas hold.*

(a) *If* $r_- \leq r_+$*, then*

$$s_{base} = \begin{cases} 2\sqrt{r_+} & \text{if } c_1 \leq 2\sqrt{r_+}, \\ \frac{c_1}{2} - \sqrt{r_+ - r_-} + \frac{r_-}{\frac{c_1}{2} - \sqrt{r_+ - r_-}} & \text{if } 2\sqrt{r_+} < c_1 \leq 2(\sqrt{r_-} + \sqrt{r_+ - r_-}), \\ 2\sqrt{r_-} & \text{if } c_1 > 2(\sqrt{r_-} + \sqrt{r_+ - r_-}). \end{cases} \tag{2.27}$$

(b) *If* $r_- > r_+$*, then*

$$s_{base} = \begin{cases} 2\sqrt{r_+} & \text{if } c_1 \leq 2\sqrt{r_+}, \\ c_1 & \text{if } 2\sqrt{r_+} < c_1 \leq 2\sqrt{r_-}, \\ 2\sqrt{r_-} & \text{if } c_1 > 2\sqrt{r_-}. \end{cases} \tag{2.28}$$

Proof. See [44, Proposition 1.7, Theorem 1, Remark 1.6] for the first part of the statement and [44, Theorems 6(iv) and 7] for (2.27) and (2.28). □

For the case when $g(y)$ does not satisfy (2.25), we may compare the solution u of (1.1) with the solution $\tilde{u}$ of the same problem with g replaced by the truncation $\min\{g, \max\{g(\pm\infty)\}\}$, to deduce that the spreading speed is always bounded from below by s_{base}. However, we will show in Corollary 4.3 below that if $\Lambda_1 > \max\{g(\pm\infty)\}$, then this lower bound is not optimal.

Corollary 2.15. *Suppose* $\sup g > \max\{g(\pm\infty)\}$*. For each* $c_1 > 0$*,*

$$\underline{c}_* \geq s_{base},$$

where $\underline{c}_*$ *is the minimal spreading speed and* s_{base} *be given in Theorem 2.14.*

Remark 2.16. In case (2.25) holds, the spreading speed can then be determined as soon as an explicit solution $\hat{\rho}_{base}$ of (2.26)-(2.12) (in the sense of Ishii) can be constructed. This gives an alternative verification of the formula (2.27) (in case $g(-\infty) \leq g(+\infty)$) and formula (2.28) (in case $g(+\infty) < g(-\infty)$) based on the viscosity solution in sense of Ishii [44].

3 Applications and Explicit Formulas

As applications, we apply Theorem 2.9 to treat (1.1), which concerns the case when there is a single environmental shifting speed c_1. We will derive explicit formulas for the spreading speed in terms of c_1, $g(+\infty), g(-\infty)$ and Λ_1, where Λ_1 is the principal eigenvalue of (2.4) defined by (2.5). To simplify the notations, we denote for the remainder of this section

$$r_- = g(-\infty), \quad \text{and} \quad r_+ = g(+\infty).$$

Thanks to standard properties of the principal eigenvalue Λ_1 (see Proposition 4.2(a)) we have

$$\Lambda_1 \in [\max\{r_-, r_+\}, \infty).$$

Remark 3.1. For $g(y) = r_-\chi_{\{y<0\}} + r_m\chi_{\{0\le y<L\}} + r_+\chi_{\{y\ge L\}}$ with $r_m > \max\{r_-, r_+\}$, there exists $\underline{L} \ge 0$ such that

$$\Lambda_1 > \max\{r_-, r_+\} \quad \text{if and only if} \quad L > \underline{L},$$

where $\underline{L} = 0$ when $r_- = r_+$ and

$$\underline{L} = \frac{1}{\sqrt{r_m - \max\{r_-, r_+\}}}\text{arccot}\left(\sqrt{\frac{r_m - \max\{r_-, r_+\}}{|r_- - r_+|}}\right) > 0 \quad \text{when } r_- \ne r_+.$$

See [32, Theorem 2.1] for the precise statement.

The following theorem says that the spreading speed c_* is enhanced according to the specific profile of $g(\cdot)$.

Theorem 3.2. *Let u be a solution of* (1.1) *with compactly supported, nonnegative, nontrivial initial data, then the rightward spreading speed satisfies*

$$c_* = \overline{c}_* = \underline{c}_* = \begin{cases} 2\sqrt{r_+} & \text{if } c_1 \le 2\sqrt{r_+}, \\ c_1 & \text{if } 2\sqrt{r_+} < c_1 \le 2\sqrt{\Lambda_1}, \\ \frac{c_1}{2} - \sqrt{\Lambda_1 - r_-} + \frac{r_-}{\frac{c_1}{2} - \sqrt{\Lambda_1 - r_-}} & \text{if } 2\sqrt{\Lambda_1} < c_1 \le 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-}), \\ 2\sqrt{r_-} & \text{if } c_1 > 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-}). \end{cases}$$

In particular, the mapping $c_1 \mapsto c_*$ is in general non-monotone and not more regular than Lipschitz-continuous, see panels (b), (d) and (f) of Figure 2 for illustration.

Remark 3.3. The case $c_1 \in (2\sqrt{r_+}, 2\sqrt{\Lambda_1})$ is contained in [8]. In fact, under this assumption, they proved the existence of a family of forced waves. Moreover, it is proved that solutions with sufficiently fast decaying initial data (including those that are compactly supported) converge locally uniformly to the unique minimal forced wave.

Remark 3.4. In case that $\Lambda_1 = r_- > r_+$, then case (iii) in Theorem 3.2 is eliminated. In case that $\Lambda_1 = r_+ > r_-$, then case (ii) in Theorem 3.2 is eliminated. In case that $\Lambda_1 = r_- = r_+$, then cases (ii) and (iii) in Theorem 3.2 are eliminated. In our former paper [44], we showed that $c_* = s_{base}$ when $\sup g \le \max\{r_\pm\}$ with three cases: $r_+ > r_-$, $r_+ < r_-$ and $r_+ = r_-$. Next corollary extends the validity of $c_* = s_{base}$ to all g such that $\Lambda_1 = \max\{r_\pm\}$, which may or may not satisfy condition $\sup g \le \max\{r_\pm\}$ due to Remark 3.1; see Figure 1.

Corollary 3.5. *Let u be a solution of* (1.1) *with compactly supported, nonnegative, nontrivial initial data.*

(a) *If $\Lambda_1 = r_+ \ge r_-$, then*

$$c_* = \overline{c}_* = \underline{c}_* = s_{base} \quad \text{with } s_{base} \text{ being given in (2.27).}$$

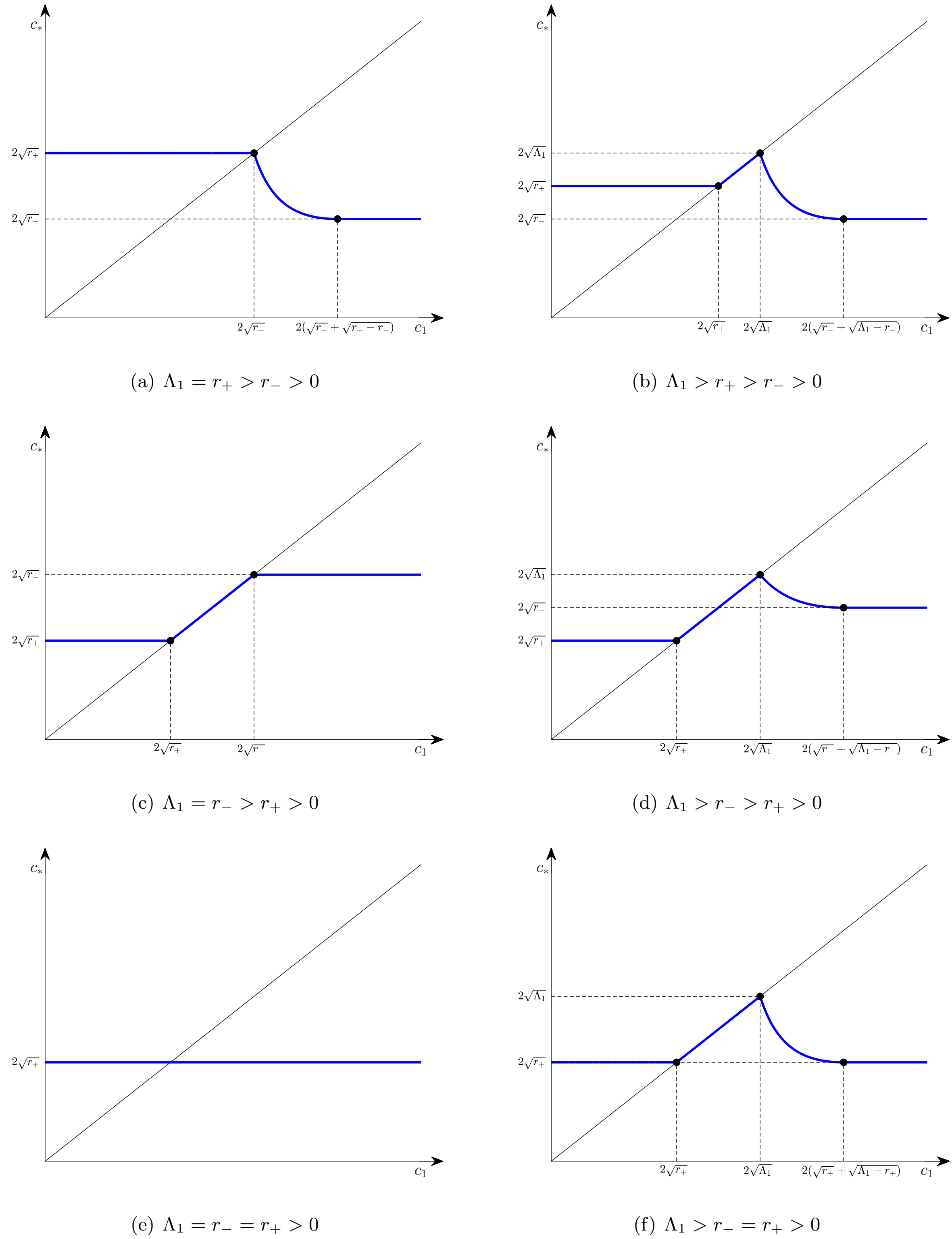
(b) *If $\Lambda_1 = r_- > r_+$, then*

$$c_* = \overline{c}_* = \underline{c}_* = s_{base} \quad \text{with } s_{base} \text{ being given in (2.28).}$$

(c) *If $c_1 \le 0$, then $c_* = 2\sqrt{r_+}$ (as $R(s) \equiv r_+$ for $s > 0$), see also [39].*

See Figure 2(a), (c) and (e) for the dependence of c_* on the shifting speed c_1.

Remark 3.6. We remark that while this paper is devoted to the rightward spreading speed, analogous formulas for the leftward spreading speed can be obtained via the transformation $x \to -x$. We do not consider the case $c_1 \le 0$ due to assertion (c) in Corollary 3.5.



(a) $\Lambda_1 = r_+ > r_- > 0$

(b) $\Lambda_1 > r_+ > r_- > 0$

(c) $\Lambda_1 = r_- > r_+ > 0$

(d) $\Lambda_1 > r_- > r_+ > 0$

(e) $\Lambda_1 = r_- = r_+ > 0$

(f) $\Lambda_1 > r_- = r_+ > 0$

Figure 2: The dependence of spreading speed c_* on c_1. Here $r_\pm = g(\pm\infty)$. The case $c_* = c_1$ is also indicated in [8]. Nonlocal pulling is illustrated by the curved part of the blue lines in panels (a), (b), (d) and (f). The part where c_* coincides either with the KPP speed of the limiting system at $\pm\infty$ is indicated by the horizontal part of the blue lines in (a)-(f).

Proof of Theorem 3.2. We will give an explicit formula for the unique FL-solution of (2.9) satisfying (2.12).

Case (i): $0 < c_1 \leq 2\sqrt{r_+}$. Define

$$\rho_1(s) := \max\{s^2/4 - r_+, 0\}.$$

It is easy to see that $\rho_1(s)$ is nonnegative and satisfies the first equation of (2.9) in the sense of Ishii[1]. We only need to verify the second equation of (2.9). Clearly, ρ_1 is automatically a FL-subsolution as $\rho_1(c_1) = 0$. Note that $F_A \geq A$ with $A = \Lambda_1 - \frac{c_1^2}{4} \geq r_+ - \frac{c_1^2}{4} \geq 0$. Therefore we have (regardless of the test function $\psi \in C^1_{pw}$)

$$\rho_1(c_1) + F_A(\psi'_+(c_1+), \psi'_-(c_1-)) \geq A \geq 0.$$

This implies that ρ_1 is also a FL-supersolution, and hence, $\rho_1(s)$ is a FL-solution of (2.9). Since ρ_1 also satisfies (2.12), $\hat{\rho}_A = \rho_1$ by uniqueness, It then follows from the definition of c_* in (2.15) that $c_* = 2\sqrt{r_+}$.
Case (ii): $2\sqrt{r_+} < c_1 \leq 2\sqrt{\Lambda_1}$. Define

$$\rho_2(s) := \begin{cases} s^2/4 - r_+, & s \geq c_1 + \sqrt{c_1^2 - 4r_+}, \\ \max\{\frac{c_1+\sqrt{c_1^2-4r_+}}{2}(s - c_1), 0\} & 0 \leq s \leq c_1 + \sqrt{c_1^2 - 4r_+}. \end{cases} \tag{3.1}$$

One could directly check that $\rho_2 \in C^1((0, c_1)) \cap C^1((c_1, +\infty))$ and satisfies the first equation of (2.9). Since $\rho_2(c_1) = 0$ and $A \geq 0$, we infer that ρ_2 is a FL-solution of (2.9) due to the same reason as in Case (i). Therefore, by locating the free boundary point in (3.1), we get $c_* = c_1$.
Case (iii): $2\sqrt{\Lambda_1} < c_1 \leq 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-})$. Set

$$\mu_+ = \frac{c_1}{2} + \sqrt{\Lambda_1 - r_+}, \qquad \mu_- = \frac{c_1}{2} - \sqrt{\Lambda_1 - r_-} > 0. \tag{3.2}$$

Define

$$\rho_3(s) := \begin{cases} s^2/4 - r_+, & s \geq 2\mu_+, \\ \mu_+ s - (\mu_+^2 + r_+), & c_1 \leq s \leq 2\mu_+, \\ \max\{\mu_- s - (\mu_-^2 + r_-), 0\}, & 0 \leq s \leq c_1, \end{cases}$$

where μ_+ and μ_- are as in (3.2). Noting that

$$\rho_3(c_1) = -A = \frac{c_1^2}{4} - \Lambda_1 > 0.$$

It is easy to check that ρ_3 satisfies (2.9) in the classical sense except for two non-differentiable points $s_1 = c_1$ and $s_2 = \frac{r_-}{\mu_-} + \mu_-$.The FL-subsolution property at $s = s_2$ holds since $\rho_3(s_2) = 0$. To show it is indeed a FL-subsolution, it suffices to consider the case that $\rho_3 - \psi$ attains a global maximum at $s_1 = c_1$ for some test function $\psi \in C^1_{pw}$. It then follows from $\rho(s) - \psi(s) \leq \rho(c_1) - \psi(c_1)$ for any s close to c_1 that $\psi'(c_1+) \geq \rho_3'(c_1+) = \mu_+ \geq \frac{c_1}{2}$, and $\psi'(c_1-) \leq \rho_3'(c_1-) = \mu_- \leq \frac{c_1}{2}$. This implies

$$\rho_3(c_1) + H^-(c_1+, \psi'(c_1+)) = \rho_3(c_1) + H(c_1+, \frac{c_1}{2}) = \frac{c_1^2}{4} - \Lambda_1 - \frac{c_1^2}{4} + r_+ = r_+ - \Lambda_1 \leq 0,$$

and

$$\rho_3(c_1) + H^+(c_1-, \psi'(c_1-)) = \rho_3(c_1) + H(c_1-, \frac{c_1}{2}) = \frac{c_1^2}{4} - \Lambda_1 - \frac{c_1^2}{4} + r_- = r_- - \Lambda_1 \leq 0,$$

[1]By the results of Subsection 5.2, the Ishii solution ρ_1 automatically qualifies as a FL-solution with $A = A_0$. Since all solutions with $A \leq A_0$ coincides, it follows that ρ_1 is an FL-supersolution for any $A \in \mathbb{R}$.

where we used $\rho_3(c_1) = -A = \frac{c_1^2}{4} - \Lambda_1$ and $\Lambda_1 \geq \max\{r_+, r_-\}$. Finally, we obtain

$$\rho_3(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \leq 0,$$

by taking the minimum of the above and of $\rho_3(c_1) + A = 0$. So, ρ_3 is a FL-subsolution of (2.9).

Next, we verify that ρ_3 is a FL-supersolution. It suffices to check two non-differentiable points $s_1 = c_1$ and $s_2 = \frac{r_-}{\mu_-} + \mu_-$. Also, observe from $c_1 \leq 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-})$ that $\mu_- \leq \sqrt{r_-}$, and hence

$$s_2 = \mu_- + \frac{r_-}{\mu_-} \geq 2\mu_-. \tag{3.3}$$

Suppose first that a test function $\psi \in C^1_{pw}$ touches ρ_3 from below at c_1, then

$$\rho_3(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \geq \rho_3(c_1) + A = 0.$$

Suppose next that a test function $\psi \in C^1_{pw}$ touches ρ_3 from below at $s_2 = \frac{r_-}{\mu_-} + \mu_-$, it then follows that $0 \leq \psi'(s_2) \leq \mu_-$ (note that $\psi'(s_2)$ exists).

$$\rho_3(s_2) + H(s_2, \psi'(s_2)) = -s_2\psi'(s_2) + [\psi'(s_2)]^2 + r_- \geq -s_2\mu_- + \mu_-^2 + r_- = 0,$$

where the first inequality is due to $\psi'(s_2) \in [0, \mu_-]$, and that $p \mapsto -s_2 p + p^2 + r_+$ is monotone decreasing in $[0, \mu_-]$ (thanks to (3.3)). We can then conclude that $\rho_3(s)$ is a FL-solution of (2.9) in $(0, \infty)$, and hence, $c_* = s_2 = \frac{r_-}{\mu_-} + \mu_-$.

Case (iv): $c_1 > 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-})$. Define

$$\rho_4(s) := \begin{cases} s^2/4 - r_+, & s \geq 2\mu_+, \\ \mu_+ s - (\mu_+^2 + r_+), & c_1 \leq s \leq 2\mu_+, \\ \mu_- s - (\mu_-^2 + r_-), & 2\mu_- \leq s \leq c_1, \\ \max\{\frac{s^2}{4} - r_-, 0\}, & 0 \leq s \leq 2\mu_-. \end{cases}$$

Noting that ρ_4 is a classical solution except at two points c_1 and $s_3 = 2\sqrt{r_-}$. For $s = c_1$, we could argue similarly to that in the case (iii) to obtain that the junction condition for super- and subsolution hold true. For $s_3 = 2\sqrt{r_-}$, $\rho_4(s_3) = 0$ implies that the junction condition for subsolution hold at $s = s_3$. Now suppose that a test function $\psi \in C^1_{pw}$ touches ρ_4 from below at s_3. Then again $\psi'(s_3)$ exists, satisfies $0 \leq \psi'(s_3) \leq \mu_-$ and

$$\rho_4(s_3) + H(s_3, \psi'(s_3)) = -s_3\psi'(s_3) + [\psi'(s_3)]^2 + r_- \geq r_- - \frac{s_3^2}{4} = 0,$$

where we used $\min_p(-s_3 p + p^2) = -|s_3|^2/4$ and $s_3 = 2\sqrt{r_-}$. As a consequence, ρ_4 is a FL-solution and $c_* = s_3 = 2\sqrt{r_-}$. □

Proof of Corollary 3.5. By part (b) of Theorem 2.9 implies that $c_* = s_{base}$. Then it is a direct consequence of [44, Theorems 6(iv) and 7]. □

4 Preliminary Results

We will give some preliminary results in this section in preparation of the proof of the main result (Theorem 2.9) in the next section.

To study the behavior of u at the leading edge, we consider the rate function

$$w^\epsilon(t, x) = -\epsilon \log u^\epsilon(t, x) = -\epsilon \log u(t/\epsilon, x/\epsilon). \tag{4.1}$$

We first observe that $\hat{w}(t, x) := \lim_{\epsilon \to 0^+} w^\epsilon(t, x)$, if exists, is 1-homogeneous, that is, $\forall \alpha > 0$, $\hat{w}(\alpha t, \alpha x) = \alpha \hat{w}(t, x)$ for every (t, x).

Lemma 4.1. *Suppose $w^\epsilon \to \hat{w}$ in $C_{loc}((0,\infty)\times[0,\infty))$, as $\epsilon \to 0^+$, then $\hat{w}(t,x) = t\hat{\rho}(x/t)$ for some function $\hat{\rho}$.*

Proof. Fix a constant $\alpha > 0$, then

$$\hat{w}(\alpha,\alpha s) = \lim_{\epsilon\to 0^+} -\epsilon \log u\left(\frac{\alpha}{\epsilon},\frac{\alpha s}{\epsilon}\right) = \alpha \lim_{(\epsilon/\alpha)\to 0^+} -(\epsilon/\alpha)\log u\left(\frac{1}{\epsilon/\alpha},\frac{s}{\epsilon/\alpha}\right) = \alpha\hat{w}(1,s).$$

The lemma follows if we take $\hat{\rho}(s) := \hat{w}(1,s)$, $s = x/t$ and $\alpha = t$. □

Suppose the limit function $\hat{w}(t,x) = t\hat{\rho}(x/t)$ exists, and define

$$\hat{s} = \sup\{s \geq 0 : \hat{\rho}(s) = 0\}, \tag{4.2}$$

then we immediately have

$$u^\epsilon(t,x) = \exp\left(-\frac{w^\epsilon(t,x)}{\epsilon}\right) = \exp\left(-\frac{\hat{w}(t,x)+o(1)}{\epsilon}\right) = \exp\left(-\frac{t\hat{\rho}(x/t)+o(1)}{\epsilon}\right) \to 0,$$

for each $x/t > \hat{s}$, thanks to the definition of $\hat{s}$.

Furthermore, it can be shown (e.g. [51, Lemma 3.1] or [23, Sect. 4]) that $u^\epsilon(t,x)$ is positive, bounded away from zero in the interior of $\{(t,x) : \hat{w}(t,x) = 0\}$, i.e., $\underline{c}_* \geq \hat{s}$, where $\underline{c}_*$ is the minimal spreading speed given in (2.1). Hence, the study of the spreading speed c_* reduces to the unqiueness of $\hat{\rho}$ and the determination of the free boundary point $\hat{s}$ given in (4.2). Next, we collect the properties of the eigenvalue problem (2.4) as well as a few technical results for Hamilton-Jacobi equations with general Hamiltonians $H(s,p)$.

4.1 The eigenvalue problem associated with $g(y)$

Observe from (2.27) and (2.28) that the value of s_{base} depends only on the values of $g(\pm\infty)$ but not on the specific profile of g. The next questions are if and when the invasion is enhanced by the specific profile of g. The answer is completely determined by the eigenvalue Λ_1 given by (2.5). This and several other notions of principal eigenvalues are analyzed in [15]. Here, we recall some basic properties of Λ_1 and the associated positive eigenfunction.

Proposition 4.2. *Let Λ_1 be given by (2.5).*

(a) *Then $\Lambda_1 \geq \max\{g(-\infty), g(+\infty)\}$ and the eigenvalue problem (2.4) has a positive solution in $C^2_{loc}(\mathbb{R})$ if and only if $\Lambda \in [\Lambda_1,\infty)$.*

(b) *If, in addition, $\Lambda_1 > \max\{g(-\infty), g(+\infty)\}$, then Λ_1 is a simple eigenvalue of (2.4) and the following statements hold.*

   (i) *Let $\lambda_\pm := \sqrt{\Lambda_1 - g(\pm\infty)}$ and $\Phi_1(y)$ be the positive eigenfunction corresponding to $\Lambda = \Lambda_1$, then*
   $$\Phi_1(y) = \exp(-\lambda_\pm|y| + o(y)) \qquad \text{as } y \to \pm\infty,$$
   i.e. for any sufficiently small $\eta > 0$, there exist positive numbers $\overline{C}_\eta, \underline{C}_\eta$, such that
   $$\begin{cases} \underline{C}_\eta e^{-(\lambda_+ + \eta)y} \leq \Phi_1(y) \leq \bar{C}_\eta e^{-(\lambda_+ - \eta)y} & \text{if } y \geq 0, \\ \underline{C}_\eta e^{(\lambda_- + \eta)y} \leq \Phi_1(y) \leq \bar{C}_\eta e^{(\lambda_- - \eta)y} & \text{if } y \leq 0. \end{cases} \tag{4.3}$$

   (ii) *Suppose (2.4) has a positive eigenfunction $\tilde{\Phi} \in C^2_{loc}(\mathbb{R})$ for some $\tilde{\Lambda} \in \mathbb{R}$ such that $\tilde{\Phi} \to 0$ as $|y| \to \infty$, then $\tilde{\Lambda} = \Lambda_1$ and $\tilde{\Phi} \in \operatorname{span}\{\Phi_1\}$.*

(c) *If* $\Lambda_1 = \max\{g(-\infty), g(+\infty)\}$. *Then for any* $\eta > 0$, *there exists* $g_\eta : \mathbb{R} \to \mathbb{R}$, *such that*

$$\begin{cases} g_\eta(x) = g(x) \quad \text{for all} \quad |x| \gg 1, \\ g(x) \le g_\eta(x) \le g(x) + \eta \quad \text{for all} \quad x \in \mathbb{R}, \\ \Lambda_1^\eta := \Lambda_1(g_\eta) \quad \text{satisfies} \quad \Lambda_1^\eta > \max\{g(\pm\infty)\}. \end{cases}$$

For the convenience of the reader, we provide the proof of Proposition 4.2 in Appendix A.

The following result describes the effect of Λ_1 in enhancing the spreading speed c_*.

Corollary 4.3. *Let* Λ_1 *be given by* (2.5) *and let* s_{base} *be given by Section 2.6.*

(a) *If* $\Lambda_1 = \max\{g(\pm\infty)\}$, *then for each* $c_1 > 0$, *we have* $c_* = s_{base}$.

(b) *If* $\Lambda_1 > \max\{g(\pm\infty)\}$, *then for some* $c_1 > 0$, *we have* $c_* > s_{base}$.

Proof. Statement (a) follows from Theorem 2.9. Statement (b) is a direct consequence of Theorem 3.2. Namely, if $c_1 = 2\sqrt{\Lambda_1}$, then we have $c_* = 2\sqrt{\Lambda_1} > 2\sqrt{\max\{g(\pm\infty)\}} \ge s_{base}$. □

4.2 The continuity of subsolutions

We discuss the weak continuity condition for sub-solutions, which are half-relaxed limits of solutions to reaction-diffusion equations. This property first appeared in [6].

Lemma 4.4. *Suppose* $\underline{\rho}$ *is nonnegative and satisfies* $\underline{\rho}(0) = 0$, *and satisfies*

$$\min\{\rho, \rho + H(s, \rho')\} \le 0 \qquad \text{in } [0, \infty)$$

in viscosity sense (of Ishii).

(a) *If* $H(s, 0) > 0$ *for each* $s > 0$, *then* $\underline{\rho}$ *is nondecreasing.*

(b) *If* $\lim\limits_{|p|\to\infty} \inf\limits_{s\in K} H(s,p) \to \infty$ *for each compact set* $K \subset [0, \infty)$, *then* $\underline{\rho} \in \mathrm{Lip}_{loc}([0, \infty))$. *In particular, it satisfies the weak continuity condition:*

$$\underline{\rho}(c_1) = \limsup_{s\to c_1+} \underline{\rho}(s) \quad \text{and} \quad \underline{\rho}(c_1) = \limsup_{s\to c_1-} \underline{\rho}(s). \tag{4.4}$$

Proof. Part (a) is due to [44, Lemma 2.9]. For Part (b), fix a bounded interval $K = [0, \bar{s}]$ with $\bar{s} > 0$, and let $M > 0$ be given such that

$$H(s, p) > 0 \quad \text{for all } s \in [0, \bar{s} + 1],\ |p| \ge M.$$

Fix any point $s_0 \in [0, \bar{s}]$, we claim that

$$\underline{\rho}(s) - \underline{\rho}(s_0) \le M|s - s_0| \quad \text{for all } s \in [0, \bar{s}]. \tag{4.5}$$

For this purpose, define

$$\Psi(s) = \begin{cases} \alpha + M|s - s_0| & \text{for } s \in [0, \bar{s}], \\ \alpha + M\left[|\bar{s} - s_0| + \frac{1}{\bar{s}+1-s} - 1\right] & \text{for } s \in [\bar{s}, \bar{s} + 1), \end{cases}$$

for any $\alpha > 0$, then Ψ is continuously differentiable except at s_0, such that

$$|\Psi'(s)| \ge M \quad \text{for } s \in [0, \bar{s} + 1) \setminus \{s_0\}. \tag{4.6}$$

Next, take the minimal α such that Ψ touches $\underline{\rho}$ from above at some point $s_1 \in [0, \bar{s}+1)$, which is possible since $\underline{\rho}$ is upper semicontinuous, and thus bounded on $[0, \bar{s} + 1]$.

If $s_1 = s_0$, then $\alpha = \rho(s_0)$ and we obtain (4.5). Suppose to the contrary that $s_1 \neq s_0$. We first observe that $\underline{\rho}(s_1) > 0$, which follows from $\Psi(s_0) < \Psi(s)$ for $s \in [0, \bar{s}+1)$ and

$$\underline{\rho}(s_1) = \Psi(s_1) > \Psi(s_0) \geq \underline{\rho}(s_0) \geq 0.$$

Now $\underline{\rho}(s_1) > 0$, so the definition of viscosity subsolution implies that

$$0 \geq \underline{\rho}(s_1) + H(s_1, \Psi'(s_1)).$$

But the right hand side is strictly positive thanks to (4.6) and the choice of M. This is a contradiction. Therefore, (4.5) is proved.

Since M depends only on $K = [0, \bar{s}]$ but not on s_0, we can reverse the role of s and s_0 in (4.5) to conclude that

$$|\underline{\rho}(s) - \underline{\rho}(s_0)| \leq M|s - s_0| \quad \text{for all } s, s_0 \in [0, \bar{s}].$$

This proves the Lipschitz continuity of $\underline{\rho}$ in any compact subset of $[0, \infty)$. □

4.3 Critical slope lemmas

The following two lemmas describe the effect of the equation in the interior of the domain on strengthening the boundary condition, and is crucial in deriving the FL-conditions in the strong sense at $s = c_1$ later. Let U be an open interval in $\mathbb{R}$ containing c_1, and recall that

$$C^1_{pw}(U) = C(U) \cap C^1(U \cap (-\infty, c_1]) \cap C^1(U \cap [c_1, \infty)).$$

(HH) Assume $p \mapsto H(s,p)$ is convex and coercive, and $H(c_1+, p)$ and $H(c_1-, p)$ exist.

Lemma 4.5. *Assume that* **(HH)** *holds. Let* $\underline{\rho} : U \to [0, \infty)$ *satisfy the following:*

(i) $\underline{\rho}$ *is a viscosity subsolution of*

$$\min\{\rho, \rho + H(s, \rho')\} = 0 \quad \text{in } \{s \in U : \ s > c_1\}.$$

(ii) $\underline{\rho}$ *satisfies the weak continuity condition* (4.4).

(iii) $\underline{\rho}(c_1) > 0$.

Suppose there is a test function $\varphi \in C^1_{pw}(U)$ *that touches* $\underline{\rho}$ *from above only at* c_1. *Let* $p_+ = \varphi'(c_1+)$, *and*

$$\overline{p}_+ := \inf\left\{\overline{p} \in \mathbb{R} : \ \exists r > 0, \ \varphi(s) + \overline{p}(s - c_1) \geq \underline{\rho}(s) \ \text{ for } 0 \leq s - c_1 < r\right\}. \tag{4.7}$$

Then $-\infty < \bar{p}_+ \leq 0$ *and*

$$\underline{\rho}(c_1) + H(c_1+, p_+ + \bar{p}_+) \leq 0.$$

The proof is an adaptation of [40, Lemmas 2.9 and 2.10], where we use the weak continuity condition. For the convenience of the reader, we provide the proof in Section B.

Remark 4.6. Suppose, in addition, that $\underline{\rho}$ is a viscosity subsolution of $\min\{\rho, \rho + H(s, \rho')\} = 0$ in $\{s \in U : \ s < c_1\}$. Then

$$\underline{\rho}(c_1) + H(c_1-, -p_- - \bar{p}_-) \leq 0,$$

where $-p_- = \varphi'(c_1-)$ and $\overline{p}_- \in (-\infty, 0]$ is given by

$$\overline{p}_- := \inf\left\{\overline{p} \in \mathbb{R} : \ \exists r > 0, \ \varphi(s) - \overline{p}(s - c_1) \geq \underline{\rho}(s) \ \text{ for } -r < s - c_1 \leq 0\right\}. \tag{4.8}$$

Lemma 4.7. *Assume that* **(HH)** *is valid. Suppose $\overline{\rho}$ is a viscosity supersolution of*

$$\rho + H(s,\rho') = 0 \quad \text{in } \{s \in U: \ s > c_1\}.$$

and there is a test function $\varphi \in C^1_{pw}(U)$ that touches $\overline{\rho}$ from below only at c_1. Let $p_+ = \varphi'(c_1+)$, and

$$\underline{p}_+ := \sup\left\{\underline{p} \in \mathbb{R}: \ \exists r > 0,\ \varphi(s) + \underline{p}(s - c_1) \leq \overline{\rho}(s) \ \text{ for } 0 \leq s - c_1 < r\right\}. \tag{4.9}$$

If $\underline{p}_+ < \infty$, then

$$\overline{\rho}(c_1) + H(c_1+, p_+ + \underline{p}_+) \geq 0, \quad \text{with} \quad \underline{p}_+ \geq 0.$$

Note that no weak continuity condition is needed as we do not claim the finiteness of $\underline{p}_+$. The proof of Lemma 4.7 is also included in Appendix B.

Remark 4.8. Suppose, in addition, that $\overline{\rho}$ is a viscosity supersolution of $\min\{\rho, \rho + H_-(s,\rho')\} = 0$ in $\{s \in U: \ s < c_1\}$. Let $-p_- = \varphi'(c_1-)$ and

$$\underline{p}_- := \sup\left\{\underline{p} \in \mathbb{R}: \ \exists r > 0,\ \varphi(s) - \underline{p}(s - c_1) \leq \overline{\rho}(s) \ \text{ for } -r < s - c_1 \leq 0\right\}. \tag{4.10}$$

If $\underline{p}_- < +\infty$, then

$$\overline{\rho}(c_1) + H(c_1-, -p_- - \underline{p}_-) \geq 0 \quad \text{and} \quad \underline{p}_- \geq 0.$$

5 Proof of Main Results

Lemma 5.1. *Let u be a nonnegative, nontrivial solution of* (1.1) *and assume* $\inf g > \delta_0 > 0$. *Then for any $\eta \in (0, 2\sqrt{\delta_0})$*

$$\liminf_{t\to\infty} \inf_{|x| < (2\sqrt{\delta_0} - \eta)t} u(t,x) \geq \delta_0. \tag{5.1}$$

Proof. Choose $\delta_1 \in (\delta_0, \inf g)$, then u is a supersolution to

$$\tilde{u}_t - \tilde{u}_{xx} = \tilde{u}(\delta_1 - \tilde{u}) \quad \text{in } (0,\infty) \times \mathbb{R}$$

It is a classical result that (5.1) holds for $\tilde{u}$, which is the solution of the above equation with compactly supported initial data u_0. By the comparison principle, (5.1) also holds for u. $\square$

Fix a solution $u(t,x)$ of (1.1), then the rate function w^ϵ given in (4.1) satisfies

$$\begin{cases} w^\epsilon_t - \epsilon w^\epsilon_{xx} + |w^\epsilon_x|^2 + g\left(\frac{x - c_1 t}{\epsilon}\right) - e^{-w^\epsilon/\epsilon} = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ w^\epsilon(0,x) = \begin{cases} -\epsilon \log u_0(x/\epsilon) & \text{if } x/\epsilon \in \mathrm{Int}(\mathrm{supp}\, u_0), \\ +\infty & \text{otherwise.} \end{cases} \end{cases} \tag{5.2}$$

Consider the half-relaxed limits [6]:

$$w^*(t,x) = \limsup_{\substack{\epsilon\to 0 \\ (t',x')\to(t,x)}} w^\epsilon(t',x') \quad \text{and} \quad w_*(t,x) = \liminf_{\substack{\epsilon\to 0 \\ (t',x')\to(t,x)}} w^\epsilon(t',x') \tag{5.3}$$

Lemma 5.2. *Let w^* and w_* be given as above. Then $w^*(t,x) = t\rho^*(x/t)$ and $w_*(t,x) = t\rho_*(x/t)$, for some upper semicontinuous function ρ^* and lower semicontinuous function ρ_*.*

Proof. The existence of ρ^* and ρ_* is similar to Lemma 4.1 and is omitted. The semicontinuity are due to the half-relaxed limits in the definition of w^*, w_*. $\square$

Lemma 5.3. *$\rho^*(s) \geq \rho_*(s) \geq 0$ for all $s \geq 0$ and $\rho^*(0) = 0$. Moreover, $\rho_*(s)/s \to +\infty$ as $s \to +\infty$.*

Proof. By the maximum principle, one can establish uniform upper bound of u, i.e. $u(t,x) \leq M_0 := \max\{\sup|u_0|, \sup g\}$, so that $w^\epsilon(t,x) \geq -\epsilon \log M_0$. This implies $w^* \geq w_* \geq 0$ and hence $\rho^* \geq \rho_* \geq 0$.

To show $\rho^*(0) = 0$, it suffices to prove $w^*(t,0) = 0$ for all $t > 0$. By Lemma 5.1, we have

$$w^*(t,0) = \limsup_{\substack{\epsilon\to 0\\(t',x')\to(t,0)}} w^\epsilon(t',x') \leq \limsup_{\substack{\epsilon\to 0\\(t',x')\to(t,0)}} -\epsilon \log u\left(\tfrac{t'}{\epsilon}, \tfrac{x'}{\epsilon}\right) \leq -\lim_{\epsilon\to 0} \epsilon \log \delta_0 = 0.$$

Finally, by a similar argument to that in [44, Lemma B.3], we have $w_*(0,x) = +\infty$ for all $x > 0$. It then follows from lower semicontinuity of w_* that

$$\liminf_{s\to+\infty} \frac{\rho_*(s)}{s} = \liminf_{s\to+\infty} w_*\left(\frac{1}{s}, 1\right) \geq w_*(0,1) = +\infty.$$

This completes the proof. □

5.1 Verification of flux-limited solutions property

The main result of this subsection is the following.

Proposition 5.4. *Let ρ^* and ρ_* be given in Lemma 5.2. Then ρ^* (resp. ρ_*) is a FL-subsolution (resp. FL-supersolution) of* (2.9) *with flux limiter given by* (2.13)*, that is, $A = \Lambda_1 - \frac{|c_1|^2}{4}$.*

We divide the proof of Proposition 5.4 into the verification of FL-subsolution and supersolution.

Lemma 5.5. *Let ρ^* be given by Lemma 5.2. Then*

(a) $\rho^* \in \mathrm{Lip}_{loc}([0,\infty))$;

(b) ρ^* *satisfies the weak continuity condition* (4.4);

(c) ρ^* *is a FL-subsolution of* (2.9) *with $A = \Lambda_1 - \frac{c_1^2}{4}$.*

Proof. By construction $\rho^* : [0,\infty) \to [0,\infty)$ is upper semicontinuous. It is standard (see, e.g. [50, Lemma 2.3] or [2, Corollary 3.1]) to show that $w^*(t,x) = t\rho^*(x/t)$ is a viscosity subsolution to

$$\begin{cases} \min\{w, w_t + |w_x|^2 + g(-\infty)\} = 0 & \text{in } \{(t,x) : 0 < x < c_1 t\}, \\ \min\{w, w_t + |w_x|^2 + g(+\infty)\} = 0 & \text{in } \{(t,x) : x > c_1 t > 0\}, \\ \min\{w, w_t + |w_x|^2 + \inf_{y\in\mathbb{R}} g(y)\} = 0 & \text{in } (0,+\infty)\times(0,+\infty). \end{cases} \tag{5.4}$$

From the third equation, we deduce as in [50, Lemma 2.3] that, in viscosity sense,

$$\min\{\rho^*, \rho^* - s(\rho^*)' + |(\rho^*)'|^2 + \inf_{\mathbb{R}} g\} \leq 0 \quad \text{in } (0,\infty). \tag{5.5}$$

Since also $\rho^*(0) = 0$ (thanks to Lemma 5.3), we infer from Lemma 4.4 that $\rho^* \in \mathrm{Lip}_{loc}([0,\infty))$. This proves assertion (a). Assertion (a) implies (b).

The proof of (c) is inspired by [34]. From the first two equations of (5.4), we deduce that,

$$\min\{\rho^*, \rho^* + H(s,(\rho^*)')\} \leq 0 \quad \text{for } s \in (0,\infty)\setminus c_1,$$

in viscosity sense with $H(s,p)$ given in (2.23). It remains to show that ρ^* is a subsolution to the second equation of (2.9). For this purpose, let $\psi \in C^1_{pw}$ and suppose ψ touches ψ from above strictly at $s = c_1$ (see Remark 2.5) and so that $\psi(c_1) = \rho^*(c_1) > 0$. Denote

$$A = \Lambda_1 - \frac{c_1^2}{4}, \qquad \lambda = -\rho^*(c_1) = -\psi(c_1), \qquad p_+ = \psi'(c_1+), \qquad p_- = -\psi'(c_1-).$$

(Note the negative sign in the definition of p_-.) We want to show

$$-\lambda + \max\left\{A, H^-(c_1+, p_+), H^+(c_1-, -p_-)\right\} \le 0. \tag{5.6}$$

(Observe that if $A \ge 0$, then any nonnegative (sub)solution $\rho^*(s)$ satisfying the junction condition $\min\{\rho, \rho + F_A(\rho'(c_1+), \rho'(c_1-))\} \le 0$ must vanish at the point c_1, i.e. the case $\rho^*(c_1) > 0$ is null.) We first claim that

$$H^-(c_1+, p_+) \le \lambda, \quad \text{and} \quad H^+(c_1-, -p_-) \le \lambda. \tag{5.7}$$

By Lemma 4.5,

$$H(c_1+, p_+ + \bar{p}_+) \le \lambda \quad \text{for some } \bar{p}_+ \in (-\infty, 0]. \tag{5.8}$$

Hence, using the fact that H^- is decreasing in p and $H^- \le H$,

$$H^-(c_1+, p_+) \le H^-(c_1+, p_+ + \bar{p}_+) \le H(c_1+, p_+ + \bar{p}_+) \le \lambda.$$

Arguing similarly, we also have $H^+(c_1-, -p_-) \le \lambda$. This proves (5.7).

It remains to show $A \le \lambda$, where $A = \Lambda_1 - \frac{c_1^2}{4}$. Suppose by contradiction $A > \lambda$. Then by (5.7), we have

$$A > \max\{\min_p H(c_1\pm, p)\} = \max\{g(\pm\infty)\} - \frac{c_1^2}{4}. \tag{5.9}$$

In particular, $\Lambda_1 > \max\{g(\pm\infty)\}$. Define

$$\phi_0(x) = \mu_+ \max\{x, 0\} - \mu_- \min\{x, 0\}, \tag{5.10}$$

where

$$\mu_+ = \frac{c_1}{2} + \sqrt{\Lambda_1 - g(+\infty)}, \quad \text{and} \quad \mu_- = \frac{-c_1}{2} + \sqrt{\Lambda_1 - g(-\infty)}. \tag{5.11}$$

Note that $\mu_\pm$ are also determined uniquely (thanks to (5.9)) by

$$\begin{cases} H(c_1+, \mu_+) = A & \text{and} \quad \mu_+ \ge \operatorname{argmin} H(c_1+, \cdot), \\ H(c_1-, -\mu_-) = A & \text{and} \quad -\mu_- \le \operatorname{argmin} H(c_1-, \cdot). \end{cases} \tag{5.12}$$

By (5.8) and that $\lambda < A = H(c_1+, \mu_+)$, we have $H(c_1+, p_+ + \bar{p}_+) < H(c_1+, \mu_+)$ and thus $p_+ + \bar{p}_+ < \mu_+$ (here we have used the fact that μ_+ is the larger root of $H(c_1+, p) = A$). By definition of $\bar{p}_+$ in (4.7), we deduce that there exists a small right neighborhood $(c_1, c_1 + r)$ of c_1 (with $0 < r < \min\{1, c_1\}$) such that (by Lipschitz continuity of ρ^*)

$$\frac{\rho^*(c_1)}{2} < \rho^*(s) \le -\lambda + \phi_0(s - c_1) \quad \text{for } c_1 \le s < c_1 + r,$$

with the second inequality being an equality iff $s = c_1$. By arguing similarly, along with the definition of $\bar{p}_-$ in (4.8), we have

$$\frac{\rho^*(c_1)}{2} < \rho^*(s) \le -\lambda + \phi_0(s - c_1) \quad \text{for } c_1 - r < s \le c_1,$$

with the second equality holds iff $s = c_1$. In other words, $-\lambda + \phi_0(s - c_1)$ is also a test function touching ρ^* from above at c_1 in $(c_1 - r, c_1 + r)$. Hence, letting

$$Q_r := \{(t, x) : x/t \in [c_1 - r, c_1 + r],\ |t - 1| < r\}, \tag{5.13}$$

we get

$$w^*(t, x) = t\rho^*(x/t) \le \varphi^0(t, x) := \frac{A - \lambda}{4}(t - 1)^2 + t\left(-\lambda + \phi_0\left(\frac{x - c_1 t}{t}\right)\right) \quad \text{in } Q_r \tag{5.14}$$

with equality iff $(t,x)=(1,c_1)$. We can then choose $\delta\in(0,(1-r)\rho^*(c_1))$ small such that

$$w^*(t,x)+\delta\le\varphi^0(t,x)\quad\text{on }\partial Q_r. \tag{5.15}$$

Next, define

$$\varphi^\epsilon(t,x):=\frac{A-\lambda}{4}(t-1)^2-t\lambda-\epsilon\log\Psi\left(\frac{x-c_1t}{\epsilon}\right)\qquad\text{for }(t,x)\in Q_r,$$

where $\Psi(y)=e^{-\frac{c_1y}{2}}\Phi_1(y)$ and Φ_1 is the positive eigenfunction given in Proposition 4.2. Thanks to Proposition 4.2(b)(i), one has $\varphi^\epsilon\to\varphi^0$ in C_{loc} since

$$-\epsilon\log\left[e^{-\frac{c_1}{2}\frac{x-c_1t}{\epsilon}}\Phi_1\left(\frac{x-c_1t}{\epsilon}\right)\right]\to\phi_0(x-c_1t)=t\phi_0\left(\frac{x-c_1t}{t}\right)\qquad\text{locally uniformly.}$$

Hence, we deduce from (5.15) that

$$w^\epsilon(t,x)+\delta/2\le\varphi^\epsilon(t,x)\quad\text{on }\partial Q_r, \tag{5.16}$$

for sufficiently small ϵ. Next, we observe that $\varphi^\epsilon-\delta/2$ satisfies

$$\varphi^\epsilon_t-\epsilon\varphi^\epsilon_{xx}+|\varphi^\epsilon_x|^2+g\left(\frac{x-c_1t}{\epsilon}\right)-e^{-\frac{(2\varphi^\epsilon-\delta)}{2\epsilon}}=\frac{A-\lambda}{2}(t-1)-\lambda+A+o(1)>0,$$

where we used $|t-1|\le r<1$ in the strict inequality, and $e^{-\frac{2\varphi^\epsilon-\delta}{2\epsilon}}=o(1)$ in the last equality follows from

$$\begin{aligned}&\varphi^\epsilon(t,x)-\frac{\delta}{2}=\varphi^0(t,x)-\frac{\delta}{2}+o(1)\ge t\rho^*\left(\frac{x}{t}\right)-\frac{\delta}{2}+o(1)\\&\ge(1-r)\rho^*\left(\frac{x}{t}\right)-\frac{\delta}{2}+o(1)\ge\frac{(1-r)\rho^*(c_1)}{2}+o(1)>0\text{ for }(t,x)\in Q_r.\end{aligned}$$

Hence $\varphi^\epsilon-\delta/2$ is a supersolution to the equation (5.2) of w^ϵ. In view of the boundary condition (5.16), the comparison principle yields

$$w^\epsilon(t,x)+\delta/2\le\varphi^\epsilon(t,x)=\varphi^0(t,x)+o(1)\quad\text{in }Q_r. \tag{5.17}$$

By definition of $w^*(1,c_1)=\rho^*(c_1)=-\lambda$ (recall that ψ touches ρ^* from above), there exists $(t^\epsilon,x^\epsilon)\to(1,c_1)$ such that $w^\epsilon(t^\epsilon,x^\epsilon)\to-\lambda$. Substituting $(t,x)=(t^\epsilon,x^\epsilon)$ into (5.17) and letting $\epsilon\to0$, we have

$$-\lambda+\delta/2\le\varphi^0(1,c_1)=-\lambda\quad\text{for some }\delta>0,$$

which leads to a contradiction. Therefore, $A\le\lambda$. This concludes the proof. □

Next, we show the FL-supersolution property of ρ_*.

Lemma 5.6. *The lower limit ρ_* is a FL-supersolution of* (2.9) *with $A=\Lambda_1-\frac{c_1^2}{4}$.*

Proof. Again, it is standard to check that w_* is a viscosity supersolution to the first two equations of (5.4) in the viscosity sense. This implies again, by [50, Lemma 2.3] that ρ_* is the viscosity supersolution of the first equation of (2.9).

It remains to verify it is a supersolution to the second equation of (2.9). Suppose that there is a test function $\psi\in C^1_{pw}$ that touches ρ_* from below only at $s=c_1$, and denote

$$\lambda=-\rho_*(c_1)=-\psi(c_1),\qquad p_+=\psi'(c_1+),\qquad p_-=-\psi'(c_1-).$$

By way of contradiction, assume that there exists $\eta>0$ such that

$$\max\{A+4\eta,H^-(c_1+,p_+),H^+(c_1-,-p_-)\}<\lambda\quad\text{where }A=\Lambda_1-\tfrac{|c_1|^2}{4}. \tag{5.18}$$

By Proposition 4.2(c), there exists $g_\eta \in C(\mathbb{R},\mathbb{R})$ such that $\Lambda_1^\eta \in (\max\{g(\pm\infty)\}, \Lambda_1 + \eta]$ and $0 \leq g_\eta - g \leq \eta$. Denote

$$A_0 = \max\{g(\pm\infty)\} - \tfrac{|c_1|^2}{4} \quad \text{and} \quad A^\eta = \Lambda_1^\eta - \tfrac{|c_1|^2}{4} \quad \text{such that} \quad A_0 < A^\eta \leq A + \eta.$$

Next, define $\mu_\pm^\eta$ as in (5.12) with A^η in place of A, and let (similarly as in (5.10))

$$\phi_0^\eta(x) = \mu_+^\eta \max\{x, 0\} + \mu_-^\eta \max\{-x, 0\}.$$

Let $\Psi^\eta(y) = e^{-\frac{c_1}{2}y}\Phi_1^\eta(y)$ where $\Phi_1^\eta(y)$ is a positive and bounded eigenfuction associated with Λ_1^η. We first prove the following claim.

Claim 5.7. *$-\lambda + \phi_0^\eta(s - c_1)$ touches ρ_* from below strictly at c_1 in a small neighborhood of c_1.*

Let $\underline{p}_\pm$ be given in (4.9) and (4.10). The claim is obviously true if $(\underline{p}_+, \underline{p}_-) \in (\mu_+^\eta - p_+, \infty] \times (\mu_-^\eta - p_-, +\infty]$. It remains to consider the case $0 \leq \underline{p}_+ < +\infty$ and/or $0 \leq \underline{p}_- < +\infty$. If the former holds, then we have

$$H^-(c_1+, p_+) < \lambda \leq H(c_1+, p_+ + \underline{p}_+),$$

which implies $p_+ + \underline{p}_+ \geq \operatorname{argmin} H(c_1+, \cdot)$ by convexity. Using $A^\eta \in (A_0, A + \eta]$, we have

$$A^\eta < \lambda, \qquad H(c_1+, \mu_+^\eta) = A^\eta$$

with μ_+^η being the larger root of $p \mapsto H(c_1+, p) - A^\eta$, we deduce that $p_+ + \underline{p}_+ > \mu_+^\eta$. This yields $\rho_*(s) > -\lambda + \phi_0^\eta(s - c_1)$ in a right neighborhood of c_1 (which depends on η).

In the case that $\underline{p}_- < +\infty$, we could argue similarly to get $p_- + \underline{p}_- > \mu_-^\eta$, so that $\rho_*(s) > -\lambda + \phi_0^\eta(s - c_1)$ in a left neighborhood of c_1. As a consequence, the Claim 5.7 is proved.

Now there exists $r \in (0, 1)$, such that $-\lambda + \phi_0^\eta(s - c_1)$ touches ρ_* from below strictly at c_1 in $(c_1 - r, c_1 + r)$. Letting Q_r be given in (5.13), we get

$$w_*(t, x) = t\rho_*(x/t) \geq \varphi^{0,\eta}(t, x) := \frac{A - \lambda}{4}(t - 1)^2 + t\left(-\lambda + \phi_0^\eta\left(\frac{x - c_1 t}{t}\right)\right) \quad \text{in } Q_r$$

with equality holds iff $(t, x) = (1, c_1)$. Then there exists $\delta(\eta) > 0$ such that

$$w_*(t, x) \geq \varphi^{0,\eta}(t, x) + \delta \ \text{ on } \ \partial Q_r.$$

Define

$$\varphi^{\epsilon,\eta}(t, x) = \frac{A - \lambda}{4}(t - 1)^2 - t\lambda - \epsilon \log\left[e^{-\frac{c_1}{2}\cdot\frac{x - c_1 t}{\epsilon}}\Phi^\eta\left(\frac{x - c_1 t}{\epsilon}\right)\right] \text{ for } (t, x) \in Q_r.$$

Clearly, $\varphi^{\epsilon,\eta} \to \varphi^{0,\eta}$ in C_{loc} since

$$-\epsilon \log\left[e^{-\frac{c_1}{2}\frac{x - c_1 t}{\epsilon}}\Phi^\eta(\frac{x - c_1 t}{\epsilon})\right] \to \phi_0^\eta(x - c_1 t) = t\phi_0^\eta\left(\frac{x - c_1 t}{t}\right) \quad \text{locally uniformly.}$$

Therefore,

$$w^\epsilon(t, x) \geq \varphi^{\epsilon,\eta}(t, x) + \delta/2 \ \text{ on } \ \partial Q_r.$$

for sufficiently small ϵ. Now we verify $\varphi^{\epsilon,\eta} + \delta/2$ is a subsolution of (5.2). Indeed,

$$\begin{aligned}
&\varphi_t^{\epsilon,\eta} - \epsilon\varphi_{xx}^{\epsilon,\eta} + |\varphi_x^{\epsilon,\eta}|^2 + g\left(\frac{x - c_1 t}{\epsilon}\right) - e^{-\frac{(2\varphi^{\epsilon,\eta} + \delta)}{2\epsilon}} \\
&\leq \frac{A - \lambda}{2}(t - 1) - \lambda + \Lambda_1^\eta - \frac{c_1^2}{4} + \|g - g_\eta\|_\infty, \\
&\leq \frac{A - \lambda}{2}(t - 1) - \lambda + A + \Lambda_1^\eta - \Lambda_1 + \|g - g_\eta\|_\infty \\
&\leq \frac{A - \lambda}{2}(t - 1) - \lambda + A + 2\eta \\
&\leq \frac{-\lambda + A}{2} + 2\eta < 0 \quad \text{in } Q_r.
\end{aligned}$$

It then follows from the maximum principle that, for all small ϵ,

$$w^\epsilon(t,x) - \delta/2 \geq \varphi^{\epsilon,\eta}(t,x) = \frac{A-\lambda}{4}(t-1)^2 - t\lambda + \phi_0^\eta(x-c_1t) + o(1) \quad \text{for } (t,x) \in Q_r. \tag{5.19}$$

Choose $(t^\epsilon, x^\epsilon) \to (1, c_1)$ such that $w^\epsilon(t^\epsilon, x^\epsilon) \to w_*(1, c_1) = \rho_*(c_1) = -\lambda$. Evaluating (5.19) at (t^ϵ, x^ϵ) and then letting $\epsilon \to 0$, we again deduce that $-\lambda - \delta/2 \geq \varphi^{0,\eta}(1, c_1) = -\lambda$, which is a contradiction. This concludes the proof. □

Proof of Proposition 5.4. It is a direct consequence of Lemmas 5.5 and 5.6. □

5.2 Equivalence between Ishii solution and FL solution with $A = A_0$

This section is a special case of [40, Section 7] with the general Hamilton $H(s,p)$ being discontinuous at c_1.

Proposition 5.8. *Let $H(s,p) : [(0,\infty) \setminus \{c_1\}] \times \mathbb{R}$ be convex in p and such that $H(c_1\pm, p)$ are well-defined and coercive, and* $\operatorname{argmin} H(c_1+,\cdot) = \operatorname{argmin} H(c_1-,\cdot)$. *Define*

$$\tilde{H}(s,p) = H(s,p) \quad \text{if } s \neq c_1, \quad \text{and} \quad \tilde{H}(c_1,p) := \max\{H(c_1-,p), H(c_1+,p)\}.$$

Then for any given nonnegative function ρ, it is a FL-supersolution (resp. FL-subsolution) to

$$\begin{cases} \min\{\rho, \rho + H(s,\rho')\} = 0 & \text{in } (0,\infty) \setminus \{c_1\}, \\ \min\{\rho(c_1),\ \rho(c_1) + \max\{A_0, H^-(c_1+, \rho'(c_1+)), H^+(c_1-, \rho'(c_1-))\}\} = 0, \end{cases} \tag{5.20}$$

with $A_0 := \max\{\min_{\mathbb{R}} H(c_1+,\cdot), \min_{\mathbb{R}} H(c_1-,\cdot)\}$, *if and only if it is a viscosity supersolution (resp. subsolution) in the sense of Ishii to*

$$\min\{\rho, \rho + \tilde{H}(s,\rho')\} = 0 \quad \text{in } (0,\infty), \tag{5.21}$$

where the definition of viscosity sub/supersolutions of (5.21) *in sense of Ishii is given in Definition D.1.*

Note also that FL-supersolution (resp. subsolution) with $A \leq A_0$ is equivalent to the case $A = A_0$ due to the fact that

$$\max\{A, H^-(c_1+, \rho'(c_1+)), H^+(c_1-, \rho'(c_1-))\} = \max\{H^-(c_1+, \rho'(c_1+)), H^+(c_1-, \rho'(c_1-))\}$$

provided $A \leq A_0$. For the particular Hamiltonians satisfying

$$H(c_1-, p) = -c_1p + p^2 + g(-\infty) \quad \text{and} \quad H(c_1+, p) = -c_1p + p^2 + g(+\infty). \tag{5.22}$$

that we consider in this paper, one has $A_0 = \max\{g(\pm\infty)\} - c_1^2/4$.

Proof of Proposition 5.8. Denote $\tilde{p}$ be the common value of $\operatorname{argmin} H(c_1\pm,\cdot)$. First, we show sufficiency, i.e. super/subsolution in sense of Ishii implies FL-super/subsolution.

Let ρ be a viscosity supersolution of (5.21) in the sense of Ishii. Then $\rho \geq 0$ for all s. Let $\psi \in C^1_{pw}$ be a test function touching ρ from below at $s = c_1$. Denote

$$\lambda = -\rho(c_1), \quad p_+ = \psi'(c_1+), \quad p_- = -\psi'(c_1-). \tag{5.23}$$

We need to show $F_{A_0}(p_+, -p_-) \geq \lambda$, where

$$F_{A_0}(p_+, -p_-) = \max\{A_0, H^-(c_1+, p_+), H^+(c_1-, -p_-)\}.$$

By the critical slope results (Lemma 4.7) and Remark 4.8), there exist $\underline{p}_\pm \geq 0$ such that

$$H(c_1+, p_+ + \underline{p}_+) \geq \lambda \quad \text{and} \quad H(c_1-, -p_- - \underline{p}_-) \geq \lambda. \tag{5.24}$$

(These $\underline{p}_\pm$ are given in (4.9)-(4.10). If any of them is infinite, then we simply take a large enough positive number satisfying (5.24).)

If $A_0 \geq \lambda$, then $F_{A_0}(p_+, -p_-) \geq A_0 \geq \lambda$, and we are done.

If $p_+ + \underline{p}_+ \leq \tilde{p}$ (resp. $-p_- - \underline{p}_- \geq \tilde{p}$), then we are done, since

$$H^-(c_1+, p_+) \geq H^-(c_1+, p_+ + \underline{p}_+) = H(c_1+, p_+ + \underline{p}_+) \geq \lambda \quad (\text{resp. } H^+(c_1-, -p_-) \geq \lambda).$$

Henceforth, we assume

$$\begin{cases} A_0 < \lambda \leq \min\{H(c_1+, p_+ + \underline{p}_+), H(c_1-, -p_- - \underline{p}_-)\}, \\ -\infty < -p_- - \underline{p}_- < \tilde{p} < p_+ + \underline{p}_+ < +\infty. \end{cases} \tag{5.25}$$

By the definition of the critical slopes, the second line in (5.25) means that $\rho - \tilde{\psi}(s)$ has a strict local minimum at $s = c_1$, where $\tilde{\psi} \in C^1$ is the special smooth test function

$$\tilde{\psi}(s) = \psi(c_1) + \tilde{p}(s - c_1).$$

By the (relaxed) supersolution property in the Ishii sense (see Definition D.1(b)), we have

$$A_0 = \max\{\min H(c_1\pm, \cdot)\} = \max\{H(c_1\pm, \tilde{p})\} \geq \lambda.$$

This is a contradiction with (5.25), and shows that ρ is FL-supersolution with flux limiter A_0.

Next, we show the subsolution in the sense of Ishii implies the FL-subsolution.

Denote ρ be a viscosity subsolution of (5.21) in the sense of Ishii with $\rho(c_1) > 0$ and let $\psi \in C^1_{pw}$ be a test function touching ρ from above at $s = c_1$. We need to show

$$\max\{A_0, H^-(c_1+,\ p_+), H^+(c_1-,\ -p_-)\} \leq \lambda, \tag{5.26}$$

where λ, p_+, p_- are as in (5.23).

By critical slope results in Lemma 4.5 and Remark 4.6 (ρ enjoys weak continuity property thanks to Lemma 4.4), there exist *finite* real numbers $\bar{p}_\pm \leq 0$ (given by (4.7)-(4.8)) such that

$$H(c_1+, p_+ + \bar{p}_+) \leq \lambda \quad \text{and} \quad H(c_1-, -p_- - \bar{p}_-) \leq \lambda. \tag{5.27}$$

In particular, we deduce that

$$A_0 \leq \lambda. \tag{5.28}$$

Moreover, (5.27) also implies that

$$\begin{aligned} H^-(c_1+, p_+) &\leq H^-(c_1+, p_+ + \bar{p}_+) \quad \text{since } H^-(c_1+, \cdot) \text{ is nonincreasing}, \\ &\leq H(c_1+, p_+ + \bar{p}_+) \leq \lambda. \end{aligned} \tag{5.29}$$

Similarly, we also obtain $H^+(c_1-, -p_-) \leq \lambda$. Combining with (5.28) and (5.29), we obtain (5.26). This proves that w is a FL-subsolution with $A = A_0$.

Next, we show the converse statement, i.e. FL-super/subsolution implies super/subsolution in sense of Ishii.

Let ρ be a FL-supersolution of (5.20), and let ψ be a C^1 test function touching ρ from below at $s = c_1$. Then we have $\rho \geq 0$ for all s, and

$$\max\{H(c_1+, \psi'(c_1)), H(c_1-, \psi'(c_1))\} \geq \max\{H^-(c_1+, \psi'(c_1)), H^+(c_1-, \psi'(c_1))\} \geq -\psi(c_1).$$

This proves that ρ is viscosity supersolution of (5.21) in the Ishii sense.

Finally, let ρ be a FL-subsolution of (5.20) with $A = A_0$, and ψ be a C^1 test function touching ρ from below at $s = c_1$. Then we have

$$\max\{H^-(c_1+, \psi'(c_1)), H^+(c_1-, \psi'(c_1))\} \leq -\rho(c_1).$$

Now, since $\tilde{p} = \operatorname{argmin} H(c_1+, \cdot) = \operatorname{argmin} H(c_1-, \cdot)$, we have either $\psi'(c_1) \geq \tilde{p}$ or $\psi'(c_1) < \tilde{p}$. In the former case, we have $H(c_1-, \psi'(c_1)) = H^+(c_1-, \psi'(c_1)) \leq -\rho(c_1)$. In the latter case, we have $H(c_1+, \psi'(c_1)) = H^-(c_1+, \psi'(c_1)) \leq -\rho(c_1)$. This implies that

$$\tilde{H}_*(c_1, \psi'(c_1)) = \min\{H(c_1-, \psi'(c_1)), H(c_1+, \psi'(c_1))\} \leq -\rho(c_1),$$

i.e. ρ is a viscosity subsolution of (5.21) in the sense of Ishii. □

Next, we specialize to the class of Hamiltonian defined in (2.23), and prove the first part of Theorem 2.9.

Proof of Theorem 2.9, first part. We establish Theorem 2.9 in case $\Lambda_1 \leq \max\{g(\pm\infty)\}$. Then $\Lambda_1 = \max\{g(\pm\infty)\}$ (thanks to Proposition 4.2(a)). By Proposition 5.4, ρ^* (resp. ρ_*) is a FL-subsolution (resp. FL-supersolution) of (2.9) with

$$A = A_0 := \max\{g(\pm\infty)\} - \frac{c_1^2}{4}.$$

Thanks to Proposition 5.8, ρ^* and ρ_* are viscosity sub- and supersolution of (2.26) in the Ishii sense. Moreover, it follows from Lemma 5.3 that

$$\rho^*(0) = \rho_*(0) = 0, \quad \text{and} \quad \rho_*(s)/s \to +\infty \quad \text{as } s \to +\infty. \tag{5.30}$$

Hence, we may apply the comparison principle [44, Proposition 2.11] for viscosity solutions in the Ishii sense to deduce that

$$\rho^*(s) \leq \rho_*(s) \quad \text{for all } s \geq 0.$$

Since also $\rho^* \geq \rho_*$ by construction (see (5.3)), we conclude that $\rho^* \equiv \rho_*$. We define $\hat{\rho}_{A_0}$ to be the common value. This proves the existence and uniqueness of $\hat{\rho}_{A_0}$ stated in Proposition 2.6. (Note that this also settles the case $A \leq A_0$, as they yield the same equation (2.9).)

Furthermore, $w^\epsilon(t,x) \to t\hat{\rho}_{A_0}(x/t)$ in $C_{loc}((0,\infty)\times(0,\infty))$. Let $s_{base} = \sup\{s \geq 0 : \hat{\rho}_{A_0}(s) = 0\}$, then $\hat{\rho}_{A_0}(s) > 0$ for $s > s_{base}$. Since $w^\epsilon(t,x)$ is bounded from below by a positive constant in each compact subset of $\{(t,x) : t > 0, x > s_{base}t\}$, this gives

$$u^\epsilon(t,x) = e^{-\frac{w^\epsilon(t,x)}{\epsilon}} \to 0 \quad \text{locally uniformly for } \{(t,x) : t > 0, x > s_{base}t\},$$

i.e. $\overline{c}_* \leq s_{base}$, where $\overline{c}_*$ is the maximal spreading speed given in (2.1).

Next, we observe that $\hat{\rho}_{A_0}$ is nonnegative and monotone increasing (Lemma 4.4(a)), so that $\hat{\rho}_{A_0}(s) = 0$ for $s \in [0, s_{base}]$ and hence

$$w^\epsilon(t,x) \to 0 \quad \text{in } C_{loc}(\{(t,x) : t > 0, 0 \leq x < s_{base}t\}).$$

It then follows as in [51, Lemma 3.1] that

$$\liminf_{\epsilon\to0} \inf_K u^\epsilon(t,x) \geq \inf g > 0$$

for each compact subset $K \subset \{(t,x) : t > 0, 0 \leq x < s_{base}t\}$. For each $\eta > 0$, we may take $K = \{(1,s) : \inf g/2 \leq s \leq s_{base} - \eta\}$, and deduce

$$\liminf_{t\to\infty} \inf_{\frac{\inf g}{2}t \leq x \leq (s_{base}-\eta)t} u(t,x) = \liminf_{\epsilon\to0} \inf_K u^\epsilon(t,x) > 0 \quad \text{for any } \eta > 0.$$

Since $\eta > 0$ is arbitrary, this implies $\underline{c}_* \geq s_{base}$. Combining with $\overline{c}_* \leq s_{base}$, we obtain $\overline{c}_* = \underline{c}_* = s_{base}$. This concludes the proof of Theorem 2.9 in the case $\Lambda_1 = \max\{g(\pm\infty)\}$. □

We continue our proof of Theorem 2.9 in the general case. Having verified that w^* and w_* are FL-subsolution and FL-supersolution of (2.9), one may apply the arguments in [40] to obtain a comparison principle. Here, however, we will follow the arguments due to Lions and Souganidis [48] to show that they are in fact viscosity sub- and supersolutions of certain Kirchhoff junction conditions, and establish the more general comparison principle (see Appendix C). The concept of FL-sub/supersolutions was originally introduced in [40, 41], in which the authors established the comparison principle based on the construction of certain "vertex test functions".

5.3 Verification of Kirchhoff junction conditions

Let $B \in \mathbb{R}$ be given. Consider the Hamilton-Jacobi equation with Kirchhoff junction condition:

$$\begin{cases} \min\{\rho, \rho + H(s, \rho')\} = 0 \quad \text{for } s \neq c_1, \\ \min\{\rho(c_1), \min\{\rho(c_1) + H(c_1\pm, \rho'(c_1\pm)), \rho'(c_1-) - \rho'(c_1+) - B\}\} \leq 0, \\ \min\{\rho(c_1), \max\{\rho(c_1) + H(c_1\pm, \rho'(c_1\pm)), \rho'(c_1-) - \rho'(c_1+) - B\}\} \geq 0. \end{cases} \tag{5.31}$$

The definition of viscosity solution to the above problem also involves the use of piecewise C^1 test functions.

Definition 5.9. (a) We say that $\underline{\rho}$ is a viscosity subsolution of (5.31) provided (i) $\underline{\rho}$ is upper semicontinuous, and (ii) if $\underline{\rho} - \psi$ has a local maximum point at some s_0 such that $\psi \in C^1_{pw}$ and $\underline{\rho}(s_0) > 0$, then

$$\underline{\rho}(s_0) + H(s_0, \psi'(s_0)) \leq 0 \quad \text{in case } s_0 \neq c_1,$$

$$\min\{\underline{\rho}(c_1) + H(c_1\pm, \psi'(c_1\pm)), \psi'(c_1-) - \psi'(c_1+) - B\} \leq 0 \quad \text{in case } s_0 = c_1.$$

(b) We say that $\overline{\rho}$ is a viscosity supersolution of (5.31) provided (i) $\overline{\rho}$ is lower semicontinuous, (ii) $\overline{\rho} \geq 0$ for all s, and (iii) if $\overline{\rho} - \psi$ has a local minimum point at some s_0 such that $\psi \in C^1_{pw}$ then

$$\overline{\rho}(s_0) + H(s_0, \psi'(s_0)) \geq 0 \quad \text{in case } s_0 \neq c_1,$$

$$\max\{\overline{\rho}(c_1) + H(c_1\pm, \psi'(c_1\pm)), \psi'(c_1-) - \psi'(c_1+) - B\} \geq 0 \quad \text{in case } s_0 = c_1.$$

(c) We say that ρ is a viscosity solution of (5.31) provided it is a viscosity subsolution and supersolution of (5.31).

Next, for each flux-limiter $A \geq A_0$, where $A_0 = \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$, we associate a Kirchhoff junction condition parameter B as follows:

$$B = -\mu_+ - \mu_-,$$

where μ_+, μ_- are uniquely determined in terms of A by

$$H^+(c_1+, \mu_+) = A \quad \text{and} \quad H^-(c_1-, -\mu_-) = A. \tag{5.32}$$

By recalling the definition of $H^+(c_1\pm, p)$ and $H^-(c_1\pm, p)$ in (2.7) and (2.8), we deduce

$$\mu_+ = \frac{1}{2}(c_1 + \sqrt{c_1^2 + 4(A - g(+\infty))}), \quad \mu_- = \frac{1}{2}(-c_1 + \sqrt{c_1^2 + 4(A - g(-\infty))})$$

Lemma 5.10. *Let $A \geq A_0 := \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$, and define $\mu_\pm$ in terms of A as in* (5.32). *If $\underline{\rho}$ is a FL-subsolution to* (2.9), *then it is a viscosity subsolution of the problem* (5.31) *with Kirchhoff junction condition with parameter $B = -\mu_+ - \mu_-$.*

Proof. It remains to show that $\underline{\rho}$ is a subsolution to the second equation of (5.31). For this purpose, let $\psi \in C^1_{pw}$ and suppose $\underline{\rho} - \psi$ has a strict global maximum point at c_1, and that $\psi(c_1) = \underline{\rho}(c_1) > 0$. Denote

$$\lambda = -\underline{\rho}(c_1) = -\psi(c_1), \qquad p_+ = \psi'(c_1+), \qquad p_- = -\psi'(c_1-). \tag{5.33}$$

Suppose

$$H(c_1+, p_+) > \lambda \quad \text{and} \quad H(c_1-, -p_-) > \lambda, \tag{5.34}$$

we need to show that

$$-p_- - p_+ + \mu_+ + \mu_- \leq 0. \tag{5.35}$$

Thanks to the critical slope lemma (Lemma 4.5), $H(c_1+, p_+ + \bar{p}_+) \leq \lambda$ for some $\bar{p}_+ < 0$, it follows by convexity that $p_+ \geq \operatorname{argmin} H(c_1+, \cdot)$. Similarly, we have $-p_- \leq \operatorname{argmin} H(c_1-, \cdot)$.

By the definition of FL-subsolution (see Defintion 2.4), it follows that $\lambda \geq A$. This, together with the fact that μ_+ (resp. $-\mu_-$) is the larger (resp. smaller) root of $p \mapsto H(c_1\pm, p) - A$, implies

$$p_+ \geq \mu_+ \quad \text{and} \quad -p_- \leq -\mu_-. \tag{5.36}$$

Therefore, we obtain $-p_+ - p_- + \mu_+ + \mu_- \leq 0$. □

Lemma 5.11. *Let $A \geq A_0$. If $\overline{\rho}$ is a FL-supersolution of* (2.9)*, then it is a viscosity supersolution of the problem* (5.31) *with Kirchhoff condition with parameter $B = -\mu_+ - \mu_-$.*

Proof. It remains to verify the third condition of (5.31). Assume that there is a test function $\psi \in C^1_{pw}$ that touches $\overline{\rho}$ from below only at $s = c_1$. Suppose, with the same notation in (5.33), that

$$H(c_1+, p_+) < \lambda \quad \text{and} \quad H(c_1-, -p_-) < \lambda, \tag{5.37}$$

we need to show

$$-p_- - p_+ + \mu_+ + \mu_- \geq 0. \tag{5.38}$$

Since $\overline{\rho}$ is a FL-supersolution, we have $A \geq \lambda$. Upon considering (5.37), and also μ_+ (resp. $-\mu_-$) being the larger (resp. smaller) root of $H^+(c_1+, p) = A$ (resp. $H^-(c_1-, p) = A$), we deduce[2]

$$p_+ < \mu_+ \quad \text{and} \quad -p_- > -\mu_-. \tag{5.39}$$

This implies (5.38). □

Corollary 5.12. *ρ^* (resp. ρ_*) given by Lemma 5.2 is a viscosity subsolution (resp. supersolution) of* (5.31) *with $B = -\mu_+ - \mu_-$ where $\mu_\pm$ are associated with $A = \Lambda_1 - \frac{c_1^2}{4}$ via* (5.12)*.*

Proof. Fix $A = \Lambda_1 - \frac{c_1^2}{4} > \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$. Define μ_+, μ_- by (5.12). By Proposition 5.4, ρ^* and ρ_* are FL-subsolution and supersolution of (2.9) with $A = \Lambda_1 - \frac{c_1^2}{4}$, respectively. By Lemmas 5.10 and 5.11, they are viscosity sub- and supersolutions of (5.31) with $B = -\mu_+ - \mu_-$. □

5.4 Proof of main results

Proof of Proposition 2.6. Recall that $A_0 = \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$. Let $\underline{\rho}$ and $\overline{\rho}$ be a pair of FL-subsolution and FL-supersolution of (2.9) for some $A \geq A_0$, such that (2.11) holds.

If $A \leq A_0$ holds, then by Proposition 5.8, $\underline{\rho}$, $\overline{\rho}$ is a pair of viscosity sub- and supersolution of (2.26) in the sense of Ishii. The comparison principle follows from [44, Proposition 2.11].

Henceforth, we assume that $A > A_0$. Then it follows from Lemmas 5.10 and 5.11 that $\underline{\rho}$ and $\overline{\rho}$ are a pair of sub- and supersolutions of (5.31). The comparison principle then follows from Theorem C.5. □

[2]Note that we do not need $\bar{p}_+ < +\infty$ here, comparing with the proof of the previous verification for junction subsolution. This asymmetry in the arguments of super and subsolutions is due to the fact that H is convex and coercive.

Proof of Corollary 2.7 and Theorem 2.9. For a given function g, denote $A = \Lambda_1 - \frac{c_1^2}{4}$, where Λ_1 is given in (2.5). By Proposition 4.2(a), $A \geq A_0 := \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$.

For the case $A = A_0$, in view of Proposition 5.8, the problem is equivalent to (2.26) and (2.12). The existence and uniqueness of $\hat{\rho}_{A_0}$ follows from [44, Proposition 1.7(b)]. The convergence $w^\epsilon(t,x) \to t\hat{\rho}(x/t)$ in C_{loc} and determination of spreading speed is given in the first part of the proof of Theorem 2.9 (see Subsection 5.2).

For the case $A > A_0$, it follows that ρ^* and ρ_* are viscosity sub- and supersolutions of (5.31) and satisfy $\rho^*(0) = \rho_*(0) = 0$ and $\rho_*(s)/s \to +\infty$ as $s \to +\infty$ (see Lemma 5.3). Hence, Theorem C.5 implies $\rho^* \leq \rho_*$. Arguing similarly as in the proof of the first part of Theorem 2.9 (in Subsection 5.2), we conclude that (i) (2.9)-(2.12) has a unique FL-solution $\hat{\rho}_A(s)$; (ii) $w^\epsilon(t,x) \to t\hat{\rho}_A(x/t)$ in C_{loc}; (iii) the spreading speed is given by $c_* = \hat{s}_A = \sup\{s \geq 0 : \hat{\rho}_A(s) = 0\}$. □

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A Proof of Proposition 4.2

Proof of Proposition 4.2. For (a), observe that if $\phi'' + g(y)\phi \leq \Lambda\phi$ in $\mathbb{R}$ for some positive $\phi \in C^2_{loc}(\mathbb{R})$ and $\Lambda < g(+\infty)$, then $\phi'' < 0$ for $y \gg 1$. Since $\phi > 0$ in $\mathbb{R}$, we deduce that $\phi' > 0$ for $y \gg 1$, and hence $\phi'(+\infty) \in [0, +\infty)$ and $\phi(+\infty) \in (0, +\infty]$ both exist. However, this means $\limsup_{y\to\infty} \phi''(y) \leq (\Lambda - g(+\infty))\phi(+\infty) < 0$, which contradicts $\phi'(+\infty) \geq 0$. This proves $\Lambda_1 \geq g(+\infty)$. Similarly, we can show that $\Lambda_1 \geq g(-\infty)$. Next, we apply [15, Theorem 1.4] to infer that (2.4) has a positive solution if and only if $\Lambda \in [\Lambda_1, \infty)$. (Note that $\Lambda_1 = -\lambda_1(L, \mathbb{R})$ in the notation of [15].) This proves (a).

For (b)(i), we first apply [15, Proposition 1.11(ii)] to deduce that if $\Lambda_1 > \max\{g(\pm\infty)\}$, then Λ_1 is a simple eigenvalue and Φ_1 converges exponentially to zero as $|y| \to \infty$. To establish (4.3), it suffices to prove the estimation of accurate decay rate of Φ_1 at $+\infty$. Since $\Lambda_1 > g(+\infty)$, there exists $a_0 > 0$ such that $\Lambda_1 > g(y)$ for any $y \geq a_0$. Now we define

$$\overline{\lambda}(a) := \sup_{y \geq a} \sqrt{\Lambda_1 - g(y)}, \quad \underline{\lambda}(a) := \inf_{y \geq a} \sqrt{\Lambda_1 - g(y)}, \quad \forall y \geq a_0.$$

Fix $a \geq a_0$. Then for any $\epsilon > 0$ and $M > 0$, it is easy to check that $\bar{\Phi}^M_\epsilon(y) = Me^{-\underline{\lambda}(a)y} + \epsilon$ is a supersolution of (2.4) on $[a, +\infty)$, that is,

$$(\bar{\Phi}^M_\epsilon)'' + (g(y) - \Lambda_1)\bar{\Phi}^M_\epsilon \leq 0 \text{ on } [a, +\infty).$$

Next, we take $M_1 = e^{\underline{\lambda}(a)a}\Phi_1(a)$ and $\epsilon_0 > 0$ such that $\bar{\Phi}^{M_1}_{\epsilon_0}(y) > \Phi_1(y)$ on $[a, +\infty)$. Using the sliding argument or strong maximum principle, we infer that $\bar{\Phi}^{M_1}_\epsilon(y) > \Phi_1(y)$ on $[a, +\infty)$ for any $\epsilon \in (0, \epsilon_0]$. Letting $\epsilon \to 0^+$, we have $M_1 e^{-\underline{\lambda}(a)y} \geq \Phi_1(y)$ on $[a, +\infty)$.

Similarly, for any $M > 0$ and $\epsilon \in (0, e^{-a\bar{\lambda}(a)})$, $\underline{\Phi}^M_\epsilon(y) := \max\{0, M(e^{-\overline{\lambda}(a)y} - \epsilon)\}$ is a subsolution of (2.4) in $(a, +\infty)$, that is,

$$(\underline{\Phi}^M_\epsilon)'' + (g(y) - \Lambda_1)\underline{\Phi}^M_\epsilon \geq 0 \text{ on } [a, +\infty).$$

Then we choose $M_2 = e^{\bar{\lambda}(a)a}\Phi_1(a)$ and $\epsilon_0 > 0$ such that $\underline{\Phi}_{\epsilon_0}^{M_2}(y) < \Phi_1(y)$ on $[a, +\infty)$. By the sliding argument again, it follows that $\underline{\Phi}_{\epsilon}^{M_2}(y) < \Phi_1(y)$ on $[a, +\infty)$ for any $\epsilon \in (0, \epsilon_0]$. Letting $\epsilon \to 0^+$, we have $M_2 e^{-\bar{\lambda}(a)y} \leq \Phi_1(y)$ on $[a, +\infty)$. Consequently, there exist $\underline{C} > 0, \bar{C} > 0$(dependent on a), such that

$$\underline{C}e^{-\bar{\lambda}(a)y} \leq \Phi_1(y) \leq \bar{C}e^{-\underline{\lambda}(a)y}, \quad \forall y \geq 0.$$

Noting that $\overline{\lambda}(a)$ and $\underline{\lambda}(a)$ are continuous on $[a_0, +\infty)$ with $\overline{\lambda}(+\infty) = \underline{\lambda}(+\infty) = \lambda_+$. Therefore, for any sufficiently small $\eta > 0$, there exists $a > a_0$ such that $\lambda_+ - \eta \leq \underline{\lambda}(a) \leq \overline{\lambda}(a) \leq \lambda_+ + \eta$. This implies the first inequality in (4.3) is valid.

For (b)(ii), suppose $(\tilde{\Lambda}, \tilde{\Phi})$ is an eigenpair of (2.4). By the first assertion of the Proposition, either $\tilde{\Lambda} = \Lambda_1$ or $\tilde{\Lambda} > \Lambda_1$. On the one hand, if $\tilde{\Lambda} = \Lambda_1$, then we can immediately conclude by the fact that Λ_1 is simple (by [15, Proposition 1.1(ii)]).

On the other hand, if $\tilde{\Lambda} > \Lambda_1$, then one can prove that $\tilde{\Phi}(y) \sim e^{-\sqrt{\tilde{\Lambda} - g(+\infty)}y + o(y)}$ as $y \to +\infty$. It then follows that $\tilde{\Phi}(y)/\Phi_1(y) \to 0$ as $y \to +\infty$. By repeating the argument, we also obtain $\tilde{\Phi}(y)/\Phi_1(y) \to 0$ as $y \to -\infty$. We can then touch $\Phi_1(y)$ from below with $k\tilde{\Phi}(y)$ to obtain, from the strong maximum principle, that $\Phi_1(x) = k\tilde{\Phi}(x)$ for some $k > 0$. In this case $\tilde{\Lambda} = \Lambda_1$, a contradiction.

For (c), if g is piecewise constant, then one can use Remark 3.1 to conclude. In the general case, fix an arbitrary $\eta > 0$ we choose, for each $k \in \mathbb{N}$, a continuous function such that

$$\begin{cases} g_\eta^k(x) = g(x), & |x| \geq k+1, \\ g(x) \leq g_\eta^k(x) \leq g(x) + \eta, & k \leq |x| \leq k+1, \\ g_\eta^k(x) = g(x) + \eta, & |x| \leq k. \end{cases}$$

If $\Lambda_1(g_\eta^k) > \Lambda_1(g)$ for some k, we are done. Suppose to the contrary that $\Lambda_1(g_\eta^k) \equiv \Lambda_1(g)$ for all k, and let $\Phi_k \in C^2_{loc}$ be a positive eigenfunction of $\Lambda_1(g_\eta^k)$. By Harnack inequality, there exists a positive number $C(R)$ independent of k such that

$$\frac{1}{C(R)} \leq \frac{\Phi_k(x)}{\Phi_k(0)} \leq C(R) \quad \text{for } k \in \mathbb{N},\ |x| \leq R.$$

Normalizing by $\Phi_k(0) = 1$, we see that $\{\Phi_k\}$ is bounded in $C^2([-R, R])$ for each R. It follows that (up to a subsequence) Φ_k converges in $C^1_{loc}(\mathbb{R})$ to a positive eigenfunction $\tilde{\Phi} \in C^2_{loc}(\mathbb{R})$ satisfying

$$\tilde{\Phi}'' + (g(x) + \eta)\tilde{\Phi} = \Lambda_1(g)\tilde{\Phi} \quad \text{in } \mathbb{R}.$$

By assertion (a), it follows that $\Lambda_1(g) \geq \Lambda_1(g + \eta)$, which is impossible since $\Lambda_1(g + \eta) = \Lambda_1(g) + \eta$. □

B Proof of Lemmas 4.5 and 4.7

Following the same procedure in [40, Lemmas 2.9 and 2.10], it suffices to prove Lemma 4.7 without weak continuity condition and then show the finiteness of $\overline{p}_+$ in Lemma 4.5 with weak continuity condition.

Proof of Lemma 4.7. By the definition of $\underline{p}_+$, we see that $\underline{p}_+ \geq 0$. For any sufficiently small $\epsilon > 0$, there exists $r_\epsilon \in (0, \epsilon)$ such that

$$\overline{\rho}(s) \geq \varphi(s) + (\underline{p}_+ - \epsilon)(s - c_1) \quad \text{for } 0 \leq s - c_1 \leq r_\epsilon$$

and there exists $s_\epsilon \in (c_1, c_1 + \frac{r_\epsilon}{2})$ such that

$$\overline{\rho}(s_\epsilon) < \varphi(s_\epsilon) + (\underline{p}_+ + \epsilon)(s_\epsilon - c_1).$$

Now construct a smooth function $\Psi : \mathbb{R} \to [-1, 0]$ such that

$$\Psi(s) = \begin{cases} 0 & \text{for } s \in (-1/2, 1/2), \\ -1 & \text{for } s \notin (-1, +1) \end{cases}$$

and define

$$\Phi(s) = \varphi(s) + 2\epsilon \Psi_{r_\epsilon}(s) + \begin{cases} (\underline{p}_+ + \epsilon)(s - c_1) & \text{if } s \in U \cap (c_1, +\infty), \\ 0 & \text{if } s \in U \cap (-\infty, c_1] \end{cases}$$

with $\Psi_{r_\epsilon}(s) = r_\epsilon \Psi((s - c_1)/r_\epsilon)$. It then follows that $\Phi(c_1) = \varphi(c_1) = \overline{\rho}(c_1)$ and

$$\begin{cases} \Phi(c_1 + r_\epsilon) = \varphi(c_1 + r_\epsilon) + (\underline{p}_+ - \epsilon) r_\epsilon \leq \overline{\rho}(c_1 + r_\epsilon), \\ \Phi(s_\epsilon) = \varphi(s_\epsilon) + (\underline{p}_+ + \epsilon)(s_\epsilon - c_1) > \overline{\rho}(s_\epsilon). \end{cases}$$

This implies that there exists a point $\bar{s}_\epsilon \in (c_1, c_1 + r_\epsilon)$ such that $\overline{\rho} - \Phi$ attains a local minimum at $\bar{s}_\epsilon$. Therefore, by the definition of viscosity supersolution and $H(\cdot, p)$ is convex in p, we obtain

$$\Phi(\bar{s}_\epsilon) + \sup_{s \in (c_1, c_1 + r_\epsilon)} H(s, \Phi'(\bar{s}_\epsilon)) \geq \overline{\rho}(\bar{s}_\epsilon) + H^*(\bar{s}_\epsilon, \Phi'(\bar{s}_\epsilon)) \geq 0,$$

which yields

$$\Phi(\bar{s}_\epsilon) + \sup_{s \in (c_1, c_1 + r_\epsilon)} H(s, \varphi'(\bar{s}_\epsilon) + 2\epsilon \Psi'_{r_\epsilon}(\bar{s}_\epsilon) + \underline{p}_+ + \epsilon) \geq 0.$$

Letting $\epsilon \to 0^+$, we reach

$$\varphi(c_1) + H(c_1+, \varphi'(c_1+) + \underline{p}_+) \geq 0.$$

Now the conclusion follows immediately from the fact that $\bar{\rho}(c_1) = \varphi(c_1)$. □

Proof of Lemma 4.5. We only show that $\bar{p}_+ > -\infty$. Without loss of generality, we might assume that $\varphi(c_1) = \underline{\rho}(c_1) > 0$. Suppose by contradiction that $\bar{p}_+ = -\infty$, then there exists $p_n \to -\infty$ and $r_n > 0$ such that

$$\varphi(s) + p_n(s - c_1) \geq \underline{\rho}(s) \ \text{ for } \ 0 \leq s - c_1 < r_n.$$

Modifying φ if necessary (e.g., $\varphi + (s - c_1)^2$), we could further assume that

$$\varphi(s) + p_n(s - c_1) > \underline{\rho}(s) \ \text{ for } \ 0 < s - c_1 \leq r_n. \tag{B.1}$$

For fixed n, since $\underline{\rho}$ satisfies the weak continuity condition (4.4), it then follows that there exists $s_m \in (c_1, c_1 + r_n)$ such that $s_m \to c_1$ and $\underline{\rho}(s_m) \to \underline{\rho}(c_1)$ as $m \to +\infty$. Define

$$\Psi_m(s) = \varphi(s) + p_n(s - c_1) + \frac{(s_m - c_1)^2}{s - c_1}, \quad s > c_1.$$

For each $m \in \mathbb{N}$, there exists $\hat{s}_m \in (c_1, c_1 + r_n]$ such that $\Psi_m - \underline{\rho}$ attains the minimum at $\hat{s}_m$. Then

$$o(1) = \Psi_m(s_m) - \underline{\rho}(s_m) \geq \Psi_m(\hat{s}_m) - \underline{\rho}(\hat{s}_m) \geq \varphi(\hat{s}_m) + p_n(\hat{s}_m - c_1) - \underline{\rho}(\hat{s}_m). \tag{B.2}$$

Suppose $\hat{s}_m \to s_0 \neq c_1$ (up to a subsequence) as $m \to \infty$. Then letting $m \to +\infty$ (in a subsequence) in (B.2), we have $\underline{\rho}(\hat{s}_m) \to \liminf_{m \to \infty} \underline{\rho}(\hat{s}_m)$ and hence

$$0 \geq \varphi(s_0) + p_n(s_0 - c_1) - \liminf_{m \to \infty} \underline{\rho}(\hat{s}_m) > \limsup_{m \to \infty} \underline{\rho}(\hat{s}_m) - \liminf_{m \to \infty} \underline{\rho}(\hat{s}_m) \geq 0,$$

where the strict inequality follows from (B.1) and the upper semicontinuity of $\underline{\rho}$. This is a contradiction. Therefore, we conclude that $\hat{s}_m \to c_1$ and $\underline{\rho}(\hat{s}_m) \to \underline{\rho}(c_1) > 0$ as $m \to \infty$. Now we might assume $\underline{\rho}(\hat{s}_m) > 0$ for each $m \in \mathbb{N}$, by the definition of viscosity subsolution, we obtain

$$\underline{\rho}(\hat{s}_m) + H_* \left(\hat{s}_m, \varphi'(\hat{s}_m) + p_n - \frac{(s_m - c_1)^2}{(\hat{s}_m - c_1)^2} \right) \leq 0.$$

Note that $\inf_{s \in (c_1, \hat{s}_m]} H(s,p) \leq H_*(\hat{s}_m, p)$. Then we pass to the limit as $m \to +\infty$ in the inequality above and get

$$\underline{\rho}(c_1) + H(c_1+, \varphi'(c_1+) + p_n^0) \leq 0,$$

where $p_n^0 = p_n - \limsup_{m \to \infty} \frac{(s_m - c_1)^2}{(\hat{s}_m - c_1)^2} \in [-\infty, 0]$. It then follows from $\liminf_{p \to -\infty} H(c_1+, p) \geq 0$ that $p_n^0 > -\infty$ and p_n^0 is bounded from below by a constant C which only depends on $H(c_1+, p)$, $\underline{\rho}(c_1)$ and $\varphi'(c_1+)$. But this also implies $p_n \geq C$, which leads to a contradiction. This completes the proof of the finiteness of $\overline{p}_+$. □

C Comparison Principle for problem with Kirchhoff condition

The comparison principle for FL-solutions was first proved by Imbert and Monneau [40, 41]. Subsequently, Lions and Souganidis gave an alternative proof by transforming it to an equivalent Kirchhoff junction condition [49, 48]. We combine the arguments of the latter and of [44] to prove a comparison result that allows for solutions that grows superlinearly.

Let $\mathcal{P} = \{c_i\}_{i=1}^n$ for some $0 < c_1 < ... < c_n$ and $B_i \in \mathbb{R}$ for all i. We establish a comparison principle for viscosity sub- and supersolutions of the Hamilton-Jacobi equation

$$\min\{\rho, \rho + H(s, \rho')\} = 0 \quad \text{in } (0, \infty) \setminus \mathcal{P} \tag{C.1}$$

with the following Kirchhoff junction condition at c_i

$$\rho'(c_i-) - \rho'(c_i+) - B_i = 0 \quad \text{for } i = 1, ..., n, \tag{C.2}$$

and boundary conditions $\rho(0) = 0$ and $\rho(s)/s \to +\infty$ as $s \to +\infty$. Here, we assume the following for the Hamiltonian function $H(s,p)$.

(A1) For any given $L > 0$ and each $s_0 \in (\frac{1}{L}, L) \setminus \mathcal{P}$, there exist $\delta_0 = \delta_0(L)$ and $h_0 \in \{\pm 1\}$ such that

$$H^*(s,p) - H_*(s',p) \leq \omega_L(|s - s'|(1 + |p|))$$

for all s, s' such that $0 < |s - s_0| + |s' - s_0| < \delta_0$ and $(s' - s)h_0 < 0$. Here $\omega_L : [0, \infty) \to [0, \infty)$ is a modulus of continuity for each $L > 0$, i.e. it is continuous with $\omega_L(0) = 0$, and H^* (resp., H_*) is the upper (resp., lower) semicontinuous envelope of H with respect to the first variable, that is,

$$H^*(s,p) = \limsup_{s' \to s} H(s', p) \quad (\text{resp.,} \quad H_*(s,p) = \liminf_{s' \to s} H(s', p)). \tag{C.3}$$

(A2) $p \mapsto H(\cdot, p)$ is convex, $\inf_{s > 0} H(s, 0) > 0$ and $\lim_{|p| \to \infty} \left[\inf_{s \in K} H(s,p) \right] \to \infty$ for each compact set $K \subset (0, \infty)$.

(A3) The one-sided limits $H(c_i \pm, p)$ are well defined.

(A4) There exists $\overline{s} > \max_{1 \leq i \leq n} \{c_i\}$ such that $H(s,p) = -sp + \tilde{H}(s,p) + R(s)$ for $s \geq \overline{s}$ and $\tilde{H}(s,p)$ is non-increasing in $s \in [\overline{s}, \infty)$. Moreover, $R \in L^\infty$ is locally monotone in $s \in [\overline{s}, \infty)$.

Remark C.1. The definition of local monotonicity of $R(s)$ is stated in (H3') of Subsection 2.4; see also [17]. Note also that (A1) implies that $H(s,p)$ is locally monotone as a function of two variables.

Remark C.2. For our purpose, we will take $H(s,p) = -sp + p^2 + R(s)$, where R equals to positive constants on $(0,c_1)$ and on (c_1,∞). It is obvious that (A1)-(A4) hold.

To define what it means by a viscosity solution to (C.1)–(C.2), we recall the space of piecewise C^1 test functions

$$C^1_{pw} = C^0_{loc}((0,\infty)) \cap C^1((0,c_1]) \cap C^1([c_1,c_2]) \cap ... \cap C^1([c_{n-1},c_n]) \cap C^1_{loc}([c_n,\infty)).$$

and use the notations in (C.3).

Definition C.3. (a) We say that $\underline{\rho}$ is a viscosity subsolution to (C.1)–(C.2) if $\underline{\rho}$ is upper semicontinuous on $(0,+\infty)$ and it satisfies, in the viscosity sense,

$$\begin{cases} \min\{\rho, \rho + H(s,\rho')\} \le 0 \quad \text{for } s \in (0,\infty) \setminus \mathcal{P}, \\ \min\{\rho(c_i), \min\{\rho'(c_i-) - \rho'(c_i+) - B_i, \rho(c_i) + H(c_i\pm, \rho'(c_i\pm))\} \le 0 \text{ for each } 1 \le i \le n, \end{cases}$$

that is, whenever $\underline{\rho} - \psi$ has a strict local maximum point at s_0 for some $\psi \in C^1_{pw}$ and $\underline{\rho}(s_0) > 0$, we have

$$\underline{\rho}(s_0) + H_*(s_0, \psi'(s_0)) \le 0 \quad \text{if } s_0 \in (0,\infty) \setminus \mathcal{P},$$

$$\min\{\psi'(c_i-) - \psi'(c_i+) - B_i, \underline{\rho}(c_i) + H(c_i\pm, \psi'(c_i\pm))\} \le 0 \quad \text{if } s_0 = c_i.$$

(b) We say that $\overline{\rho}$ is a viscosity supersolution to (C.1)–(C.2) if $\overline{\rho}$ is lower semicontinuous on $(0,+\infty)$ and it satisfies, in the viscosity sense,

$$\begin{cases} \min\{\rho, \rho + H(s,\rho')\} \ge 0 \quad \text{for } s \in (0,\infty) \setminus \mathcal{P}, \\ \min\{\rho(c_i), \max\{\rho'(c_i-) - \rho'(c_i+) - B_i, \rho(c_i) + H(c_i\pm, \rho'(c_i\pm))\}\} \ge 0 \text{ for each } 1 \le i \le n, \end{cases}$$

that is, $\overline{\rho} \ge 0$ for all $s > 0$ and whenever $\underline{\rho} - \psi$ has a strict local minimum point at s_0 for some $\psi \in C^1_{pw}$, we have

$$\overline{\rho}(s_0) + H^*(s_0, \psi'(s_0)) \ge 0 \quad \text{if } s_0 \in (0,\infty) \setminus \mathcal{P},$$

$$\max\{\psi'(c_i-) - \psi'(c_i+) - B_i, \overline{\rho}(c_i) + H(c_i\pm, \psi'(c_i\pm)\} \ge 0 \quad \text{if } s_0 = c_i.$$

(c) We say that $\overline{\rho}$ is a viscosity solution to (C.1) –(C.2), if it is both a viscosity subsolution and viscosity supersolution.

Remark C.4. The above setting includes the case with general $r(x,t)$ with infinitely many shifts (in that case $\tilde{H}_i(p) = p^2$ and R is locally monotone except possibly at c_i), as well as the case when there is finitely many shifts, but periodic homogenization in between (in that case $R \equiv 0$).

Theorem C.5. *Let $\underline{\rho}$ and $\overline{\rho}$ be, respectively, viscosity sub- and supersolutions of* (C.1)–(C.2), *such that*

$$\underline{\rho}(0) \le 0 \le \overline{\rho}(0), \quad \underline{\rho}(s) < \infty \quad \textit{for all } s \ge 0, \quad \textit{and} \quad \frac{\overline{\rho}(s)}{s} \to +\infty \quad \textit{as } s \to +\infty.$$

Then $\underline{\rho}(s) \le \overline{\rho}(s)$ in $[0,+\infty)$.

Remark C.6. The above theorem directly implies the corresponding comparison principle for FL-super/subsolutions (see Proposition 2.6). This is because $\overline{\rho}$ and $\underline{\rho}$ being a pair FL-super/subsolution (for an arbitrarily given $A \in \mathbb{R}$) implies the corresponding super/subsolution property with Kirchhoff junction condition (for some $B = B(A)$); See Subsection 5.3.

Proof of Theorem C.5. First of all, we may assume without loss that $\underline{\rho}$ is nonnegative, $\underline{\rho}(0) = 0$ and $\underline{\rho} \in \mathrm{Lip}_{loc}([0,\infty))$. This can be achieved by replacing $\underline{\rho}$ by $\max\{\underline{\rho}, 0\}$. Since 0 and $\underline{\rho}$ (and hence also $\max\{\underline{\rho}, 0\}$ is automatically a viscosity subsolution (in Ishii sense) to $\min\{\rho, \rho + H(s,\rho')\} = 0$, Lemma 4.5 is applicable and it is therefore locally Lipschitz continuous.

Suppose to the contrary that

$$\sup_{s\geq 0}[\underline{\rho}(s) - \overline{\rho}(s)] > 0. \tag{C.4}$$

Step #1. We may assume without loss of generality that $\underline{\rho}(s) - \overline{\rho}(s) \to -\infty$ as $s \to +\infty$.

If $\limsup_{s\to\infty} \frac{\underline{\rho}(s)}{s} < \infty$, then we are done. Otherwise, we proceed as in [44, Proposition 2.11]. First, observe that $\underline{\rho} \in \mathrm{Lip}_{loc}((0,\infty))$. By Rademacher's theorem, it is differentiable in some $[0,\infty)\setminus\mathcal{S}$ where $\mathcal{S}$ has zero Lebesgue measure. Hence, we may choose $s_k \in [\overline{s}, \infty)$, $s_k \to \infty$ such that $\underline{\rho}$ is differentiable at s_k, and

$$\inf_k \underline{\rho}(s_k) > 0 \quad \text{ and } \quad R(s_k) \to \limsup_{s\to\infty} R(s).$$

Note that the latter is a consequence of local monotonicity of R thanks to (A4). Next, define

$$\underline{\rho}_k(s) := \begin{cases} \underline{\rho}(s) - \nu_k & \text{for } 0 \leq s \leq s_k, \\ \underline{\rho}(s_k) - \nu_k + \underline{\rho}'(s_k)(s - s_k) & \text{for } s > s_k, \end{cases}$$

where $\nu_k = \sup_{[s_k,\infty)} R - R(s_k)$ (note that $\nu_k \to 0$). Observe that $\underline{\rho}_k$ is a viscosity subsolution in $[0,\infty)$ with linear growth as $s \to \infty$. Indeed, $\underline{\rho}_k$ is a viscosity subsolution of (C.1) in $[0, s_k)$ on the one hand, and a classical subsolution of (C.1) in $[s_k, \infty)$ on the other hand, since

$$\begin{aligned}
\underline{\rho}_k + H(s, \underline{\rho}_k') &= \underline{\rho}_k - s\underline{\rho}_k' + \tilde{H}(s, \underline{\rho}_k') + R(s) \\
&= [a_k - \nu_k + b_k(s - s_k)] - sb_k + \tilde{H}(s, b_k) + R(s) \\
&\leq a_k - \nu_k - s_k b_k + \tilde{H}(s_k, b_k) + R(s) \\
&= (a_k - s_k b_k + \tilde{H}(s_k, b_k) + R(s_k)) + (R(s) - R(s_k) - \nu_k) \\
&\leq \underline{\rho}(s_k) + H(s_k, \underline{\rho}'(s_k)),
\end{aligned}$$

where we adopted the notation $a_k = \underline{\rho}(s_k)$, $b_k = \underline{\rho}'(s_k)$ and $s > s_k$.

We can then replace $\underline{\rho}$ by $\underline{\rho}_k$, if necessary. Note that (C.4) still holds provided k is sufficiently large, since $\nu_k \to 0$ and $s_k \to +\infty$.

In the rest of the proof, we will show the comparison result is valid, i.e.

$$\max\{\underline{\rho}_k(s), 0\} \leq \overline{\rho}(s) \quad \text{ in } [0,\infty), \quad \text{ for all sufficiently large } k.$$

Granted, then we can take $k \to \infty$ to deduce that $\underline{\rho} \leq \overline{\rho}$.

Step #2. For $\lambda \in (0,1)$, define $W(s) = \lambda\underline{\rho}(s) - \overline{\rho}(s)$. Then choose $\lambda \nearrow 1$, $0 < s_0 < \overline{s}_0$ such that

$$\eta_0 := W(s_0) = \max_{s\in[0,\infty)} W(s) > \max\left\{ \sup_{[\overline{s}_0 - 1,\infty)} W, (1-\lambda) \sup_{\substack{s\in[0,\overline{s}_0] \\ |p|\leq 2|\overline{s}_0|}} |H(s,p)| \right\}. \tag{C.5}$$

For given $1 \leq i \leq n$, we consider two cases. Either (i) there is a sequence $\lambda_j \nearrow 1$ such that $s_0 \neq c_i$ for all j, or (ii) there exists a sequence $\lambda_j \nearrow 1$ such that $s_0 = c_i$ for all j. We first consider case (i), $s_0 \neq c_i$.

Step #3. Next, define

$$\Psi_\alpha(s,t) = \lambda\underline{\rho}(s + \alpha^{-1/2}h_0) - \overline{\rho}(t) - \frac{\alpha}{2}|s-t|^2 - \frac{1-\lambda}{2}|s - s_0|^2, \tag{C.6}$$

where h_0 is given in (A1).

Claim C.7. *There exists $\bar{\alpha} > 0$ such that if $\alpha > \bar{\alpha}$, then the following statements hold.*

(i) Ψ_α *has an interior local maximum* (s_1, t_1) *in* $(\frac{s_0}{2}, \frac{\bar{s}_0+s_0}{2}) \times (0, \bar{s}_0)$.

(ii) $\Psi_\alpha(s_1, t_1) \geq \Psi_\alpha(s_0, s_0) = \eta_0 + o(1) > 0$.

(iii) $\alpha|s_1 - t_1|^2 \to 0$ *as* $\alpha \to \infty$.

(iv) $(s_1, t_1) \to (s_0, s_0)$ *as* $\alpha \to \infty$.

Denote $Q = [\frac{s_0}{2}, \frac{\bar{s}_0+s_0}{2}] \times [0, \bar{s}_0]$. Clearly, Ψ_α is upper semicontinuous in Q, therefore, it attains the maximum at $(s_1, t_1) \in Q$. By (C.5) and $\underline{\rho} \in \mathrm{Lip}_{loc}((0, \infty))$, it follows that

$$\sup_Q \Psi_\alpha \geq \Psi_\alpha(s_0, s_0) = \eta_0 + \lambda(\underline{\rho}(s_0 + \alpha^{-1/2}h_0) - \underline{\rho}(s_0)) = \eta_0 + o(1)$$

where $o(1)$ is considered with respect to $\alpha \to +\infty$. This proves (ii). For (iii), first observe that $\alpha|s_1 - t_1|^2 = O(1)$ which is a direct consequence of statement (ii). In particular, $(s_1, t_1) \to (\hat{s}, \hat{s})$ for some $\hat{s} \in [\frac{s_0}{2}, \frac{\bar{s}_0+s_0}{2}]$. By (ii), we can write

$$\frac{\alpha}{2}|s_1 - t_1|^2 \leq -W(s_0) + [W(s_1) + \overline{\rho}(s_1)] - \overline{\rho}(t_1) - \frac{1-\lambda}{2}|s_1 - s_0|^2 + o(1).$$

Since $s \mapsto W(s) + \overline{\rho}(s)$ and $t \mapsto -\overline{\rho}(t)$ are both upper semicontinuous, we can take $\alpha \to \infty$ to obtain

$$0 \leq \limsup_{\alpha\to\infty} \frac{\alpha}{2}|s_1 - t_1|^2 \leq -W(s_0) + W(\hat{s}) - \frac{1-\lambda}{2}|\hat{s} - s_0|^2 \leq 0,$$

where the last inequality follows from the fact that W attains global maximum at s_0. This proves (iii) and (iv). Finally, statement (iv) yields (s_1, t_1) must be an interior point of Q. This proves (i).

Step #4. Fixing $t = t_1$, observe that for $\alpha > \bar{\alpha}$, $\underline{\rho}(s_1 + \alpha^{-1/2}h_0) \geq \eta_0 + \overline{\rho}(t_1) + o(1) > 0$ since $\overline{\rho} \geq 0$, and $s \mapsto \underline{\rho}(s) - \varphi(s)$ attains a local maximum at $s = \hat{s}_1 := s_1 + \alpha^{-1/2}h_0$, where

$$\varphi(s) = \frac{1}{\lambda}\left[\overline{\rho}(t_1) + \frac{\alpha}{2}|s - \alpha^{-1/2}h_0 - t_1|^2 + \frac{1-\lambda}{2}|s - \alpha^{-1/2}h_0 - s_0|^2\right].$$

By definition of viscosity subsolution, we have

$$\underline{\rho}(\hat{s}_1) + H_*\left(\hat{s}_1, \frac{\alpha(s_1 - t_1)}{\lambda} + \frac{1-\lambda}{\lambda}(s_1 - s_0)\right) \leq 0. \tag{C.7}$$

Using the convexity of H_*, we have

$$\lambda H_*\left(\hat{s}_1, \frac{\alpha(s_1 - t_1)}{\lambda} + \frac{1-\lambda}{\lambda}(s_1 - s_0)\right) + (1-\lambda)H_*\left(\hat{s}_1, -(s_1 - s_0)\right) \geq H_*\left(\hat{s}_1, \alpha(s_1 - t_1)\right).$$

Substitute into (C.7)

$$\lambda\underline{\rho}(\hat{s}_1) + H_*\left(\hat{s}_1, \alpha(s_1 - t_1)\right) \leq (1-\lambda)H_*(\hat{s}_1, -(s_1 - s_0)) \leq (1-\lambda)\sup_{\substack{s\in[0,\bar{s}_0]\\|p|\leq 2|\bar{s}_0|}} H(s, p). \tag{C.8}$$

Next, we fix $s = s_1$ and observe that $t \mapsto \overline{\rho}(t) - \psi(t)$ has an interior local minimum at $t = t_1$, where

$$\psi(t) = \lambda\underline{\rho}(s_1) - \frac{\alpha}{2}|t - s_1|^2 - \frac{1-\lambda}{2}|s_1 - s_0|^2.$$

Hence,

$$\overline{\rho}(t_1) + H^*(t_1, \alpha(s_1 - t_1)) \geq 0. \tag{C.9}$$

Since $s_0 \notin \mathcal{P}$, it follows that $t_1 \notin \mathcal{P}$ for sufficiently large α. Combining (C.8) and (C.9), we have

$$\lambda\underline{\rho}(\hat{s}_1) - \overline{\rho}(t_1) \le H^*(t_1, \alpha(s_1 - t_1)) - H_*(\hat{s}_1, \alpha(s_1 - t_1)) + (1-\lambda) \sup_{\substack{s\in[0,\bar{s}_0]\\ |p|\le 2|\bar{s}_0|}} H(s,p). \quad \text{(C.10)}$$

Step #5. Observe that $\lambda\underline{\rho}(\hat{s}_1) - \overline{\rho}(t_1) \ge \Psi_\alpha(s_1, t_1) \ge \eta_0 + o(1)$. Using also $|h_0| = 1$ (by (A1)) and $\alpha|s_1 - t_1|^2 = o(1)$ (by Claim C.7(iii)), we have

$$(t_1 - \hat{s}_1)h_0 = [t_1 - s_1 - \alpha^{-1/2}h_0]h_0 \le |t_1 - s_1| - \alpha^{-1/2} < 0.$$

We can then apply (A1) to (C.10) to get

$$o(1) + \eta_0 \le \omega_L(|t_1 - s_1 - \alpha^{-1/2}h_0|(\alpha|t_1 - s_1| + 1)) + (1-\lambda) \sup_{s\in[0,\bar{s}_0], |p|\le 2|\bar{s}_0|} H(s,p). \quad \text{(C.11)}$$

Using $\alpha|s_1 - t_1|^2 = o(1)$ and

$$|t_1 - s_1 - \alpha^{-1/2}h_0|(\alpha|t_1 - s_1| + 1) \le (\alpha^{1/2}|t_1 - s_1| + 1)(\alpha^{1/2}|t_1 - s_1| + \alpha^{-1/2}),$$

we let $\alpha \to \infty$ in (C.11) to obtain

$$\eta_0 \le (1-\lambda) \sup_{s\in[0,\bar{s}_0], |p|\le 2|\bar{s}_0|} H(s,p),$$

which is a contradiction. This concludes the proof when $W(s) = \lambda\underline{\rho}(s) - \overline{\rho}(s)$ attains a local maximum at s_0, such that $s_0 \ne c_i$ for all i.

It remains to consider the case (ii) (see Step #2), when there is $\lambda_j \nearrow 1$ such that $\lambda_j\underline{\rho} - \overline{\rho}$ has a global maximum at $s_0 = c_i$ for some i. Since this holds for a sequence of $\lambda_j \nearrow 1$, we reduce to the case that $\underline{\rho} - \overline{\rho}$ attains the global maximum at c_i for some i. For convenience, let's assume $s_0 = c_1$. Next, define

$$a = \underline{\rho}(c_1) \quad \text{and} \quad b = \overline{\rho}(c_1)$$

and assume to the contrary that $a > b \ge 0$. (We will show that $a \le b$ so there is a contradiction.)

Step #6. We claim that the critical slopes of $\underline{\rho}$, given as follows, are finite.

$$p_- = \sup\{\bar{p} \in \mathbb{R} : \ \exists r > 0, \ \underline{\rho}(c_1) + \bar{p}(s - c_1) \ge \underline{\rho}(s) \ \text{ for } -r < s - c_1 \le 0\},$$

$$p_+ = \sup\{\bar{p} \in \mathbb{R} : \ \exists r > 0, \ \underline{\rho}(c_1) - \bar{p}(s - c_1) \ge \underline{\rho}(s) \ \text{ for } 0 \le s - c_1 < r\}.$$

Indeed, they are finite because $\underline{\rho}$ is locally Lipschitz continuous. Moreover, we have

$$\begin{cases} a + H(c_1-, p_-) \le 0, \quad a + H(c_1+, -p_+) \le 0, \quad \text{and} \\ \min\{p'_- + p'_+ - B_1, a + H(c_1\pm, \mp p'_\pm)\} \le 0 \ \text{ for } (p'_-, p'_+) \in (-\infty, p_-] \times (-\infty, p_+], \end{cases} \quad \text{(C.12)}$$

where the former is due to Lemma 4.5 and Remark 4.6 (note that $\underline{\rho} \in \mathrm{Lip}_{loc}$, so there exists at least one constant test function in C^1_{pw}), while the latter holds by considering the test function $\psi(s) = \underline{\rho}(c_1) + p'_- \min\{s - c_1, 0\} - p'_+ \max\{s - c_1, 0\}$, which touches $\underline{\rho}$ from above at c_1.

Next, define

$$p^*_- = \limsup_{s\to c_1-} \frac{\underline{\rho}(s) - \underline{\rho}(c_1)}{s - c_1} \quad \text{and} \quad p^*_+ = -\liminf_{s\to c_1+} \frac{\underline{\rho}(s) - \underline{\rho}(c_1)}{s - c_1}. \quad \text{(C.13)}$$

Note that

$$p^*_- \ge p_- \quad \text{and} \quad p^*_+ \ge p_+, \quad \text{(C.14)}$$

since $p_- = \liminf_{s\to c_1-} \frac{\underline{\rho}(s)-\underline{\rho}(c_1)}{s-c_1}$ and $p_+ = -\limsup_{s\to c_1+} \frac{\underline{\rho}(s)-\underline{\rho}(c_1)}{s-c_1}$.

Step #7. We improve (C.12) to

$$\begin{cases} a + H(c_1-, p_-^*) \le 0, \quad a + H(c_1+, -p_+^*) \le 0, \quad \text{and} \\ \min\{p'_- + p'_+ - B_1, a + H(c_1\pm, \mp p'_\pm)\} \le 0 \ \text{for}\ (p'_-, p'_+) \in (-\infty, p_-^*] \times (-\infty, p_+^*]. \end{cases} \tag{C.15}$$

Indeed, since $\underline{\rho}$ is locally Lipschitz, $\underline{\rho}'$ exists a.e. and $\underline{\rho}(s)-\underline{\rho}(c_1) = \int_{c_1}^s \underline{\rho}'(t)\,dt$, the definition of p_+^* implies that, for each $\delta > 0$, the set $\{s \in (c_1, c_1+\delta) : \ \underline{\rho}'(s) < -p_+^* + \delta\}$ has positive measure. This implies that there is a sequence $s_k \searrow c_1$ such that $\underline{\rho}$ is differentiable at s_k and also that $\limsup_{k\to\infty} \underline{\rho}'(s_k) \le -p_+^*$. Hence, letting $k \to \infty$ in $\underline{\rho}(s_k) + H_*(s_k, \underline{\rho}'(s_k)) \le 0$, we obtain

$$a + H(c_1+, \liminf_{k\to\infty} \underline{\rho}'(s_k)) \le 0.$$

Noting that H is convex in p variable, $\liminf_{k\to\infty} \underline{\rho}'(s_k) \le -p_+^* \le -p_+$, and using the first part of (C.12), we deduce

$$a + H(c_1+, -p'_+) \le 0 \quad \text{for all}\ p'_+ \in [p_+, p_+^*]. \tag{C.16}$$

By a completely similar argument, we also have

$$a + H(c_1-, p'_-) \le 0 \quad \text{for all}\ p'_- \in [p_-, p_-^*]. \tag{C.17}$$

Combining (C.16) and (C.17) into (C.12), we obtain (C.15).

Step #8. We claim that the critical slopes of $\overline{\rho}$, defined as follows, are well-defined but possibly equals $-\infty$.

$$q_- = \inf\{\bar{q} \in \mathbb{R} : \ \exists r > 0,\ \overline{\rho}(c_1) + \bar{q}(s-c_1) \le \overline{\rho}(s) \ \text{ for } -r < s - c_1 \le 0\}$$

$$q_+ = \inf\{\bar{q} \in \mathbb{R} : \ \exists r > 0,\ \overline{\rho}(c_1) - \bar{q}(s-c_1) \le \overline{\rho}(s) \ \text{ for } 0 \le s - c_1 < r\}.$$

Indeed,

$$\underline{\rho}(s) - \overline{\rho}(s) \le \underline{\rho}(c_1) - \overline{\rho}(c_1) \quad \text{for all } s, \quad \text{with equality holds at } s = c_1, \tag{C.18}$$

i.e. the locally Lipschitz function $\underline{\rho}(s) - \underline{\rho}(c_1) + \overline{\rho}(c_1)$ touches $\overline{\rho}(s)$ from below at $s = c_1$. This shows that q_- and q_+ are well-defined in $\mathbb{R} \cup \{-\infty\}$.

Next, we observe that

$$q_- \le p_-^* \quad \text{and} \quad q_+ \le p_+^*, \tag{C.19}$$

which is due to (C.13), (C.18), and

$$q_- = \limsup_{s\to c_1-} \frac{\overline{\rho}(s)-\overline{\rho}(c_1)}{s-c_1} \quad \text{and } q_+ = -\liminf_{s\to c_1+} \frac{\overline{\rho}(s)-\overline{\rho}(c_1)}{s-c_1}.$$

Step #9. Suppose $q_\pm > -\infty$, then we have

$$\begin{cases} b + H(c_1-, q_-) \ge 0, \quad b + H(c_1+, -q_+) \ge 0 \quad \text{and} & \text{(C.20a)} \\ \max\{q'_- + q'_+ - B_1, b + H(c_1\pm, \mp q'_\pm)\} \ge 0, \ \text{for}\ (q'_-, q'_+) \in [q_-, \infty) \times [q_+, \infty), & \text{(C.20b)} \end{cases}$$

where the former holds by virtue of the critical slope lemma (Lemma 4.7 and Remark 4.8), and the latter holds by considering the C^1_{pw} test function $\psi(s) = \overline{\rho}(c_1) + q'_- \min\{s - c_1, 0\} - q'_+ \max\{s - c_1, 0\}$.

If $q_- = -\infty$ (resp. $q_+ = -\infty$) then take q_- (resp. q_+) large and negative enough (but finite) to satisfy both (C.19) and (C.20a). Then, for any $(q'_-, q'_+) \in [q_-, \infty) \times [q_+, \infty)$, the test function $\psi(s) = \bar{\rho}(c_1) + q'_- \min\{s - c_1, 0\} - q'_+ \max\{s - c_1, 0\}$ touches $\overline{\rho}$ at $s = c_1$. Then it follows that (C.20b) holds.

Step #10.

In view of (C.15), (C.19), and (C.20), we may apply the Lemma C.8 with ($H_1(p) = H(c_1-, p)$ and $H_2(p) = H(c_1+, -p)$, $p \in \mathbb{R}$, $p_1 = p^*_-$, $p_2 = p^*_+$, $q_1 = q_-$ and $q_2 = q_+$) to conclude that $a \le b$. This is a contradiction to the assumption that $a > b$. The proof is now complete. □

The following key lemma is due to Lions and Souganidis [48].

Lemma C.8. *Assume that $H_1, ..., H_m \in C(\mathbb{R})$, $p_1, ..., p_m, q_1, ..., q_m \in \mathbb{R}$, and $a, b \in \mathbb{R}$ are such that, for all $i = 1, ..., m$,*

1. *$p_i \ge q_i$, $a + H_i(p_i) \le 0 \le b + H_i(q_i)$ for all i,*
2. *$\min\left(\sum_i p'_i - B, \min_i(a + H_i(p'_i))\right) \le 0$ for each $p'_i \in (-\infty, p_i]$,*
3. *$\max\left(\sum_i q'_i - B, \max_i(b + H_i(q'_i))\right) \le 0$ for each $q'_i \in [q_i, \infty)$.*

Then $a \le b$.

Remark C.9. By replacing p_1 by $p_1 + B$ and q_1 by $q_1 + B$, and redefining

$$H_1(\cdot) \quad \text{to be} \quad H_1(\cdot + B),$$

one can reduce Lemma C.8 to the case $B = 0$, which is exactly [48, Lemma 3.1].

D Definition of viscosity solution in sense of Ishii

Definition D.1. Let $\hat{\rho} : (0, \infty) \to \mathbb{R}$.

(a) We say that $\hat{\rho}$ is a viscosity subsolution of

$$\min\{\rho, \rho + \tilde{H}(s, \rho')\} = 0 \qquad \text{for } s \ge 0 \tag{D.1}$$

in the sense of Ishii provided (i) $\hat{\rho}$ is upper semicontinuous, (ii) if $\hat{\rho} - \psi$ has a local maximum point at $s_0 > 0$ such that $\psi \in C^1$ and $\hat{\rho}(s_0) > 0$, then

$$\hat{\rho}(s_0) + \tilde{H}_*(s, \psi'(s_0)) \le 0,$$

where $\tilde{H}_*(s, p)$ is the lower semi-continuous envelope of $\tilde{H}(s, p)$, i.e.

$$\tilde{H}_*(s, p) = \begin{cases} -sp + p^2 + g(-\infty) & \text{for } s < c_1, \\ \tilde{H}(c_1-, p) \wedge \tilde{H}(c_1+, p) = -c_1 p + p^2 + \min\{g(-\infty), g(+\infty)\} & \text{for } s = c_1, \\ -sp + p^2 + g(+\infty) & \text{for } s > c_1. \end{cases}$$

(b) We say that $\hat{\rho}$ is a viscosity supersolution of (D.1) in the sense of Ishii provided (i) $\hat{\rho}$ is lower semicontinuous, (ii) $\hat{\rho} \ge 0$ for all $s > 0$, (iii) if $\hat{\rho} - \psi$ has a local minimum point at $s_0 > 0$ such that $\psi \in C^1$, then

$$\hat{\rho}(s_0) + \tilde{H}^*(s, \psi'(s_0)) \ge 0,$$

where $\tilde{H}^*(s, p)$ is the upper semi-continuous envelope of $H(s, p)$, i.e.

$$\tilde{H}^*(s, p) = \begin{cases} -sp + p^2 + g(-\infty) & \text{for } s < c_1, \\ \tilde{H}(c_1-, p) \vee \tilde{H}(c_1+, p) = -c_1 p + p^2 + \max\{g(-\infty), g(+\infty)\} & \text{for } s = c_1, \\ -sp + p^2 + g(+\infty) & \text{for } s > c_1. \end{cases}$$

(c) We say that $\hat{\rho}$ is a viscosity solution of (D.1) in the sense of Ishii if it is both subsolution and supersolution of (2.26) in the sense of Ishii.

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