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PATTERN FORMATION IN A CROSS-DIFFUSION SYSTEM

Yuan Lou

Institute for Mathematical Sciences, Renmin University of China
Haidian District, Beijing, 100872, China
and
Department of Mathematics, Ohio State University
Columbus, OH 43210, USA

Wei-Ming Ni

Center for PDE, East China Normal University
Minhang, Shanghai, 200241, China
and
School of Mathematics, University of Minnesota
Minneapolis, MN 55455, USA

Shoji Yotsutani

Department of Applied Mathematics and Informatics
Ryukoku University, Seta, Otsu, 520-2194, Japan

Abstract. In this paper we study the Shigesada-Kawasaki-Teramoto model [17] for two competing species with cross-diffusion. We prove the existence of spectrally stable non-constant positive steady states for high-dimensional domains when one of the cross-diffusion coefficients is sufficiently large while the other is equal to zero.

1. **Introduction.** The movement of organisms generally depend upon the densities of their conspecifics and competitors. It is well known that density-dependent dispersal plays an important role in population dynamics and affects the spatial distribution of populations [1, 16]. To understand the spatial segregation of competing species, Shigesada et al. [17] proposed a mathematical model for two species, in which the transition probability of each species depend only on the densities of both species at the departure point. The Shigesada-Kawasaki-Teramoto model (abbreviated as SKT henceforth) is a strongly coupled quasilinear parabolic system and it has been studied extensively for the last three decades; See [2, 5, 4, 6, 7, 8, 9, 11, 12, 13, 14, 18, 20, 21, 22, 23] and references therein. In this paper we will focus on the following model which is a special case of the SKT model:

$$\begin{cases} u_t = \Delta\left[(d_1+\alpha v)u\right] + u(a_1 - b_1 u - c_1 v) \quad \text{in } \Omega\times(0,\infty),\\ v_t = d_2\Delta v + v(a_2 - b_2 u - c_2 v) \quad \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1}$$

where $u(x,t)$ and $v(x,t)$ represent the densities of two species at location x and time t. We assume that Ω is a bounded open domain in Euclidean space $\mathbb{R}^N$

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with smooth boundary $\partial\Omega$, and ν is the outward unit normal vector on $\partial\Omega$. The boundary conditions for u and v mean that there is no net flux across the boundary for either population.

The coefficients d_1, d_2 are the random dispersal rates of species, a_1, a_2 are their intrinsic growth rates, b_1, c_2 account for their intraspecific competition and b_2, c_1 are their interspecific competition coefficients. We shall assume that d_i, a_i, b_i, c_i are positive constants throughout the paper. The parameter α is referred as the cross-diffusion coefficient and it measures the population pressure from species v towards species u. System (1) is a special case of the SKT model which assumes that the movement rate of species v is also a linear function of the density of its competitor (i.e., species u).

System (1) for $\alpha = 0$ has been extensively studied as well. It follows from the work of Kishimoto and Weinberger [3] that system (1) without cross-diffusion (i.e., $\alpha = 0$) has no stable non-constant positive steady states, provided that Ω is convex. On the other hand, it was shown in [10] that system (1) for $\alpha = 0$ can possess stable non-constant positive steady states for some non-convex domains and suitable coefficients d_i, a_i, b_i, c_i.

One natural question arises: Does system (1) with large α have stable non-constant positive steady states for general domains? For sufficiently large α, formally we have $[d_1/\alpha + v(x,t)]u(x,t) \approx \tau(t)$ for some function $\tau(t)$, so it is natural to expect that the dynamics of (1) is related with that of the shadow system

$$\begin{cases} \displaystyle\int_\Omega \left(\frac{\tau}{v}\right)_t = \tau \int_\Omega \frac{1}{v}\left(a_1 - \frac{b_1\tau}{v}\right) - c_1\tau|\Omega|, \quad t > 0, \\ v_t = d_2\Delta v + v(a_2 - \dfrac{b_2\tau}{v} - c_2 v) \quad \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{2}$$

When $N = 1$, i.e., Ω is an interval, the existence of non-constant positive steady states of (2) has been studied in Lou et al. [9]. Among other things, it is shown in [9] that for $\Omega = (0,1)$, if $a_1/a_2 > b_1/b_2$, then (2) has non-constant positive steady states for any d_2 slightly less than a_2/π^2. This is in strong contrast with the case $d_2 > a_2/\pi^2$, where (2) has no non-constant positive steady states for any values a_i, b_i and c_i. In a recent work [15], Ni et al. are able to derive more precise estimates of these steady solutions of (2) as $d_2 \to a_2/\pi^2$, which enables them to construct non-constant positive steady states of system (1) and further show that these positive steady states are asymptotically stable.

The goal of this paper is to extend some of the results of Ni et al. [15] to higher dimensional domains. To this end, we first introduce some notation. Let $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_k \le ...$ denote the eigenvalues of the linear eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi \quad \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Denote the corresponding eigenfunction of λ_k by φ_k, normalized by $\min_{\bar\Omega} \varphi_k = -1$. In particular, $\varphi_0 \equiv -1$ in Ω. However, for $k \ge 1$, φ_k is not uniquely determined by $\min_{\bar\Omega} \varphi_k = -1$ since $-\varphi_k/\max_{\bar\Omega} \varphi_k$ is also an eigenfunction of λ_k which satisfies $\min_{\bar\Omega} \frac{-\varphi_k}{\max_{\bar\Omega} \varphi_k} = -1$. If λ_k is a simple eigenvalue and $k \ge 1$, then there are exactly two eigenfunctions of λ_k with the global minimum value -1; i.e., φ_k and $-\varphi_k/\max_{\bar{\Omega}} \varphi_k$.

Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that $N \leq 4$, λ_k is simple for some $k \geq 1$, and $a_1/a_2 > b_1/b_2$.*

(a) (Existence) *There exists $\delta > 0$ such that for every $d_2 \in (a_2/\lambda_k - \delta, a_2/\lambda_k)$, if α is sufficiently large, then (1) has two non-constant positive steady state solutions in $C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$, denoted by $(u^*_{\pm,k}, v^*_{\pm,k})$.*

(b) (Asymptotic behavior) $(u^*_{\pm,k}, v^*_{\pm,k})$ *satisfy*

$$\lim_{\alpha\to\infty} u^*_{\pm,k} = \frac{\tau_{\pm,k}}{v_{\pm,k}}, \quad \lim_{\alpha\to\infty} v^*_{\pm,k} = v_{\pm,k}$$

uniformly in $\bar{\Omega}$. Here $(v_{\pm,k}, \tau_{\pm,k})$ are non-constant positive steady states of system (2) which satisfy

$$\begin{aligned} &\lim_{d_2\to a_2/\lambda_k} \frac{v_{\pm,k}}{d_2 - a_2/\lambda_k} = -\frac{\lambda_k}{c_2} \frac{\int_\Omega \varphi_k^2}{\int_\Omega \varphi_k^2(2+\mu_{\pm,k}\varphi_k)}[1+\mu_{\pm,k}\varphi_k] \quad \text{in } L^\infty(\Omega), \\ &\lim_{d_2\to a_2/\lambda_k} \frac{\tau_{\pm,k}}{d_2 - a_2/\lambda_k} = -\frac{a_2\lambda_k}{b_2c_2} \frac{\int_\Omega \varphi_k^2}{\int_\Omega \varphi_k^2(2+\mu_{\pm,k}\varphi_k)}, \end{aligned} \tag{3}$$

where $\mu_{+,k} > 0 > \mu_{-,k}$ are the roots of

$$\frac{\int_\Omega (1+\mu\varphi_k)^{-2}\,dx}{\int_\Omega (1+\mu\varphi_k)^{-1}\,dx} = \frac{a_1/a_2}{b_1/b_2}. \tag{4}$$

(c) (Stability) *For every $d_2 \in (a_2/\lambda_1 - \delta, a_2/\lambda_1)$, if α is sufficiently large, $(u^*_{\pm,1}, v^*_{\pm,1})$ are spectrally stable in $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$.*

Remark 1. For $k \geq 2$, it is shown in Ni et al. [15] that if $N = 1$, then for d_2 close to a_2/λ_k, $(u^*_{\pm,k}, v^*_{\pm,k})$ are unstable when α is sufficiently large.

Remark 2. For $N \geq 5$, Theorem 1.1 remains to hold provided that (4) has positive and negative roots, e.g., when a_1/a_2 is slightly larger than b_1/b_2.

Remark 3. Numerical computations suggest that the branches of positive steady states of (1) which contain $\{(u^*_{\pm,1}, v^*_{\pm,1}, d_2)\}$ can be extended up to $d_2 = 0$, but they may no longer be stable for some smaller values of d_2 under suitable assumptions on a_i, b_i, c_i. Figure 1 shows positive steady states for the case

$$\Omega = (0,1)\times(0,15/16) \subset R^2, \quad d_1 = 1,\ \alpha = 10^7, \quad b_1 = 1,\ c_1 = 2,\ a_2 = b_2 = c_2 = 1,$$

$$A := a_1/a_2,\ B := b_1/b_2 = 1,\ C := c_1/c_2 = 2.$$

It holds that $\lambda_1(\Omega) = \pi^2$, and we take $\varphi_1 = \cos(\pi x)$. Two graphs near $d_2 = a_2/\lambda_1(\Omega)$ represent $u^*_{+,1}$ and $v^*_{+,1}$. As d_2 decreases, we numerically obtain a branch of positive steady states connecting to $u^*_{+,1}$ and $v^*_{+,1}$. As $d_2 \to 0$, it seems that small spiky solutions appear for the case $(B+C)/2 < A < (B+3C)/4$, and large spiky solutions appear for the case $(B+3C)/4 \leq A$. We suspect from various numerical computations that small spiky solutions would be unstable and large spiky solutions would be stable. We note that numbers $(B+C)/2$ and $(B+3C)/4$ appear in the papers [6, 7] in multi-dimensional case and [9, 15, 19] in one-dimensional case. It seems that they are very important numbers to investigate the existence and the stability of steady state solutions.

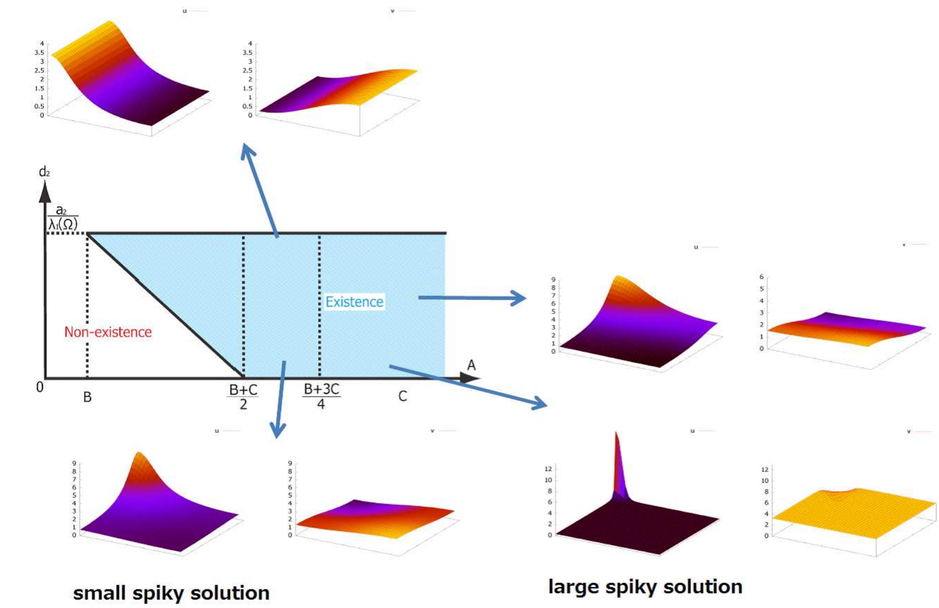

FIGURE 1. Numerically obtained steady states

Our paper is organized as follows: Section 2 is devoted to the existence of non-constant positive steady states of system (1) and parts (a) and (b) of Theorem 1.1 are proved there. In Section 3 we prove part (c) of Theorem 1.1.

2. **Existence of non-constant positive steady states.** In this section we prove parts (a) and (b) of Theorem 1.1. We first establish some qualitative properties concerning the eigenfunctions φ_k in Subsection 2.1, which will play critical roles in later analysis. The existence of non-constant positive steady states of (2) is given in Subsection 2.2. Finally in Subsection 2.3 we prove the existence of non-constant positive steady states of (1).

2.1. **Preliminary results.** We recall that φ_k is an eigenfunction of the eigenvalue λ_k. For each $k \geq 1$, define

$$g_k(\mu) := \frac{\int_\Omega (1+\mu\varphi_k)^{-2}\,dx}{\int_\Omega (1+\mu\varphi_k)^{-1}\,dx}, \qquad -\frac{1}{\max_{\bar\Omega}\varphi_k} < \mu < -\frac{1}{\min_{\bar\Omega}\varphi_k}. \tag{5}$$

We shall show that $g_k(\mu) > 1$ for any $\mu \neq 0$. Given any number $\eta > 1$, we are interested in whether $g_k(\mu) = \eta$ has exactly one positive root and one negative root.

Lemma 2.1. *For each $k \geq 1$, function $g_k(\mu)$ satisfies $g_k(0) = 1$, $\mu g_k'(\mu) > 0$ for any $\mu \neq 0$. In particular, $g_k(\mu) > 1$ for $\mu \neq 0$, $g_k'(\mu) > 0$ for $\mu > 0$, and $g_k'(\mu) < 0$ for $\mu < 0$.*

Proof. It is obvious that $g_k(0) = 1$. We first show that $g_k(\mu) > 1$ for any $\mu \neq 0$. By the equation of φ_k we have

$$\int_\Omega \frac{1}{(1+\mu\varphi_k)^2}\,dx - \int_\Omega \frac{1}{1+\mu\varphi_k}\,dx$$

$$\begin{aligned}
&= -\mu \int_\Omega \frac{\varphi_k}{(1+\mu\varphi_k)^2}\, dx \\
&= \frac{\mu}{\lambda_k} \int_\Omega \frac{\Delta\varphi_k}{(1+\mu\varphi_k)^2}\, dx \qquad (6)\\
&= \frac{2\mu^2}{\lambda_k} \int_\Omega \frac{|\nabla\varphi_k|^2}{(1+\mu\varphi_k)^3}\, dx \geq 0,
\end{aligned}$$

where the last inequality is strict for any $\mu \neq 0$ as φ_k is non-constant for any $k \geq 1$. This proves that $g_k(\mu) > 1$ for any $\mu \neq 0$.

By direct calculation,

$$\begin{aligned}
g_k'(\mu) = \left(\int_\Omega \frac{dx}{1+\mu\varphi_k}\right)^{-2} \Bigg[&-2 \int_\Omega \frac{\varphi_k\, dx}{(1+\mu\varphi_k)^3} \int_\Omega \frac{dx}{1+\mu\varphi_k} \\
&+ \int_\Omega \frac{dx}{(1+\mu\varphi_k)^2} \int_\Omega \frac{\varphi_k\, dx}{(1+\mu\varphi_k)^2} \Bigg].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mu g_k'(\mu) \left(\int_\Omega \frac{dx}{1+\mu\varphi_k}\right)^2 \\
&= -2 \int_\Omega \frac{\mu\varphi_k\, dx}{(1+\mu\varphi_k)^3} \int_\Omega \frac{dx}{1+\mu\varphi_k} + \int_\Omega \frac{dx}{(1+\mu\varphi_k)^2} \int_\Omega \frac{\mu\varphi_k\, dx}{(1+\mu\varphi_k)^2} \\
&= -2 \left[\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2} - \int_\Omega \frac{dx}{(1+\mu\varphi_k)^3}\right] \int_\Omega \frac{dx}{1+\mu\varphi_k} \\
&\quad + \int_\Omega \frac{dx}{(1+\mu\varphi_k)^2} \left[\int_\Omega \frac{dx}{1+\mu\varphi_k} - \int_\Omega \frac{dx}{(1+\mu\varphi_k)^2}\right] \qquad (7)\\
&= -\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2} \int_\Omega \frac{dx}{1+\mu\varphi_k} + 2 \int_\Omega \frac{dx}{(1+\mu\varphi_k)^3} \int_\Omega \frac{dx}{1+\mu\varphi_k} \\
&\quad - \left[\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2}\right]^2 \\
&\geq 2 \int_\Omega \frac{dx}{(1+\mu\varphi_k)^3} \int_\Omega \frac{dx}{1+\mu\varphi_k} - 2 \left[\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2}\right]^2,
\end{aligned}$$

where the last inequality follows from the fact that $g_k(\mu) \geq 1$ for any μ.

By Cauchy-Schwartz inequality,

$$\int_\Omega \frac{dx}{(1+\mu\varphi_k)^3} \int_\Omega \frac{dx}{1+\mu\varphi_k} \geq \left[\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2}\right]^2, \qquad (8)$$

where the inequality is strict if and only if $\mu \neq 0$. This together with (7) completes the proof. □

By Lemma 2.1 we see that there exists some $\delta > 0$ small such that for any $\eta \in (1, 1+\delta)$, $g_k(\mu) = \eta$ has exactly one positive and one negative root. Next we study when $g_k(\mu) = \eta$ has exactly one positive root and one negative root for any $\eta > 1$.

Lemma 2.2. *Suppose that $N \leq 4$. Then for each $k \geq 1$, function $g_k(\mu)$ satisfies*

$$\lim_{\mu \to -1/\max_{\bar\Omega} \varphi_k} g_k(\mu) = \lim_{\mu \to -1/\min_{\bar\Omega} \varphi_k} g_k(\mu) = +\infty. \qquad (9)$$

Proof. We argue by contradiction. Suppose that the limit in (9) fails for $\mu \to -1/\min_{\bar\Omega}\varphi_k$. Since $g_k' > 0$ for $\mu > 0$, there exists some positive constant C_1 such that $g_k(\mu) \le C_1$ for any $0 < \mu < -1/\min_{\bar\Omega}\varphi_k$. Hence, for any $0 < \mu < -1/\min_{\bar\Omega}\varphi_k$,

$$\begin{aligned}\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2} &= g_k(\mu)\int_\Omega \frac{dx}{1+\mu\varphi_k}\\ &\le C_1 \int_\Omega \frac{dx}{1+\mu\varphi_k} \\ &\le C_1\left[\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2}\right]^{1/2}|\Omega|^{1/2}.\end{aligned} \tag{10}$$

Therefore we get

$$\int_\Omega \frac{dx}{(1+\mu\varphi_k)^2} \le C_2 := C_1^2|\Omega|, \quad \forall\ 0<\mu<-\frac{1}{\min_{\bar\Omega}\varphi_k}. \tag{11}$$

Let $x^* \in \bar\Omega$ such that $\varphi_k(x^*) = \min_{\bar\Omega}\varphi_k = -1$. We claim that $\nabla\varphi_k(x^*) = 0$. This is clear if $x^* \in \Omega$. If $x^* \in \partial\Omega$, it follows from $\nabla\varphi_k \cdot \nu = 0$ at x^*. Hence there exist positive small constants δ and γ such that

$$\varphi_k(x) \le -1 + \gamma|x-x^*|^2, \quad \forall x \in B_\delta(x^*)\cap\Omega,$$

where $B_\delta(x^*)$ denotes the open ball centered at x^* with radius δ. Therefore, for any $\mu > 0$,

$$1+\mu\varphi_k(x) \le 1-\mu+\mu\gamma|x-x^*|^2, \quad \forall x\in B_\delta(x^*)\cap\Omega.$$

This implies that

$$\int_{B_\delta(x^*)\cap\Omega} \frac{dx}{(1-\mu+\mu\gamma|x-x^*|^2)^2} \le C_2, \quad \forall\ 0<\mu<-\frac{1}{\min_{\bar\Omega}\varphi_k} = 1. \tag{12}$$

By choosing $\gamma\delta^2 < 1$, we see that for any $x \in B_\delta(x^*)\cap\Omega$, $1-\mu+\mu\gamma|x-x^*|^2$ is monotone in μ. Therefore by the Monotone Convergence Theorem, by letting $\mu \to 1-$ in (12) we find

$$\int_{B_\delta(x^*)\cap\Omega} \frac{dx}{|x-x^*|^4} \le C_2\gamma^2. \tag{13}$$

However, this is a contradiction since for $N \le 4$,

$$\int_{B_\delta(x^*)\cap\Omega} \frac{dx}{|x-x^*|^4} = +\infty. \tag{14}$$

The limit of $g_k(\mu)$ as $\mu \to -1/\max_{\bar\Omega}\varphi_k$ can be similarly treated. $\square$

The following result is a direct consequence of Lemmas 2.1 and 2.2.

Corollary 1. *Suppose that $N \le 4$ and $a_1/a_2 > b_1/b_2$. Then for each $k \ge 1$, $g_k(\mu) = (a_1/a_2)/(b_1/b_2)$ has exactly one positive and one negative root, denoted by μ_k^+ and μ_k^-, respectively. Moreover, $g_k'(\mu_k^+) > 0$ and $g_k'(\mu_k^-) < 0$.*

Remark 4. For $N \ge 5$, (4) has exactly one positive and one negative root when a_1/a_2 is slightly larger than b_1/b_2.

2.2. **Shadow system.** This subsection is devoted to the study of the system

$$\begin{cases} d_2\Delta v + v(a_2 - c_2 v) - b_2\tau = 0 \quad \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{1}{v}\left(a_1 - \frac{b_1\tau}{v}\right) = c_1|\Omega|, \\ v > 0 \quad \text{in } \Omega, \quad \tau > 0. \end{cases} \tag{15}$$

Our main result for this subsection can be stated as follows:

Theorem 2.3. *Suppose that $N \le 4$, λ_k is simple, and $a_1/a_2 > b_1/b_2$. Then there exists $\delta > 0$ such that for every $d_2 \in (a_2/\lambda_k - \delta, a_2/\lambda_k)$, system (15) has two non-constant solutions, denoted by $(v_{\pm,k}(x; d_2), \tau_{\pm,k}(d_2))$, which satisfy*

$$\begin{aligned} \lim_{d_2 \to a_2/\lambda_k} \frac{v_{\pm,k}(x; d_2)}{d_2 - a_2/\lambda_k} &= -\frac{\lambda_k}{c_2}\frac{\int_\Omega \varphi_k^2}{\int_\Omega \varphi_k^2(2 + \mu_{\pm,k}\varphi_k)}[1 + \mu_{\pm,k}\varphi_k], \\ \lim_{d_2 \to a_2/\lambda_k} \frac{\tau_{\pm,k}(d_2)}{d_2 - a_2/\lambda_k} &= -\frac{a_2\lambda_k}{b_2 c_2}\frac{\int_\Omega \varphi_k^2}{\int_\Omega \varphi_k^2(2 + \mu_{\pm,k}\varphi_k)}, \end{aligned} \tag{16}$$

where $\mu_{+,k} > 0 > \mu_{-,k}$ are the two roots of $g_k(\mu) = (a_1/a_2)/(b_1/b_2)$.

Proof. Set $w = v/\tau$. Then (w, τ) solves

$$\begin{cases} d_2\Delta w + w(a_2 - c_2\tau w) - b_2 = 0 \quad \text{in } \Omega, \\ \displaystyle\int_\Omega \frac{1}{w}\left(a_1 - \frac{b_1}{w}\right) = c_1\tau|\Omega|, \\ \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad w > 0 \quad \text{in } \Omega, \quad \tau > 0. \end{cases} \tag{17}$$

For $p > N$, set

$$\begin{aligned} W_\nu^{2,p} &= \left\{u \in W^{2,p}(\Omega) : \tfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\right\}, \\ W_{\nu,+}^{2,p} &= \left\{u \in W_\nu^{2,p}(\Omega) : u > 0 \quad \text{in } \bar\Omega\right\}. \end{aligned}$$

Define mapping $F : W_{\nu,+}^{2,p} \times R \times (0, \infty) \to L^p \times R$ by

$$F(w, \tau, d_2) = \begin{pmatrix} d_2\Delta w + w(a_2 - c_2\tau w) - b_2 \\ \displaystyle\int_\Omega \frac{1}{w}\left(a_1 - \frac{b_1}{w}\right) - c_1\tau|\Omega| \end{pmatrix} \tag{18}$$

Since $g_k(\mu_{+,k}) = (a_1/a_2)/(b_1/b_2)$, it is straightforward to check that

$$F\left(\frac{b_2}{a_2}(1 + \mu_{+,k}\varphi_k), 0, \frac{a_2}{\lambda_k}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Fréchet derivative of F at $(w, \tau, d_2) = (b_2(1 + \mu_{+,k}\varphi_k)/a_2, 0, a_2/\lambda_k)$, with respect to (w, τ), is given by

$$\begin{aligned} &D_{(w,\tau)}F\big|_{(b_2(1+\mu_{+,k}\varphi_k)/a_2, 0, a_2/\lambda_k)}\begin{pmatrix} \varphi \\ \eta \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{a_2}{\lambda_k}\Delta\varphi + a_2\varphi - \dfrac{c_2 b_2^2}{a_2^2}\eta(1 + \mu_{+,k}\varphi_k)^2 \\ \left(\dfrac{a_2}{b_2}\right)^3 \displaystyle\int_\Omega \left[\frac{-a_1 b_2\varphi}{a_2(1 + \mu_{+,k}\varphi_k)^2} + \frac{2b_1\varphi}{(1 + \mu_{+,k}\varphi_k)^3}\right] - c_1\eta|\Omega| \end{pmatrix} \end{aligned}$$

We claim that the kernel of the operator $D_{(w,\tau)}F\big|_{(b_2(1+\mu_{+,k}\varphi_k)/a_2,0,a_2/\lambda_k)}$ is trivial. To this end, we argue by contradiction: If not, suppose that there exists $(\varphi,\eta)^T$ in the kernel of $D_{(w,\tau)}F\big|_{(b_2(1+\mu_{+,k}\varphi_k)/a_2,0,a_2/\lambda_k)}$, where $\varphi \in W^{2,p}_\nu$ and $\eta \in R$. Then φ and η satisfy

$$\begin{cases} \frac{a_2}{\lambda_k}\Delta\varphi + a_2\varphi - \dfrac{c_2 b_2^2}{a_2^2}\eta(1+\mu_{+,k}\varphi_k)^2 = 0 \quad \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \\ \left(\dfrac{a_2}{b_2}\right)^3 \displaystyle\int_\Omega \left[\dfrac{-a_1 b_2 \varphi}{a_2(1+\mu_{+,k}\varphi_k)^2} + \dfrac{2b_1\varphi}{(1+\mu_{+,k}\varphi_k)^3}\right] - c_1\eta|\Omega| = 0. \end{cases} \tag{19}$$

Multiplying the first equation of (19) by φ_k and integrating in Ω we have

$$\eta \int_\Omega \varphi_k(1+\mu_{+,k}\varphi_k)^2 = 0.$$

Since $\int_\Omega \varphi_k = 0$ and $\mu_{+,k} \in (0, -1/\min_{\bar\Omega}\varphi_k)$, we have

$$\int_\Omega \varphi_k(1+\mu_{+,k}\varphi_k)^2 = \mu_{+,k}\int_\Omega \varphi_k^2(2+\mu_{+,k}\varphi_k) > 0.$$

Hence, $\eta = 0$. By (19) we have $\Delta\varphi + \lambda_k\varphi = 0$ in Ω and $\partial\varphi/\partial\nu = 0$ on $\partial\Omega$. If $\varphi \not\equiv 0$, as λ_k is assumed to be simple, $\varphi = s\varphi_k$ for some constant $s \neq 0$. Substituting $\varphi = s\varphi_k$ and $\eta = 0$ into the last equation of (19) we find

$$\int_\Omega \left[\frac{-a_1 b_2\varphi_k}{a_2(1+\mu_{+,k}\varphi_k)^2} + \frac{2b_1\varphi_k}{(1+\mu_{+,k}\varphi_k)^3}\right] = 0,$$

which can be written as

$$\frac{a_1}{a_2}\int_\Omega \frac{\varphi_k}{(1+\mu_{+,k}\varphi_k)^2} = \frac{2b_1}{b_2}\int_\Omega \frac{\varphi_k}{(1+\mu_{+,k}\varphi_k)^3}.$$

By definition of $\mu_{+,k}$,

$$\frac{a_1}{a_2}\int_\Omega \frac{1}{(1+\mu_{+,k}\varphi_k)} = \frac{b_1}{b_2}\int_\Omega \frac{1}{(1+\mu_{+,k}\varphi_k)^2}.$$

Hence we have

$$2\int_\Omega \frac{\varphi_k\,dx}{(1+\mu_{+,k}\varphi_k)^3}\int_\Omega \frac{dx}{1+\mu_{+,k}\varphi_k} = \int_\Omega \frac{dx}{(1+\mu_{+,k}\varphi_k)^2}\int_\Omega \frac{\varphi_k}{(1+\mu_{+,k}\varphi_k)^2}.$$

That is, $g_k'(\mu_{+,k}) = 0$, which is a contradiction.

By using the Fredholm Alternative we can invoke the Implicit Function Theorem to find that there exist some positive constant $\delta > 0$ and $w = w(x;d_2)$ and $\tau = \tau(d_2)$ such that $F(w(x;d_2),\tau(d_2),d_2) = 0$ for $d_2 \in (a_2/\lambda_k - \delta, a_2/\lambda_k)$, where $w(x;a_2/\lambda_k) = b_2(1+\mu_{+,k}\varphi_k)/a_2$ and $\tau(a_2/\lambda_k) = 0$. Since we are only seeking solutions for which $\tau > 0$, we need to determine the sign of $\tau'(a_2/\lambda_k)$. To this end, differentiate (17) with respect to d_2. We have

$$\Delta w + d_2\Delta(\partial w/\partial d_2) + \partial w/\partial d_2(a_2 - c_2\tau w) + w(-c_2\tau' w - c_2\tau\partial w/\partial d_2) = 0 \quad \text{in } \Omega.$$

Set $d_2 = a_2/\lambda_k$, $w = (b_2/a_2)(1+\mu_{+,k}\varphi_k)$, $\tau = 0$ and $w^* = \partial w/\partial d_2|_{d_2=a_2/\lambda_k}$. Then w^* satisfies

$$\tfrac{b_2}{a_2}\mu_{+,k}\Delta\varphi_k + \tfrac{a_2}{\lambda_k}\Delta w^* + a_2 w^* - c_2\tau'(a_2/\lambda_k)[\tfrac{b_2}{a_2}(1+\mu_{+,k}\varphi_k)]^2 = 0 \quad \text{in } \Omega, \quad \tfrac{\partial w^*}{\partial\nu}|_{\partial\Omega} = 0.$$

Multiplying the equation of w^* by φ_k and integrating the result in Ω we find that

$$\tau'(a_2/\lambda_k) = -\frac{a_2\lambda_k \int_\Omega \varphi_k^2}{b_2 c_2 \int_\Omega \varphi_k^2(2+\mu_{+,k}\varphi_k)} < 0.$$

Therefore, choosing δ smaller if necessary, $\tau(d_2) > 0$ for $d_2 \in (a_2/\lambda_k - \delta, a_2/\lambda_k)$. Moreover,

$$\lim_{d_2\to a_2/\lambda_k} \frac{\tau(d_2)}{d_2 - a_2/\lambda_k} = \tau'(a_2/\lambda_k) = -\frac{a_2\lambda_k \int_\Omega \varphi_k^2}{b_2 c_2 \int_\Omega \varphi_k^2(2+\mu_{+,k}\varphi_k)}.$$

This established the second equation of (16). Finally, letting $d_2 \to a_2/\lambda_k$ in (17) and setting $\tilde{w} = w(x; a_2/\lambda_k)$, we see that $\tilde{w}$ satisfies

$$\begin{cases} \frac{a_2}{\lambda_k}\Delta\tilde{w} + a_2\tilde{w} - b_2 = 0 \quad \text{in } \Omega, \quad \tilde{w} \ge 0 \quad \text{in } \Omega, \quad \dfrac{\partial \tilde{w}}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{1}{\tilde{w}}\left(a_1 - \frac{b_1}{\tilde{w}}\right) = 0. \end{cases}$$

Direct calculation yields that $\tilde{w} = (b_2/a_2)(1+\mu_{+,k}\varphi_k)$. Hence, $v(x;d_2)/\tau(d_2) = w(x;d_2) \to (b_2/a_2)(1+\mu_{+,k}\varphi_k)$ as $d_2 \to a_2/\lambda_k$. This together with the second equation of (16) establishes the first equation of (16). □

2.3. **Cross-diffusion system.** In this subsection we prove the existence of non-constant positive steady states of (1) and complete the proofs of parts (a) and (b) of Theorem 1.1. To this end, let (u,v) be a positive steady state of (1). Set $\epsilon = 1/\alpha$ and $w = (\epsilon d_1 + v)u$. Then the steady state problem of system (1) is equivalent to

$$\begin{cases} \Delta w + \dfrac{\epsilon w}{\epsilon d_1 + v}\left(a_1 - \dfrac{b_1 w}{\epsilon d_1 + v} - c_1 v\right) = 0 \quad \text{in } \Omega, \\ d_2\Delta v + v\left(a_2 - \dfrac{b_2 w}{\epsilon d_1 + v} - c_2 v\right) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{20}$$

Define operator $P : L^2(\Omega) \to L^2(\Omega)$ by $P[\varphi] = \varphi - \frac{1}{|\Omega|}\int_\Omega \varphi$. Then (20) can be rewritten as follows:

$$\begin{cases} \Delta w + \epsilon P\left[\dfrac{w}{\epsilon d_1 + v}\left(a_1 - \dfrac{b_1 w}{\epsilon d_1 + v} - c_1 v\right)\right] = 0 \quad \text{in } \Omega, \\ \displaystyle\int_\Omega \frac{w}{\epsilon d_1 + v}\left(a_1 - \frac{b_1 w}{\epsilon d_1 + v} - c_1 v\right) = 0, \\ d_2\Delta v + v\left(a_2 - \dfrac{b_2 w}{\epsilon d_1 + v} - c_2 v\right) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{21}$$

Set $w = \tau_{+,k}(1 + w_1)$ and $v = v_{+,k} + v_1$. Then (21) is equivalent to

$$\begin{cases} \Delta w_1 + \epsilon P\left[\dfrac{1+w_1}{\epsilon d_1 + v_{+,k} + v_1}\left(a_1 - \dfrac{b_1\tau_{+,k}(1+w_1)}{\epsilon d_1 + v_{+,k} + v_1} - c_1(v_{+,k}+v_1)\right)\right] = 0 \text{ in } \Omega, \\ \displaystyle\int_\Omega \frac{1+w_1}{\epsilon d_1 + v_{+,k} + v_1}\left(a_1 - \frac{b_1\tau_{+,k}(1+w_1)}{\epsilon d_1 + v_{+,k} + v_1} - c_1(v_{+,k}+v_1)\right) = 0, \\ d_2\Delta(v_{+,k}+v_1) + (v_{+,k}+v_1)\left(a_2 - \dfrac{b_2\tau_{+,k}(1+w_1)}{\epsilon d_1 + v_{+,k} + v_1} - c_2(v_{+,k}+v_1)\right) = 0 \text{ in } \Omega, \\ \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial v_1}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now we define $F : R \times W^{2,p}_\nu \times W^{2,p}_\nu \to L^p \times R \times L^p$ by

$$\begin{aligned} &F(\epsilon, w_1, v_1) \\ &= \begin{pmatrix} \Delta w_1 + \epsilon P\left[\dfrac{1+w_1}{\epsilon d_1 + v_{+,k} + v_1}\left(a_1 - \dfrac{b_1\tau_{+,k}(1+w_1)}{\epsilon d_1 + v_{+,k} + v_1} - c_1(v_{+,k}+v_1)\right)\right] \\ \displaystyle\int_\Omega \frac{1+w_1}{\epsilon d_1 + v_{+,k} + v_1}\left(a_1 - \frac{b_1\tau_{+,k}(1+w_1)}{\epsilon d_1 + v_{+,k} + v_1} - c_1(v_{+,k}+v_1)\right) \\ d_2\Delta(v_{+,k}+v_1) + (v_{+,k}+v_1)\left(a_2 - \dfrac{b_2\tau_{+,k}(1+w_1)}{\epsilon d_1 + v_{+,k} + v_1} - c_2(v_{+,k}+v_1)\right) \end{pmatrix} \end{aligned}$$

It is easy to check that $F(0,0,0) = (0,0,0)^T$. The Fréchet derivative $D_{(w_1,v_1)}F$ at $(\epsilon, w_1, v_1) = (0,0,0)$ is given by

$$\begin{aligned} &D_{(w_1,v_1)}F\Big|_{(\epsilon,w_1,v_1)=(0,0,0)}\begin{pmatrix}\varphi \\ \psi\end{pmatrix} \\ &= \begin{pmatrix} \Delta\varphi \\ \displaystyle\int_\Omega \frac{\varphi}{v_{+,k}}\left(a_1 - \frac{2b_1\tau_{+,k}}{v_{+,k}} - c_1 v_{+,k}\right) + \int_\Omega \frac{\psi}{v_{+,k}^2}\left(-a_1 + \frac{2b_1\tau_{+,k}}{v_{+,k}}\right) \\ d_2\Delta\psi + \psi(a_2 - 2c_2 v_{+,k}) - b_2\tau_{+,k}\varphi \end{pmatrix} \end{aligned}$$

We claim that if d_2 is sufficiently close to a_2/λ_k, then $D_{(w_1,v_1)}F\big|_{(\epsilon,w_1,v_1)=(0,0,0)}$ has the trivial kernel. If not, by passing to a sequence if necessary, we may suppose that for any d_2 sufficiently close to a_2/λ_k, there exist some $(\varphi,\psi) \neq (0,0)$ such that (φ,ψ) belongs to the kernel of $D_{(w_1,v_1)}F\big|_{(\epsilon,w_1,v_1)=(0,0,0)}$. Since φ satisfies $\Delta\varphi = 0$ in Ω and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$, then $\varphi = \eta$ for some number η. Set $z = \psi/\tau_{+,k}$. Since

$$\int_\Omega \frac{1}{v_{+,k}}\left(a_1 - \frac{b_1\tau_{+,k}}{v_{+,k}}\right) = c_1|\Omega|,$$

then (η, z) satisfies

$$\begin{cases} d_2\Delta z + z(a_2 - 2c_2 v_{+,k}) - b_2\eta = 0 \quad \text{in } \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{z}{v_{+,k}^2}\left(-a_1 + \frac{2b_1\tau_{+,k}}{v_{+,k}}\right) - b_1\eta\int_\Omega \frac{1}{v_{+,k}^2} = 0. \end{cases} \tag{22}$$

As $(\eta, z) \neq (0,0)$, we can always normalize η, z such that $|\eta| + \|z\|_{L^\infty} = 1$. By standard elliptic regularity theory we see that for any $q > 1$, $\|z\|_{W^{2,q}}$ is uniformly bounded for d_2 close to a_2/λ_k. By letting $d_2 \to a_2/\lambda_k$, passing to a subsequence if necessary, we may assume that $z \to z^*$ in $C^1(\bar{\Omega})$ and $\eta \to \eta^*$, where (z^*, η^*) satisfy

$$\begin{cases} \dfrac{a_2}{\lambda_k}\Delta z^* + a_2 z^* - b_2\eta^* = 0 \quad \text{in } \Omega, \\ \dfrac{\partial z^*}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{z^*}{(1+\mu_{+,k}\varphi_k)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,k}\varphi_k)}\right) \\ \displaystyle -\, b_1\eta^* \int_\Omega \frac{1}{(1+\mu_{+,k}\varphi_k)^2} = 0, \end{cases} \tag{23}$$

where we used $v_{+,k} \to 0$ and $v_{+,k}/\tau_{+,k} \to \frac{b_2}{a_2}(1+\mu_{+,k}\varphi_k)$ as $d_2 \to a_2/\lambda_k$.

We first show that $\eta^* \neq 0$. If $\eta^* = 0$, then z^* satisfies $\Delta z^* + \lambda_k z^* = 0$ in Ω and $\partial z^*/\partial\nu = 0$ on $\partial\Omega$. Since λ_k is assumed to be simple, $z^* = s\varphi_k$ for some real number s. If $s \neq 0$, substituting $z^* = s\varphi_k$ and $\eta^* = 0$ into the last equation of (23) we obtain

$$\int_\Omega \frac{\varphi_k}{(1+\mu_{+,k}\varphi_k)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,k}\varphi_k)}\right) = 0.$$

Then we can argue similarly as in the proof of Theorem 2.3 to deduce that $g_k'(\mu_{+,k}) = 0$, which is a contradiction. Hence, $s = 0$, i.e., $z^* \equiv 0$. But this contradicts $|\eta| + \|z\|_{L^\infty} = 1$. Therefore, $\eta^* \neq 0$.

Since λ_k is assumed to be simple and $\eta^* \neq 0$, by the first two equations of (23) we have

$$z^* = \frac{b_2}{a_2}\eta^*(1+\mu^*\varphi_k) \tag{24}$$

for some real number μ^*. We claim that $\mu^* = \mu_{+,k}$. To establish this assertion, substituting (24) into the last equation of (23) and dividing both sides by η^* we obtain

$$\int_\Omega \frac{\frac{b_2}{a_2}(1+\mu^*\varphi_k)}{(1+\mu_{+,k}\varphi_k)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,k}\varphi_k)}\right) - b_1\int_\Omega \frac{1}{(1+\mu_{+,k}\varphi_k)^2} = 0.$$

Define

$$H(\mu) := \int_\Omega \frac{\frac{b_2}{a_2}(1+\mu\varphi_k)}{(1+\mu_{+,k}\varphi_k)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,k}\varphi_k)}\right) - b_1\int_\Omega \frac{1}{(1+\mu_{+,k}\varphi_k)^2}.$$

Since $H(\mu^*) = H(\mu_{+,k}) = 0$ and

$$H'(\mu) = \int_\Omega \frac{\frac{b_2}{a_2}\varphi_k}{(1+\mu_{+,k}\varphi_k)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,k}\varphi_k)}\right) \neq 0,$$

where the last inequality follows from $g_k'(\mu_{+,k}) \neq 0$, we see that $\mu^* = \mu_{+,k}$.

Now multiplying the equation of z by φ_k and integrating the result in Ω, we have

$$\int_\Omega z\varphi_k = 2c_2\int_\Omega z\varphi_k \frac{v_{+,k}}{a_2 - d_2\lambda_k}.$$

By letting $d_2 \to a_2/\lambda_k$ and applying $z \to \frac{b_2}{a_2}\eta^*(1+\mu_{+,k}\varphi_k)$ and (16), dividing both sides by $\eta^* b_2/a_2$ we have

$$\int_\Omega \varphi_k(1+\mu_{+,k}\varphi_k) = 2\int_\Omega \varphi_k(1+\mu_{+,k}\varphi_k)^2 \cdot \frac{\int_\Omega \varphi_k^2}{\int_\Omega \varphi_k^2(2+\mu_{+,k}\varphi_k)}.$$

After further simplifications we obtain

$$\mu_{+,k}\int_\Omega \varphi_k^2 = 0,$$

which is a contradiction as $\mu_{+,k} > 0$. This shows that $D_{(w_1,v_1)}F\big|_{(\epsilon,w_1,v_1)=(0,0,0)}$ has only trivial kernel, provided that d_2 is sufficiently close to a_2/λ_k. Hence, by the Implicit Function Theorem there exists some $\delta > 0$ such that for every $d_2 \in (a_2/\lambda - \delta, a_2/\lambda)$, if α is sufficiently large, then there exists one positive solution, denoted by $(u^*_{+,k}, v^*_{+,k})$, such that

$$\lim_{\alpha\to\infty} u^*_{+,k} = \frac{\tau_{+,k}}{v_{+,k}}, \quad \lim_{\alpha\to\infty} v^*_{+,k} = v_{+,k}$$

uniformly in $\bar\Omega$. The existence of $(u^*_{-,k}, v^*_{-,k})$ can be similarly established. This together with Theorem 2.3 complete the proof of parts (a) and (b) of Theorem 1.1.

3. **Spectral stability of non-constant positive steady states.** The goal of this section is to prove part (c) of Theorem 1.1. We first study the spectral stability of positive steady states of shadow system (2) in Subsection 3.1. The proof of part (c) of Theorem 1.1 is given in Subsection 3.2, where we adopt some ideas from [15].

3.1. **Shadow system.** We are interested in the spectral stability of $(v_{\pm,1}, \tau_{\pm,1})$, positive solutions of the shadow system (2), which were constructed in previous section. For simplicity we focus on $(v_{+,1}, \tau_{+,1})$. The stability of $(v_{+,1}, \tau_{+,1})$ is determined by

$$\begin{cases} d_2\Delta\psi + (a_2 - 2c_2 v_{+,1})\psi - b_2\eta = \sigma\psi \quad \text{in } \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \\ \tau_{+,1}\displaystyle\int_\Omega \frac{\psi}{v_{+,1}^2}\left(-a_1 + \frac{2b_1\tau_{+,1}}{v_{+,1}}\right) - b_1\tau_{+,1}\eta\int_\Omega \frac{1}{v_{+,1}^2} \\ = \sigma\eta\displaystyle\int_\Omega \frac{1}{v_{+,1}} - \sigma\tau_{+,1}\int_\Omega \frac{\psi}{v_{+,1}^2}. \end{cases} \tag{25}$$

Theorem 3.1. *If d_2 is sufficiently close to a_2/λ_1, the real part of any eigenvalue of (25) is strictly negative.*

Proof. We argue by contradiction. Suppose that there exists a sequence of $\{d_{2,j}\}$ with $d_{2,j} < a_2/\lambda_1$ for every j and $\lim_{j\to\infty} d_{2,j} = a_2/\lambda_1$ such that the eigenvalue problem (25) with $d_2 = d_{2,j}$ has an eigenvalue σ_j which has non-negative real part for every $j \geq 1$. We normalize corresponding functions ψ_j and η_j so that $|\eta_j| + \|\psi_j\|_{L^2} = 1$.

Step 1. We show that $\{|\sigma_j|\}$ is a bounded sequence. Write $\sigma_j = \sigma_{1,j} + i\sigma_{2,j}$, $\psi_j = \psi_{1,j} + i\psi_{2,j}$, $\eta_j = \eta_{1,j} + i\eta_{2,j}$, where $\sigma_{1,j}, \sigma_{2,j}, \eta_{1,j}, \eta_{2,j}$ are real numbers, and $\psi_{1,j}, \psi_{2,j}$ are real valued functions. Since the real part of σ_j is non-negative, we have $\sigma_{1,j} \geq 0$. Then

$$\begin{cases} d_{2,j}\Delta\psi_{1,j} + a(x)\psi_{1,j} - b_2\eta_{1,j} = \sigma_{1,j}\psi_{1,j} - \sigma_{2,j}\psi_{2,j} \quad \text{in } \Omega, \\ d_{2,j}\Delta\psi_{2,j} + a(x)\psi_{2,j} - b_2\eta_{2,j} = \sigma_{1,j}\psi_{2,j} + \sigma_{2,j}\psi_{1,j} \quad \text{in } \Omega, \\ \dfrac{\partial\psi_{1,j}}{\partial\nu} = \dfrac{\partial\psi_{2,j}}{\partial\nu} \quad \text{on } \partial\Omega, \end{cases}$$

where $a(x) = a_2 - 2c_2 v_{+,1}$. We claim that if $|\sigma_j| \to \infty$ as $j \to \infty$, then $\|\psi_j\|_{L^2} \to 0$. To establish our assertion, we consider two cases:

Case 1. $\sigma_{1,j} \to +\infty$ as $j \to \infty$. Multiplying the equation of $\psi_{1,j}$ by $\psi_{1,j}$ and the equation of $\psi_{2,j}$ by $\psi_{2,j}$, adding two equations and integrating the result in Ω, we obtain

$$\begin{aligned} d_{2,j} \int_\Omega (|\nabla \psi_{1,j}|^2 + |\nabla \psi_{2,j}|^2) + \sigma_{1,j} \int_\Omega (\psi_{1,j}^2 + \psi_{2,j}^2) \\ = \int_\Omega \psi_{1,j}(a\psi_{1,j} - b_2\eta_{1,j}) + \int_\Omega \psi_{2,j}(a\psi_{2,j} - b_2\eta_{2,j}). \end{aligned}$$

If $\sigma_{1,j} \to +\infty$, by Cauchy-Schwartz inequality we have $\int_\Omega (\psi_{1,j}^2 + \psi_{2,j}^2) \to 0$.

Case 2. $|\sigma_{2,j}| \to \infty$ as $j \to \infty$. For this case, multiplying the equation of $\psi_{1,j}$ by $\psi_{2,j}$ and the equation of $\psi_{2,j}$ by $\psi_{1,j}$, subtracting the result and integrating it in Ω, we obtain

$$\sigma_{2,j} \int_\Omega (\psi_{1,j}^2 + \psi_{2,j}^2) = -\int_\Omega \psi_{2,j}(a\psi_{1,j} - b_2\eta_{1,j}) + \int_\Omega \psi_{1,j}(a\psi_{2,j} - b_2\eta_{2,j}).$$

Again by Cauchy-Schwartz inequality we see that if $|\sigma_{2,j}| \to \infty$, then $\int_\Omega (\psi_{1,j}^2 + \psi_{2,j}^2) \to 0$.

Therefore, if $|\sigma_j| \to \infty$ as $j \to \infty$, we have $\|\psi_j\|_{L^2} \to 0$. Since $|\eta_j| + \|\psi_j\|_{L^2} = 1$, we have $|\eta_j| \to 1$ as $j \to \infty$.

By the last equation of (25) we have

$$\sigma_j = \frac{\int_\Omega \frac{\psi_j}{(v_{+,1}/\tau_{+,1})^2} \left(-a_1 + \frac{2b_1}{v_{+,1}/\tau_{+,1}} \right) - b_1 \eta_j \int_\Omega \frac{1}{(v_{+,1}/\tau_{+,1})^2}}{\eta_j \int_\Omega \frac{1}{v_{+,1}/\tau_{+,1}} - \int_\Omega \frac{\psi_j}{(v_{+,1}/\tau_{+,1})^2}}.$$

Since $\|\psi_j\|_{L^2} \to 0$, $|\eta_j| \to 1$ and $v_{+,1}/\tau_{+,1} \to (b_2/a_2)(1 + \mu_{+,1}\varphi_1)$ as $j \to \infty$, by passing to the limit in the above equation we see that

$$\lim_{j\to\infty} \sigma_j = -\frac{a_2 b_1 \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}}{b_2 \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)}} < 0,$$

which is a contradiction since we assume that the real part of σ_j is non-negative. This proves that $\{\sigma_j\}$ is a bounded sequence.

Step 2. We claim that as $j \to \infty$, $\sigma_j \to 0$ and $(\eta_j, \psi_j) \to (1, \frac{b_2}{a_2}(1 + \mu_{+,1}\varphi_1))$ after suitable rescaling.

By elliptic regularity theory, $\|\psi_j\|_{W^{2,2}}$ is uniformly bounded. By Sobolev embedding theorem, passing to a subsequence if necessary, we may assume that $\psi_j \to \psi^*$ in $W^{1,2}$, $\eta_j \to \eta^*$, and $\sigma_j \to \sigma^*$ as $j \to \infty$, where (η^*, ψ^*) and σ^* satisfy

$$\begin{cases} \dfrac{a_2}{\lambda_1} \Delta \psi^* + a_2 \psi^* - b_2 \eta^* = \sigma^* \psi^* \quad \text{in } \Omega, \\ \dfrac{\partial \psi^*}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega \frac{\psi^*}{(1 + \mu_{+,1}\varphi_1)^2} \left(-a_1 + \frac{2b_1}{(b_2/a_2)(1 + \mu_{+,1}\varphi_1)} \right) - b_1 \eta^* \int_\Omega \frac{1}{(1 + \mu_{+,1}\varphi_1)^2} \\ \displaystyle = \frac{b_2}{a_2} \sigma^* \eta^* \int_\Omega \frac{1}{1 + \mu_{+,1}\varphi_1} - \sigma^* \int_\Omega \frac{\psi^*}{(1 + \mu_{+,1}\varphi_1)^2}. \end{cases} \tag{26}$$

We first show that $\sigma^* = 0$. Since the real part of σ_j is non-negative, we see that the real part of σ^* is also non-negative. Since $|\eta_j| + \|\psi_j\|_{L^2} = 1$ we have $|\eta^*| + \|\psi^*\|_{L^2} = 1$. In the following we consider two cases:

Case 1. $\eta^* = 0$. For this case, $\psi^* \neq 0$ and it satisfies

$$-\Delta\psi^* = \lambda_1 \frac{a_2 - \sigma^*}{a_2}\psi^* \quad \text{in } \Omega, \quad \frac{\partial\psi^*}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

If $\sigma^* \neq 0$, then σ^* must be real and positive. Hence, $\lambda_1 \frac{a_2-\sigma^*}{a_2} < \lambda_1$. Since $\psi^* \neq 0$, the only possibility is that $\sigma^* = a_2$ and ψ^* is a non-zero constant. By the last equation of (26) and $\eta^* = 0$ we have

$$\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,1}\varphi_1)}\right) = -a_2 \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2} < 0,$$

which implies that

$$\frac{a_1}{a_2}\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2} > \frac{2b_1}{b_2}\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^3}.$$

By the definition of $\mu_{+,1}$, we have

$$\frac{a_1}{a_2}\int_\Omega \frac{1}{1+\mu_{+,1}\varphi_1} = \frac{b_1}{b_2}\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}.$$

Therefore, we obtain

$$\left(\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}\right)^2 > 2\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^3}\int_\Omega \frac{1}{1+\mu_{+,1}\varphi_1},$$

which is a contradiction to Cauchy-Schwartz inequality. This proves that if $\eta^* = 0$, then $\sigma^* = 0$.

Case 2. $\eta^* \neq 0$. Integrating the equation of ψ^* in Ω we have

$$(a_2 - \sigma^*)\int_\Omega \psi^* = b_2|\Omega|\eta^* \neq 0.$$

Therefore, $\sigma^* \neq a_2$. Hence, the equation of ψ^* can be written as

$$-\Delta\psi^* = \lambda_1 \frac{a_2 - \sigma^*}{a_2}\left(\psi^* - \frac{b_2\eta^*}{a_2 - \sigma^*}\right) \quad \text{in } \Omega, \quad \frac{\partial\psi^*}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

Suppose that $\sigma^* \neq 0$. We claim that the only possibility is $\psi^* = b_2\eta^*/(a_2 - \sigma^*)$. If not, then $\lambda_1 \frac{a_2-\sigma^*}{a_2}$ is an eigenvalue of the Laplace operator with zero Neumann boundary condition. As $\sigma^* \neq 0$ and $Re(\sigma^*) \geq 0$, σ^* must be real and strictly positive. This together with $\sigma^* \neq a_2$ implies that $\lambda_1 \frac{a_2-\sigma^*}{a_2} < \lambda_1$ and $\lambda_1 \frac{a_2-\sigma^*}{a_2} \neq 0$, which is a contradiction. Therefore, $\psi^* = b_2\eta^*/(a_2 - \sigma^*)$. By the last equation of (26) and $\eta^* \neq 0$, we have

$$\begin{aligned}&\int_\Omega \frac{b_2/(a_2-\sigma^*)}{(1+\mu_{+,1}\varphi_1)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,1}\varphi_1)}\right) - b_1\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}\\ &= \sigma^*\left[\int_\Omega \frac{1}{(a_2/b_2)(1+\mu_{+,1}\varphi_1)} - \int_\Omega \frac{b_2/(a_2-\sigma^*)}{(1+\mu_{+,1}\varphi_1)^2}\right],\end{aligned}$$

which can be reduced to a quadratic equation of σ^* of the form

$$C(\sigma^*)^2 + \sigma^*(B - a_2C + b_2D) + b_2A - a_2B = 0, \tag{27}$$

where

$$
\begin{aligned}
A &= \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,1}\varphi_1)}\right),\\
B &= b_1 \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2},\\
C &= \int_\Omega \frac{1}{(a_2/b_2)(1+\mu_{+,1}\varphi_1)},\\
D &= \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}.
\end{aligned}
$$

Claim. $b_2 A > a_2 B$.

To prove this assertion, note that

$$
\begin{aligned}
\frac{b_2}{a_2}A - B = &-\frac{a_1 b_2}{a_2}\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2} + 2b_1 \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^3}\\
&- b_1 \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}.
\end{aligned}
$$

Rewrite $g_1(\mu_{+,1}) = (a_1/a_2)/(b_1/b_2)$ *as*

$$
\frac{a_1 b_2}{a_2} = b_1 \cdot \frac{\int_\Omega (1+\mu_{+,1}\varphi_1)^{-2}}{\int_\Omega (1+\mu_{+,1}\varphi_1)^{-1}}.
$$

Hence,

$$
\begin{aligned}
\frac{b_2}{a_2}A - B = \frac{b_1}{\int_\Omega (1+\mu_{+,1}\varphi_1)^{-1}}\Big[&-\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}\int_\Omega \frac{1}{1+\mu_{+,1}\varphi_1}\\
&+ 2\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^3}\int_\Omega \frac{1}{1+\mu_{+,1}\varphi_1} - \left(\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}\right)^2 \Big].
\end{aligned}
$$

Following the proof of Lemma 2.1 we find that $b_2 A > a_2 B$.

Note that

$$
\frac{a_2}{b_2}C - D = \int_\Omega \frac{1}{1+\mu_{+,1}\varphi_1} - \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2} < 0,
$$

where the last inequality follows from Lemma 2.1. Since $B > 0$, from (27) we see that $Re(\sigma^*) < 0$, which is contradiction since the real part of σ_j is non-negative. This contradiction shows that if $\eta^* \neq 0$, then $\sigma^* = 0$.

In conclusion, we always have $\sigma^* = 0$. Hence, (26) can be written as

$$
\begin{cases}
\dfrac{a_2}{\lambda_1}\Delta\psi^* + a_2\psi^* - b_2\eta^* = 0 \quad \text{in } \Omega,\\
\dfrac{\partial \psi^*}{\partial \nu} = 0 \quad \text{on } \partial\Omega,\\
\displaystyle\int_\Omega \frac{\psi^*}{(1+\mu_{+,1}\varphi_1)^2}\left(-a_1 + \frac{2b_1}{(b_2/a_2)(1+\mu_{+,1}\varphi_1)}\right)\\
\displaystyle\quad - b_1\eta^* \int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2} = 0.
\end{cases}
\tag{28}
$$

We next show that $\eta^* \neq 0$. To this end we argue by contradiction. Suppose that $\eta^* = 0$. As $(\eta^*, \psi^*) \neq (0,0)$, we see that $\psi^* = s\varphi_1$ for some $s \neq 0$. Substituting this into (28) we have

$$\int_\Omega \frac{\varphi_1}{(1+\mu_{+,1}\varphi_1)^2}\left(-a_1+\frac{2b_1}{(b_2/a_2)(1+\mu_{+,1}\varphi_1)}\right)=0,$$

which implies that $g_1'(\mu_{+,1})=0$. This contradiction implies that $\eta^*\neq 0$. Therefore

$$\psi^*=\frac{b_2}{a_2}\eta^*(1+\mu\varphi_1)$$

for some real number μ. Substituting this into (28) and dividing both sides by η^* we have

$$\int_\Omega \frac{\frac{b_2}{a_2}(1+\mu\varphi_1)}{(1+\mu_{+,1}\varphi_1)^2}\left(-a_1+\frac{2b_1}{(b_2/a_2)(1+\mu_{+,1}\varphi_1)}\right)-b_1\int_\Omega \frac{1}{(1+\mu_{+,1}\varphi_1)^2}=0.$$

By using exactly the same argument as in Subsection 2.3, we see that $\mu=\mu_{+,1}$. This completes the proof for Step 2.

Step 3. Multiplying the equation of ψ_j by φ_1 and integrating the result in Ω, after some rearrangement of terms we have

$$\int_\Omega \psi_j\varphi_1-2c_2\int_\Omega \psi_j\varphi_1\frac{v_{+,1}}{a_2-d_{2,j}\lambda_1}=\frac{\sigma_j}{a_2-d_{2,j}\lambda_1}\int_\Omega \psi_j\varphi_1. \tag{29}$$

By Step 2, $\psi_j\to(b_2/a_2)(1+\mu_{+,1}\varphi_1)$ as $j\to\infty$. Also recall that

$$\lim_{j\to\infty}\frac{v_{+,1}}{a_2-d_{2,j}\lambda_1}=\frac{1}{c_2}(1+\mu_{+,1}\varphi_1)\frac{\int_\Omega\varphi_1^2}{\int_\Omega\varphi_1^2(2+\mu_{+,1}\varphi_1)}.$$

Passing to the limit in (29) we obtain

$$\lim_{j\to\infty}\frac{\sigma_j}{a_2-d_{2,j}\lambda_1}=-1.$$

However, this contradicts our assumption that the real part of σ_j is non-negative. This completes the proof of Theorem 3.1. □

3.2. **Cross-diffusion system.** To study the dynamics of (1), we set $w=u(d_1/\alpha+v)$. Then (w,v) satisfies

$$\begin{cases}\left(\dfrac{w}{d_1/\alpha+v}\right)_t=\alpha\Delta w+\dfrac{w}{d_1/\alpha+v}\left(a_1-\dfrac{b_1w}{d_1/\alpha+v}-c_1v\right) & \text{in } \Omega^+,\\ v_t=d_2\Delta v+v\left(a_2-\dfrac{b_2w}{d_1/\alpha+v}-c_2v\right) & \text{in } \Omega^+,\\ \dfrac{\partial w}{\partial\nu}=\dfrac{\partial v}{\partial\nu}=0 & \text{on } \partial\Omega\times(0,\infty),\end{cases} \tag{30}$$

where $\Omega^+=\Omega\times(0,\infty)$.

Recall that $(u^*_{\pm,1},v^*_{\pm,1})$ are positive solutions of (1). Set $w^*_{\pm,1}:=u^*_{\pm,1}(d_1/\alpha+v^*_{\pm,1})$. We are interested in the spectral stability of $(w^*_{\pm,1},v^*_{\pm,1})$, positive solutions of system (30). For simplicity we only consider the spectral stability of $(w^*_{+,1},v^*_{+,1})$. The stability of $(w^*_{+,1},v^*_{+,1})$ is determined by

$$\begin{cases}\dfrac{\sigma\phi}{d_1/\alpha+v^*_{+,1}}-\dfrac{\sigma w^*_{+,1}\psi}{(d_1/\alpha+v^*_{+,1})^2}=\alpha\Delta\phi+a_{11}\phi+a_{12}\psi & \text{in } \Omega,\\ \sigma\psi=d_2\Delta\psi+a_{21}\phi+a_{22}\psi & \text{in } \Omega,\\ \dfrac{\partial\phi}{\partial\nu}=\dfrac{\partial\psi}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases} \tag{31}$$

where the coefficients a_{ij} are given by

$$a_{11} = \frac{1}{d_1/\alpha + v^*_{+,1}} \left(a_1 - \frac{2b_1 w^*_{+,1}}{d_1/\alpha + v^*_{+,1}} - c_1 v^*_{+,1} \right),$$

$$a_{12} = -\frac{w^*_{+,1}}{(d_1/\alpha + v^*_{+,1})^2} \left(a_1 - \frac{2b_1 w^*_{+,1}}{d_1/\alpha + v^*_{+,1}} + \frac{c_1 d_1}{\alpha} \right),$$

$$a_{21} = -\frac{b_2 v^*_{+,1}}{d_1/\alpha + v^*_{+,1}},$$

$$a_{22} = a_2 - 2c_2 v^*_{+,1} - \frac{d_1 b_2}{\alpha} \frac{w^*_{+,1}}{(d_1/\alpha + v^*_{+,1})^2}.$$

Proof of part (c) *of Theorem* 1. It suffices to show that the real part of any eigenvalue of (31) must be strictly negative. We argue by contradiction. Suppose that for every $d_2 \in (a_2/\lambda_1 - \delta, a_2/\lambda_1)$, there exists a sequence $\{\alpha_j\}$ with $\lim_{j\to\infty} \alpha_j = +\infty$ such that the eigenvalue problem (31) with $\alpha = \alpha_j$ has an eigenvalue σ_j which has non-negative real part for every $j \geq 1$. We normalize corresponding eigenfunction (ϕ_j, ψ_j) so that $\|\phi_j\|_{L^2} + \|\psi_j\|_{L^2} = 1$.

Step 1. We show that the eigenvalues σ_j are uniformly bounded. We argue by contradiction. Suppose that σ_j is unbounded and passing to a subsequence if necessary we may assume that $|\sigma_j| \to \infty$ as $j \to \infty$. Multiplying the equation for ϕ_j by $\bar{\phi}_j$ and integrating the results in Ω we have

$$\alpha_j \int_\Omega |\nabla \phi_j|^2 + Re(\sigma_j) \int_\Omega |\phi_j|^2 \leq C_1 \int_\Omega |\phi_j|^2 + C_2(|\sigma_j| + 1) \int_\Omega |\psi_j|^2,$$

where $Re(\sigma_j)$ denotes the real part of σ_j, and C_1, C_2 are positive constants independent of j. Similarly, we also have

$$|Im(\sigma_j)| \int_\Omega |\phi_j|^2 \leq C_3 \int_\Omega |\phi_j|^2 + C_4(|\sigma_j| + 1) \int_\Omega |\psi_j|^2,$$

where $Im(\sigma_j)$ denotes the imaginary part of σ_j, and C_3, C_4 are positive constants independent of j. As $\alpha_j \geq 0$, combining the above two inequalities we have

$$[Re(\sigma_j) + |Im(\sigma_j)|] \int_\Omega |\phi_j|^2 \leq (C_1 + C_3) \int_\Omega |\phi_j|^2 + (C_2 + C_4)(|\sigma_j| + 1) \int_\Omega |\psi_j|^2.$$

As we assume that $Re(\sigma_j) \geq 0$ and $|\sigma_j| \to \infty$ as $j \to \infty$, we have

$$\int_\Omega |\phi_j|^2 \leq C_5 \int_\Omega |\psi_j|^2 \tag{32}$$

for all sufficiently large j, where C_5 is some positive constant independent of j.

Multiplying the equation for ψ_j by $\bar{\psi}_j$ and integrating the result in Ω we can similarly obtain

$$[Re(\sigma_j) + |Im(\sigma_j)|] \int_\Omega |\psi_j|^2 \leq C_6 \int_\Omega |\phi_j|^2 + C_7 \int_\Omega |\psi_j|^2.$$

As we assume that $Re(\sigma_j) \geq 0$ and $|\sigma_j| \to \infty$ as $j \to \infty$, we have

$$\int_\Omega |\psi_j|^2 \leq \frac{1}{2C_5} \int_\Omega |\phi_j|^2 \tag{33}$$

for all sufficiently large j. By (32) and (33) we obtain $\phi_j = \psi_j = 0$ for sufficiently large j, but this contradicts $\|\phi_j\|_{L^2} + \|\psi_j\|_{L^2} = 1$ for all j. This contradiction shows that $\{\sigma_j\}$ is a bounded sequence.

Step 2. Since σ_j is bounded, by elliptic regularity theory we see that $\|\phi_j\|_{W^{2,2}}$ and $\|\psi_j\|_{W^{2,2}}$ are uniformly bounded. By Sobolev embedding theorem and passing to a subsequence if necessary we may assume that $\phi_j \to \phi$ and $\psi_j \to \psi$ weakly in $W^{2,2}$ and strongly in $W^{1,2}$. In particular, as $\alpha_j \to \infty$, $\Delta\phi = 0$ in Ω and $\partial\phi/\partial\nu = 0$ on $\partial\Omega$. Therefore, $\phi = \tau$ for some constant τ. Recall that as $\alpha_j \to \infty$, $w^*_{+,1} \to \tau_{+,1}$ and $v^*_{+,1} \to v_{+,1}$. Therefore, τ and ψ satisfy $|\tau| + \|\psi\|_{L^2} = 1$ and (25). Hence, (25) has an eigenvalue with non-negative real part, which contradicts Theorem 3.1. This completes the proof of (c) of Theorem 1.1. □

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REFERENCES

[1] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, Series in Mathematical and Computational Biology, John Wiley and Sons, Chichester, UK, 2003.

[2] Y. S. Choi, R. Lui and Y. Yamada, Existence of global solutions for the Shigesada-Kawasaki-Teramoto model with strongly coupled cross-diffusion, *Discrete Contin. Dyn. Syst.*, **10** (2004), 719–730.

[3] K. Kishimoto and H. F. Weinberger, The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains, *J. Differential Equations*, **58** (1985), 15–21.

[4] K. Kuto and Y. Yamada, On limit systems for some population models with cross-diffusion, *Discrete Contin. Dyn. Syst.-Series B*, **17** (2012), 2745–2769.

[5] M. Iida, M. Mimura and H. Ninomiya, Diffusion, cross-diffusion and competitive interaction, *J. Math. Biol.*, **53** (2006), 617–641.

[6] Y. Lou and W.-M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*, **131** (1996), 79–131.

[7] Y. Lou and W. M. Ni, Diffusion vs cross-diffusion: An elliptic approach, *J. Differential Equations*, **154** (1999), 157–190.

[8] Y. Lou, W.-M. Ni and Y. Wu, On the global existence of a cross-diffusion system, *Discrete Contin. Dyn. Syst.,* **4** (1998), 193–203.

[9] Y. Lou, W.-M. Ni and S. Yotsutani, On a limiting system in the Lotka-Volterra competition with cross-diffusion, *Discrete Contin. Dyn. Syst.*, **10** (2004), 435–458.

[10] H. Matano and M. Mimura, Pattern formation in competition-diffusion systems in nonconvex domains, *Publ. RIMS. Kyoto Univ.*, **19** (1983), 1049–1079.

[11] M. Mimura, Stationary pattern of some density-dependent diffusion system with competitive dynamics, *Hiroshima Math. J.*, **11** (1981), 621–635.

[12] M. Mimura, Y. Nishiura, A. Tesei and T. Tsujikawa, Coexistence problem for two competing species models with density-dependent diffusion, *Hiroshima Math. J.*, **14** (1984), 425–449.

[13] W. M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices Amer. Math. Soc.*, **45** (1998), 9–18.

[14] W. M. Ni, Qualitative properties of solutions to elliptic problems, Stationary partial differential equations. *Handb. Differ. Equ., North-Holland, Amsterdam,* **I** (2004), 157–233.

[15] W. M. Ni, Y. Wu and Q. Xu, The existence and stability of nontrivial steady states for S-K-T competition model with cross-diffusion, *Discrete Contin. Dyn. Syst.*, **34** (2014), 5271–5298.

[16] A. Okubo and S. A. Levin, *Diffusion and Ecological Problems: Modern Perspectives,* Interdisciplinary Applied Mathematics, Vol. 14, 2nd ed. Springer, Berlin, 2001.

[17] N. Shigesada, K. Kawasaki and E. Teramoto, Spatial segregation of interacting species, *J. Theor. Biol.*, **79** (1979), 83–99.

[18] Y. Wu, Existence of stationary solutions with transition layers for a class of cross-diffusion systems, *Proc. of Royal Soc. Edinburg, Sect. A,* **132** (2002), 1493–1511.

[19] Y. Wu, The instability of spiky steady states for a competing species model with cross-diffusion, *J. Differential Equations*, **213** (2005), 289–340.

[20] Y. Wu and Q. Xu, The Existence and structure of large spiky steady states for S-K-T competition system with cross-diffusion, *Discrete Contin. Dyn. Syst.,* **29** (2011), 367–385.

[21] Y. Wu and Y. Zhao, The existence and stability of traveling waves with transition layers for the S-K-T competition model with cross-diffusion, *Science in China*, **53** (2010), 1161–1184.

[22] Y. Yamada, Positive solutions for Lotka-Volterra systems with cross-diffusion, Handbook of Differential Equations, *Stationary Partial Differential Equations,* Edited by M. Chipot, Elsevier, Amsterdam, **6** (2008), 411–501.

[23] Y. Yamada, Global solutions for the Shigesada-Kawasaki-Teramoto model with cross-diffusion, *Recent Progress on Reaction-Diffusion Systems and Viscosity Solutions,* 282–299, World Sci. Publ. Hackensack, NJ, 2009.

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E-mail address: lou@math.ohio-state.edu
E-mail address: ni@math.umn.edu
E-mail address: shoji@math.ryukoku.ac.jp