# Global Dynamics of a Lotka-Volterra Competition-Diffusion-Advection System in Heterogeneous Environments

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**Abstract:** We study a Lotka-Volterra type reaction-diffusion-advection system, which describes the competition for the same resources between two aquatic species undergoing different dispersal strategies, as reflected by their diffusion and/or advection rates. For the non-advective case, this problem was solved by Dockery et al. [9], and recently He and Ni [14] provided a further classification on the global dynamics for a more general model. However, the key ideas developed in [9, 14] do not appear to work when advection terms are involved. By assuming the resource function is decreasing in the spatial variable, we establish the non-existence of coexistence steady state and perform sufficient analysis on the local stability of two semi-trivial steady states, where new techniques were introduced to overcome the difficulty caused by the non-analyticity of stationary solutions as well as the diffusion-advection type operators. Combining these two aspects with the theory of monotone dynamical systems, we finally obtain the global dynamics, which suggests that the competitive exclusion principle holds in most situations.

**Résumé:** Nous étudions un systéme de réaction-diffusion-advection de type Lotka-Volterra, qui décrit la compétition pour les mêmes ressources entre deux espéces aquatiques ayant différentes stratégies de dispersion,

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reflétées par leurs taux de diffusion et/ou d'advection. Le cas sans advection a été traité par Dockery et al. [9], et récemment He et Ni [14] ont également donné une classification de la dynamique globale pour un modéle plus général. Cependant, les idées clés développées dans [9, 14] ne semblent pas fonctionner en présence de termes d'advection. En supposant que les ressources sont décroissantes par rapport á la variable d'espace, nous montrons la non-existence d'état stationnaire avec co-existence des espéces et analysons la stabilité locale de deux états stationnaires semi-triviaux. De nouvelles techniques sont introduites pour contourner la difficulté créée par la non-analyticité des solutions stationnaires et par des opérateurs de type diffusion-advection. En combinant ces deux aspects avec la théorie des systémes dynamiques monotones, nous obtenons finalement la dynamique globale, qui suggére que le principe d'exclusion par compétition se produit dans la plupart des situations.

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# 1 Introduction

In the past few decades, reaction-diffusion equations have been frequently utilized as standard models to address problems related to spatial ecology and evolution, and one of the most successful examples is the two-species Lotka-Volterra competition-diffusion system; see for instance [12, 13, 15, 16, 24, 25] and the books [2, 31].

One well-known and widely accepted result on the evolution of dispersal, based on a reaction-diffusion system, is due to Hastings [11], where he considered two competing species with the only difference lying in their random diffusion rates, and showed that the species can invade successfully when rare if and only if it is the slower diffuser provided that the environment is spatially heterogeneous but temporally constant; see also Dockery et al. [9]. A recent remarkable work, in this research direction, was given by He and Ni [14], where they further investigated a more general model in the sense that two populations are allowed to behave differently in their dispersal rates, growth rates and competition abilities, and they successfully established a complete classification on all possible long time dynamical behaviors. This success, to a large extent, is attributed to a key a priori estimate on the co-existence steady state, which says that every co-existence steady state, if exists, is linearly stable. Such a finding is very powerful, since it, together with the theory of abstract competitive systems [18, 19], enables one to conclude that the global dynamics could be determined by the local dynamics.

However, when advection terms are incorporated in these classical Lotka-Volterra competition-diffusion systems, the global dynamics of the resulting reaction-diffusion-advection systems is far from being completely understood. Some recent progress has been made by Averill et. al [1] via a bifurcation approach. The main reason, from the viewpoint of the operator theory, is that the linearized operator of the diffusion-advection type is *no longer self-adjoint*, and hence, various methods developed in the field of diffusive Lotka-Volterra models (without drifting terms) do not continue to work well; for instance, the asymptotic behavior of the principal eigenvalue of diffusion-advection operators for small diffusion coefficient can be very different [6]. In this connection, our current work may be partially viewed as a further exploration and application of the spectral theory for non-self-adjoint operators.

In this paper, we aim to investigate the dynamical behaviors of the following two species Lotka-Volterra reaction-diffusion-advection system:

$$\begin{cases} u_t = d_1 u_{xx} - \alpha_1 u_x + u[r(x) - u - v], & 0 < x < L, t > 0, \\ v_t = d_2 v_{xx} - \alpha_2 v_x + v[r(x) - u - v], & 0 < x < L, t > 0, \\ d_1 u_x(x,t) - \alpha_1 u(x,t) = 0, & x = 0, L, t > 0, \\ d_2 v_x(x,t) - \alpha_2 v(x,t) = 0, & x = 0, L, t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & 0 < x < L, \\ v(x,0) = v_0(x) \geq, \not\equiv 0, & 0 < x < L. \end{cases} \tag{1.1}$$

Problem (1.1) can be used to describe the competition between two aquatic species, whose population density are denoted, respectively, by $u(x,t)$ and $v(x,t)$ at location $x$ and time $t > 0$, in advective environments such as a stream/river with unidirectional water flow, which is abstracted into a one-dimensional habitat and is represented by the interval $(0, L)$. Due to self-propelling and/or water turbulence, both populations are subject to diffusive movements with rate $d_i > 0$, and meanwhile, under the effect of unidirectional water flow, they are also experiencing some passive/directed movements towards the downstream end (defined by $x = L$) with rate $\alpha_i > 0$, $i = 1, 2$. The function $r(x)$ accounts for the local carrying capacity or the intrinsic growth rate at location $x$, which also reflects the spatial distribution of resources and is usually called the resource function. The no-flux boundary conditions imposed above indicate that both boundaries act as barriers and no individuals can move in or out through the habitat ends, that is, the environment is closed. We note here that the unidirectional water flow may produce different extent of drifting effects on different species, and so, in general, the resulting effective advection speeds may be different, i.e., $\alpha_1 \neq \alpha_2$. An empirical support for this comes from Trimbee and Harris [35], where they designed an experiment in a small reservoir, Guelph Lake, Ontario, which lasted 105 days, and finally they detected that *Stephanodiscus* has an advection rate of 0.4 cm/s, while *Aphanizomenon* has an advection rate of 0.2 cm/s, which illustrates that different algal species may undergo different advection speeds although they are living in the same river. For the derivation details, based on a random-walk approach, of model (1.1) and its variants, we refer the interested readers to [29, 30].

We remark here that system (1.1) above can also be applied to model the vertical motion of nutrients or plankton species in a water column [27, 37], where as most of these organisms are heavier than the water, under the action of gravity, they will sink to the ocean floor. In such a situation, $x = 0$ and $x = L$ represent the water surface and bottom, respectively, and also the no-flux boundary conditions make sense since no individuals can cross over the water surface and bottom.

Mathematically, system (1.1) can be studied in the following two cases

$$d_1 \neq d_2 \quad \text{and} \quad \alpha_1, \alpha_2 > 0, \tag{1.2}$$

and

$$d_1 = d_2, \quad \alpha_1, \alpha_2 > 0, \quad \text{and} \quad \alpha_1 \neq \alpha_2. \tag{1.3}$$

Case (1.2), biologically, reflects a more reasonable and flexible situation since two species are allowed to have different diffusion and advection rates, and case (1.3) describes a special scenario with the only difference between two competitors lying in their advection rates. This special case will enable us to explore whether strong or weak advection is more beneficial for individuals to win the competition. Without loss of generality, we may assume $0 < d_1 < d_2$ in the general case (1.2) and $0 < \alpha_1 < \alpha_2$ in the special case (1.3). Moreover, throughout this paper, we make the following hypothesis:

(H) $r(x) \in C^{1+\nu}([0, L])$ is non-constant and positive on $[0, L]$ for some $\nu \in (0, 1)$, and $r'(x) \leq, \not\equiv 0$ in $[0, L]$.

The positivity of $r(x)$ is used to guarantee the existence of semi-trivial steady states for later discussion convenience. The decreasing monotonicity of $r(x)$ comes from the biological motivation. Specifically, in the context of river ecology, it means that better resources are located along upstream direction, while in the context of water column, this assumption seems to have more compelling biological interpretations as plankton species depend on light for their metabolism which clearly decreases with depth.

To understand the global dynamics of system (1.1), in view of the monotone dynamical system theory (see, e.g., [17, 32]), we need to figure out two things: (1) the local stability of two semi-trivial steady states, and (2) whether there are co-existence steady states; if so, whether they are all stable (and thus unique). Usually, the second issue is highly challenging. The work of He and Ni [14] made an important contribution to this issue by establishing the stability of all possible co-existence steady states for a class of Lotka-Volterra competition systems without advection, but the techniques developed therein rely heavily on the self-adjoint property of the diffusion-type operator and do not appear to work for the current advective case. Moreover, the advection terms also bring new difficulty to studying the linear stability of semi-trivial steady states, since the current diffusion-advection type operator does not possess same good properties as the diffusion-type operators [9, 11]. For example, the principal eigenvalue (guaranteed by the Krein-Rutman theorem [21]) of the diffusion-advection type operator, generally, is *not* monotonic in diffusion rate or advection rate.

As indicated in the above discussion, we need to introduce new ideas and techniques to overcome the emerging difficulties caused by advection. Specifically, regarding the co-existence steady state, we develop a technical analytic approach to directly exclude the existence of any co-existence steady state for most situations of system (1.1) (indeed, we treat a more general model including different types of boundary conditions), in which the key point is to do sufficient analysis on the potential behaviors of two auxiliary functions $T$ and $S$; see Theorem 1.1 and its proof. For the linear stability of semi-trivial steady states, we introduce different kinds of ways to address this issue. For instance, we derive a very useful characterization of the principal eigenvalue in terms of the semi-trivial steady state and the principal eigenfunction (see, e.g., (4.31) and (5.10)), and then establish proper estimates on the steady state and eigenfunction to determine the linear stability, where we employed some analytic skills in PDE. Resting on these two aspects, we finally appeal to the theory of abstract competitive systems developed in [18, 19, 22] to obtain the global dynamics. It is expected that our work may be of interest to those researchers in the areas of spectral theory, reaction-diffusion equations, and dynamical systems.

## 1.1 Developments and related works

System (1.1) can be regarded as a ramification of the following single species growth model:

$$\begin{cases} u_t = du_{xx} - \alpha u_x + u[r-u], & 0<x<L, t>0, \\ du_x(0,t) - \alpha u(0,t) = 0, & t>0, \\ u(L,t) = 0, & t>0, \end{cases} \tag{1.4}$$

which was proposed by Speirs and Gurney [34] to describe certain hydrodynamical s-cenarios and to study the biological phenomenon "drift paradox". Here all parameters in (1.4) can be understood similarly as that in (1.1), except the zero Dirichlet boundary condition at the downstream end $x = L$, which means that the downstream area is hostile for organisms to survive and can be applied to depict the situation "stream to ocean". Among other things, Speirs and Gurney proved that the trivial steady state $u \equiv 0$ is unstable if and only if $\alpha < \sqrt{4dr}$ and $L > L^*$, where

$$L^* = 2d \frac{\pi - \arctan\left(\frac{\sqrt{4dr-\alpha^2}}{\alpha}\right)}{\sqrt{4dr - \alpha^2}},$$

which suggests that the persistence of single species is only likely when advection is weak relative to diffusion and the stream is long enough. In other words, sufficient random movements, to some extent, can balance the negative effects incurred by advection which always drives the species to the hostile environment—the right boundary, and so the likelihood of persistence is enhanced. See also [36] and [28] for the discussion of the same equation but with different boundary conditions at the downstream end.

Recently, there are some works on the homogeneous version of system (1.1), that is,

$$\begin{cases} u_t = d_1 u_{xx} - \alpha_1 u_x + u[r_0 - u - v], & 0 < x < L, t > 0, \\ v_t = d_2 v_{xx} - \alpha_2 v_x + v[r_0 - u - v], & 0 < x < L, t > 0, \end{cases} \tag{1.5}$$

where $r_0$ is a positive constant. For instance, to study the evolution of random diffusion, Lou and Lutscher [26] assumed $\alpha_1 = \alpha_2$ and prescribed the so-called Danckwerts boundary condition, and they proved that the slower diffuser will always be wiped out by the faster one, i.e., faster diffusion will evolve, which is in sharp contrast to the *non-advective* case [9, 11]. This finding was further generalized by Lou and Zhou [28] to a wide class of boundary conditions including the no-flux type. Later on, Lou, Xiao and Zhou [27] applied the above system to the biological situation of water column, where the advection is caused by the gravity. By assuming $d_1 = d_2$ and $\alpha_1 \neq \alpha_2$ and imposing the no-flux boundary conditions, they demonstrated that weaker advection is more beneficial for the individuals to win the competition; indeed, they finally showed that the movement without advection in homogeneous environment is evolutionarily stable. The general case, $d_1 \neq d_2$ and $\alpha_1 \neq \alpha_2$, of system (1.5) with no-flux type boundary conditions was recently studied by Zhou [38], where he obtained two main results: (i) the strategy of a combination of faster diffusion and slower advection is always favorable, which can be seen as a mixture of the main conclusions in [28] and [27]; (ii) the strategy of faster diffusion together with much stronger advection relative to diffusion is always selected against, which shows that too much strong advection can counterbalance the positive effects of larger diffusion and is thus disadvantageous for organisms.

When the spatial variations of the environment are taken into account, that is, $r = r(x)$, depending nontrivially on the spatial variable $x$, much less is known about system (1.1). The case $d_1 \neq d_2$ and $\alpha_1 = \alpha_2$ has been investigated by Lam, Lou and Lutscher [23], where, by assuming both diffusion and advection rates are sufficiently small and comparable, they studied the existence and multiplicity of evolutionarily stable strategies. Another case, $d_1 \neq d_2$ and $\alpha_1 = 0 < \alpha_2$, which means that one species undergoes random diffusion only (no advection) while the other one takes both random and biased movements, has been recently studied by Zhao and Zhou [37], where they found much richer phenomena than the corresponding homogeneous case [27].

The current work aims to treat a more general situation. To be more specific, if we look at the plane of the parameters $\alpha_1$-$\alpha_2$, then [23] addresses the dynamics on the line $\alpha_2 = \alpha_1$ but nearby the origin, and [37] deals with the dynamics on the line $\alpha_1 = 0$. Our setting (1.2) corresponds to the first quadrant of the $\alpha_1$-$\alpha_2$ plane. We will regard the advection rate $\alpha_1$ (resp. $\alpha_2$) as the variable parameter to study the global dynamics of system (1.1) under the assumption (H); see Theorem 1.3 and Figure 1 below. In the last section, we will also make a comparison between the current work and the homogeneous case [38].

## 1.2 Main results

In the sequel, we denote by $(\widetilde{u}, 0)$ and $(0, \widetilde{v})$ the two semi-trivial steady states of system (1.1). In addition, by "g.a.s" we mean that the steady state is globally asymptotically stable among all non-negative and not identically zero initial conditions.

Our first result concerns the non-existence of co-existence steady state, which plays a significant role in determining the global dynamics of system (1.1). For independent interest and potential applications, we present this result in a more general version by considering more general boundary conditions at the downstream end:

$$
\begin{cases}
d_1 u_{xx} - \alpha_1 u_x + u[r(x) - u - v] = 0, & 0 < x < L, \\
d_2 v_{xx} - \alpha_2 v_x + v[r(x) - u - v] = 0, & 0 < x < L, \\
d_1 u_x(0) - \alpha_1 u(0) = d_2 v_x(0) - \alpha_2 v(0) = 0, & \\
d_1 u_x(L) - \alpha_1 u(L) = -b\alpha_1 u(L), & \\
d_2 v_x(L) - \alpha_2 v(L) = -b\alpha_2 v(L), &
\end{cases}
\tag{1.6}
$$

where the parameter $b$ is used to measure the relative rate of population loss at the downstream end due to water flow [29]. In particular, for $b = 0$ we have the no-flux type boundary conditions; for $b = 1$, we obtain the free-flow (Neumann) boundary conditions; and for $b \to \infty$, we see the hostile boundary conditions.

**Theorem 1.1** *Assume that* (H) *holds. Then the following statements hold:*

(1) *If* $0 < d_1 \le d_2$, $\alpha_2 \ge \frac{d_2}{d_1}\alpha_1$ *and* $(d_1 - d_2)^2 + (\alpha_1 - \alpha_2)^2 \neq 0$, *then for any* $b \in [0, \infty)$, *system* (1.6) *has no positive solution;*

(2) *If* $0 < d_1 < d_2$, $\alpha_1 \ge \alpha_2$ *and* $\alpha_1 \ge 2\sqrt{r(0)d_1}$, *then for any* $b \in [0, \infty)$, *system* (1.6) *has no positive solution.*

Compared with the non-existence result [37, Theorem 1.1] ($\alpha_1 = 0 < \alpha_2$, no-flux boundary condition), part (1) of the above theorem is much more general since it holds for any $b \geq 0$. Part (2) is also new. From the biological point of view, Theorem 1.1 above reveals that when the resource function is decreasing in space, very possibly two species will not coexist eventually. We provide here a possible interpretation for this phenomenon: advective force will not be favored as it always drives individuals to move toward the downstream end, which is exactly the direction the resources are declining; if $b = 0$, although there is no loss of individuals at the downstream end, advection will cause overcrowding of the population at the downstream end with the least amount of resource, which clearly is harmful for both population to coexist; as $b$ increases, as more individuals will be washed out of the habitat by the water flow at the downstream end, it is even harder for two species to coexist. To establish the above theorem, we note here that the arguments developed in [38] for the homogeneous case of system (1.1) (i.e., $r(x) \equiv r_0$ for some constant $r_0 > 0$) do not work anymore, since now the steady states of system (1.6) (if exist) are not analytic in view of the Cauchy-Kowalevski theory [20]. This fact was ignored in our recent work [37], but here we will introduce new analytic skills to fill this gap.

Our second result concerns the linear stability of semi-trivial steady states $(\widetilde{u}, 0)$ and $(0, \widetilde{v})$.

**Theorem 1.2** *Assume that* (H) *holds. Then the following statements hold:*

(1) *Given* $0 < d_1 < d_2$. *For each* $\alpha_1 \geq 2\sqrt{r(0)d_1}$, *there exists a unique* $\alpha_2^{**} = \alpha_2^{**}(d_1, d_2, \alpha_1, r) > 0$ *such that* $(\widetilde{u}, 0)$ *is linearly stable for* $\alpha_2 > \alpha_2^{**}$ *and linearly unstable for* $\alpha_2 < \alpha_2^{**}$. *Furthermore,* $\alpha_2^{**}$ *satisfies*

$$\lim_{\alpha_1 \to \infty} \frac{\alpha_2^{**}}{\alpha_1} = \frac{d_2}{d_1}. \tag{1.7}$$

(2) *Given* $0 < d_1 < d_2$. *For each* $\alpha_2 \geq 2\sqrt{r(0)d_2}$, *there exists a unique* $\alpha_1^{**} = \alpha_1^{**}(d_1, d_2, \alpha_2, r) > 0$ *such that* $(0, \widetilde{v})$ *is linearly stable for* $\alpha_1 > \alpha_1^{**}$ *and linearly unstable for* $\alpha_1 < \alpha_1^{**}$. *Furthermore,* $\alpha_1^{**}$ *satisfies*

$$\lim_{\alpha_2 \to \infty} \frac{\alpha_1^{**}}{\alpha_2} = \frac{d_1}{d_2}. \tag{1.8}$$

The asymptotic behaviors described in (1.7) and (1.8) will help us determine the behaviors of the other two important critical values $\alpha_2^*$ and $\alpha_1^*$, which will be given in Theorem 1.3 below. Besides the above theorem, we will also display other types of sufficient conditions for the local stability; see subsection 4.3.

We now state the global dynamics of system (1.1). See Theorem 1.3 for case (1.2) and Theorem 1.4 for case (1.3).

**Theorem 1.3** *Assume that* (H) *holds and* $0 < d_1 < d_2$. *Then the following statements are valid:*

(1) *For each* $\alpha_1 > 0$, *there exists a critical number* $\alpha_2^* = \alpha_2^*(d_1, d_2, \alpha_1, r) \in [0, \frac{d_2}{d_1}\alpha_1)$ *such that for any* $\alpha_2 \in (\alpha_2^*, \infty)$, $(\widetilde{u}, 0)$ *is g.a.s. Moreover,* $\alpha_2^*$ *satisfies*

$$\alpha_2^* \equiv 0 \text{ for } \alpha_1 \in (0, \epsilon_0) \qquad \text{and} \qquad \lim_{\alpha_1 \to \infty} \frac{\alpha_2^*}{\alpha_1} = \frac{d_2}{d_1}, \tag{1.9}$$

*where* $\epsilon_0$ *is a small positive number depending on* $d_1, d_2, r(x)$.

(2) *For each* $\alpha_2 > 0$, *there exists a critical number* $\alpha_1^* = \alpha_1^*(d_1, d_2, \alpha_2, r) \in (\frac{d_1}{d_2}\alpha_2, \infty)$ *such that for any* $\alpha_1 \in (\alpha_1^*, \infty)$, $(0, \widetilde{v})$ *is g.a.s. Moreover,* $\alpha_1^*$ *satisfies*

$$\begin{cases} \alpha_1^* \in (\frac{d_1}{d_2}\alpha_2,\ 2\sqrt{r(0)d_1}) & \text{for} \quad \alpha_2 \in (0, 2\sqrt{r(0)d_1}), \\ \alpha_1^* \in (\frac{d_1}{d_2}\alpha_2,\ \alpha_2) & \text{for} \quad \alpha_2 \in [2\sqrt{r(0)d_1}, \infty). \end{cases} \tag{1.10}$$

(3) *For each* $\alpha_1 > 0$ *such that* $\alpha_2^*(\alpha_1) > 0$, *there exists* $\alpha_2 \in (0, \alpha_2^*(\alpha_1)]$ *such that system* (1.1) *admits a co-existence steady state; Similarly, for each* $\alpha_2 > 0$, *there exists* $\alpha_1 \in (0, \alpha_1^*(\alpha_2)]$ *such that system* (1.1) *admits a co-existence steady state.*

We make some comments on the above theorem. For given $0 < d_1 < d_2$ and $r(x)$ satisfying assumption (H), we can regard $\alpha_2^*$ above as a function of $\alpha_1$ in $(0, \infty)$ and $\alpha_1^*$ above as a function of $\alpha_2$ in $(0, \infty)$. Next, we use the parameter plane of $\alpha_1$-$\alpha_2$ to explain the above results. Let $\Theta := \{(\alpha_1, \alpha_2) : \alpha_1 > 0,\ \alpha_2 > 0\}$ denote the first quadrant of $\alpha_1$-$\alpha_2$ plane. Then statement (1) above is equivalent to saying that species $u$ will win the competition when $(\alpha_1, \alpha_2) \in \Sigma_1$, where

$$\Sigma_1 := \{(\alpha_1, \alpha_2) \in \Theta : \quad \alpha_2 \ > \ \alpha_2^*(\alpha_1)\},$$

which, particularly, includes the region

$$\{(\alpha_1, \alpha_2) \in \Theta : \ \alpha_2 \geq \frac{d_2}{d_1}\alpha_1\}$$

and the small square $(0, \epsilon_0) \times (0, \epsilon_0)$. Moreover, the limit in (1.9) illustrates that this result is *optimal* when $\alpha_1 \gg 1$. In contrast, statement (2) above indicates that species $v$ will become a superior when $(\alpha_1, \alpha_2) \in \Sigma_2$, where

$$\Sigma_2 := \{(\alpha_1, \alpha_2) \in \Theta : \quad \alpha_1 \ > \ \alpha_1^*(\alpha_2)\},$$

which, in view of (1.10), particularly contains the region

$$\{(\alpha_1, \alpha_2) \in \Theta : \quad \alpha_1 \geq 2\sqrt{r(0)d_1}, \quad \alpha_1 \geq \alpha_2\}.$$

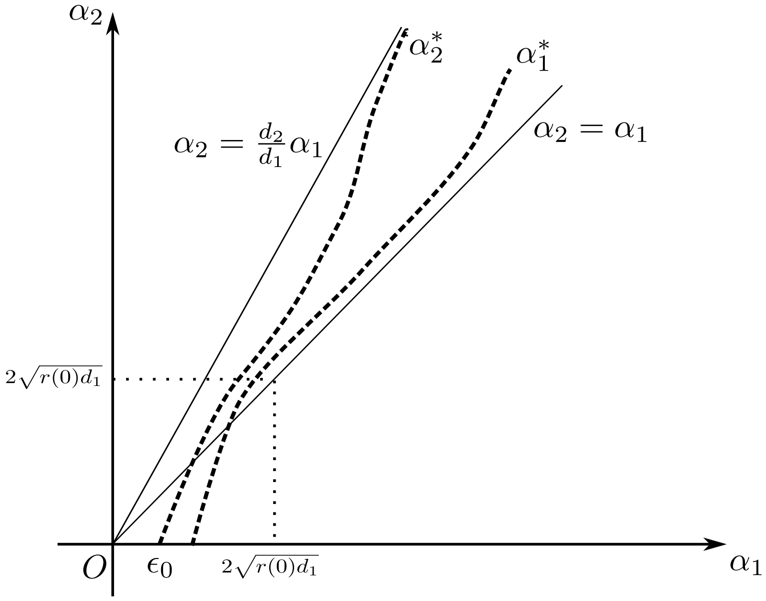


Figure 1: Illustration of $\alpha_2^*(\alpha_1)$ and $\alpha_1^*(\alpha_2)$

For reader's convenience, we provide here, based on the estimates in (1.9) and (1.10), two illuminating graphs of $\alpha_2^*(\alpha_1)$ and $\alpha_1^*(\alpha_2)$; see Figure 1 below. Basically speaking, in the region above the curve of $\alpha_2 = \alpha_2^*(\alpha_1)$, $(\widetilde{u}, 0)$ is g.a.s (statement (1)); in the region below the curve of $\alpha_1 = \alpha_1^*(\alpha_2)$, $(0, \widetilde{v})$ is g.a.s (statement (2)); and in the region between these two curves, we observe the co-existence steady state (statement (3)). The stability and uniqueness of these co-existence steady states is an open problem. We suspect that the curve $\alpha_1 = \alpha_1^*$ also approaches the line $\alpha_2/\alpha_1 = d_2/d_1$ asymptotically as $\alpha_2 \to \infty$.

**Theorem 1.4** *Assume that* (H) *holds,* $d_1 = d_2 := d > 0$*. If* $0 < \alpha_1 < \alpha_2$*, then* $(\widetilde{u}, 0)$ *is g.a.s. Similarly, if* $0 < \alpha_2 < \alpha_1$*, then* $(0, \widetilde{v})$ *is g.a.s.*

Theorem 1.4 implies that, as $d_1 \to d_2$, the two curves $\alpha_1 = \alpha_1^*$ and $\alpha_2 = \alpha_2^*$ will converge to the line $\alpha_1 = \alpha_2$. It also makes precise the following intuition: when two species only differ in their advection rates and the resource is distributed decreasingly across space, the competitor with stronger advection will be completely wiped out since it would be more likely driven to the most unfavorable region: the downstream end.

The remainder of this paper is organized as follows. In Section 2, we establish the non-existence result Theorem 1.1, which plays an important role in later analysis. Section 3 is devoted to the investigation of a useful auxiliary problem, where the diffusion and advection rates of two populations are supposed to be proportional. In

Section 4, we mainly discuss the linear stability of two semi-trivial steady states and verify Theorem 1.2. Based on these preparations, we exhibit the proof of the global dynamics described in Theorems 1.3 and 1.4 in Section 5, and finally, a discussion is included in Section 6.

## 2 Non-existence of co-existence steady state

In this section, we will focus on the elliptic system (1.6) and aim to establish the non-existence result as described in Theorem 1.1. To this end, we need to make some a priori estimates on the positive solution of system (1.6).

### 2.1 A priori estimates

Suppose that $(u, v)$ is a positive solution of system (1.6) and define

$$T := \frac{u_x}{u} \qquad \text{and} \qquad S := \frac{v_x}{v}. \tag{2.1}$$

Then by a series of straightforward computations, one can deduce

$$\begin{cases} -d_1 T_{xx} + \big[\alpha_1 - 2d_1 T\big] T_x + uT + vS = r'(x), & 0 < x < L, \\ -d_2 S_{xx} + \big[\alpha_2 - 2d_2 S\big] S_x + uT + vS = r'(x), & 0 < x < L, \\ T(0) = \frac{\alpha_1}{d_1} > 0, \qquad T(L) = (1-b)\frac{\alpha_1}{d_1}, \\ S(0) = \frac{\alpha_2}{d_2} > 0, \qquad S(L) = (1-b)\frac{\alpha_2}{d_2}. \end{cases} \tag{2.2}$$

Next, we present several properties about $T$ and $S$, which will be frequently used in later analysis.

The first one refers to the following useful identities whose proof can be found in [38, Lemmas 3.2 and 4.3].

**Lemma 2.1** *Assume that $d_1, d_2, \alpha_1, \alpha_2 > 0$. Choose any two points $0 \le y_1 < y_2 \le L$. Then the following identities hold:*

(1) *If $d_1 \neq d_2$, we have*

$$\begin{aligned} \Big[d_1 - d_2\Big] \int_{y_1}^{y_2} \Big[S - \frac{\alpha_2 - \alpha_1}{d_2 - d_1}\Big] \cdot \Big[T - \frac{\alpha_1}{d_1}\Big] \cdot e^{-\frac{\alpha_1}{d_1}x} \cdot u \cdot v \, dx \\ = \left\{ \Big[d_1\big(T - \frac{\alpha_1}{d_1}\big) - d_2\big(S - \frac{\alpha_2}{d_2}\big)\Big] \cdot e^{-\frac{\alpha_1}{d_1}x} \cdot u \cdot v \right\}\Bigg|_{y_1}^{y_2} \end{aligned} \tag{2.3}$$

*and*

$$\begin{aligned} \Big[d_2 - d_1\Big] \int_{y_1}^{y_2} \Big[T - \frac{\alpha_2 - \alpha_1}{d_2 - d_1}\Big] \cdot \Big[S - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \, dx \\ = \left\{ \Big[d_2\big(S - \frac{\alpha_2}{d_2}\big) - d_1\big(T - \frac{\alpha_1}{d_1}\big)\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \right\}\Bigg|_{y_1}^{y_2}; \end{aligned} \tag{2.4}$$

(2) *If $d_1 = d_2 := d$, we have*

$$\begin{aligned}\Big[\alpha_2 - \alpha_1\Big] \int_{y_1}^{y_2} \Big[T - \frac{\alpha_1}{d}\Big] \cdot e^{-\frac{\alpha_1}{d}x} \cdot u \cdot v\, dx \\ = \Big\{d \cdot \Big[\big(T - \frac{\alpha_1}{d}\big) - \big(S - \frac{\alpha_2}{d}\big)\Big] \cdot e^{-\frac{\alpha_1}{d}x} \cdot u \cdot v\Big\}\Big|_{y_1}^{y_2}\end{aligned} \tag{2.5}$$

*and*

$$\begin{aligned}\Big[\alpha_1 - \alpha_2\Big] \int_{y_1}^{y_2} \Big[S - \frac{\alpha_2}{d}\Big] \cdot e^{-\frac{\alpha_2}{d}x} \cdot u \cdot v\, dx \\ = \Big\{d \cdot \Big[\big(S - \frac{\alpha_2}{d}\big) - \big(T - \frac{\alpha_1}{d}\big)\Big] \cdot e^{-\frac{\alpha_2}{d}x} \cdot u \cdot v\Big\}\Big|_{y_1}^{y_2}.\end{aligned} \tag{2.6}$$

(3) *For any $x \in [0, L]$, we have*

$$-d_1 T_x + \alpha_1 T - d_1 T^2 = -d_2 S_x + \alpha_2 S - d_2 S^2. \tag{2.7}$$

The second property concerns the interior behaviors of $T$ and $S$ at the extreme points.

**Lemma 2.2** *Assume that (H) holds. Then the following statements are valid:*

(1) *If $T$ achieves a positive local maximum at $x_0 \in (0, L)$, then $S(x_0) < 0$;*

(2) *If $S$ achieves a positive local maximum at $x_0 \in (0, L)$, then $T(x_0) < 0$.*

**Proof:** By an inspection of the equations of $T$ and $S$ in (2.2), this result follows immediately from the maximum principle. □

To get a picture of the boundary behaviors of $T$ and $S$, we need to first study the following two auxiliary functions

$$f(x) := d_1 u_x - \alpha_1 u \quad \text{and} \quad g(x) := d_2 v_x - \alpha_2 v. \tag{2.8}$$

**Lemma 2.3** *Assume that (H) holds. Then the following statements about $f$ and $g$ are true:*

(1) *$f'$ and $g'$ have the same sign;*

(2) *$f'(0) < 0$, and thus $g'(0) < 0$ due to statement (1);*

(3) *If $b = 0$ (no-flux boundary condition), then $f'(L) > 0$, and thus $g'(L) > 0$ due to statement (1).*

**Proof:** Statement (1) holds due to $\frac{f'(x)}{u} = \frac{g'(x)}{v} = -\Big[r - u - v\Big]$.

We next prove statement (2). We claim that $f'(0) < 0$. Otherwise, we must have $f'(0) > 0$ or $f'(0) = 0$.

If $f'(0) > 0$, then $g'(0) > 0$. Since $f(L) = g(L) \leq 0$, without loss of generality, we may assume there exists $x^* \in (0, L]$ such that

$$f(0) = f(x^*) = 0, \qquad f(x) > 0 \ \text{ in } \ (0, x^*),$$

and

$$g(0) = 0 \leq g(x^*), \qquad g(x) > 0 \quad \text{in} \ \ (0, x^*).$$

This particularly implies that $T$ achieves a positive local maximum in $(0, x^*)$, say $x^{**}$, at which, $S$ is positive, contradicting Lemma 2.2. So $f'(0) > 0$ cannot happen.

If $f'(0) = 0$, then by the equation of $u$, we see

$$\Big[r(x) - u - v\Big]\Big|_{x=0} = 0,$$

which in turn gives

$$f''(0) = \Big\{ -u\Big[r'(x) - u'(x) - v'(x)\Big]\Big\}\Big|_{x=0} > 0,$$

and

$$g'(0) = 0, \qquad g''(0) = \Big\{ -v\Big[r'(x) - u'(x) - v'(x)\Big]\Big\}\Big|_{x=0} > 0,$$

where the assumption and boundary conditions are used. This tells us that both $f$ and $g$ will strictly increase when $x$ becomes positive, so we can perform a similar analysis as above to deduce a contradiction. Thus, statement (2) is proved.

Statement (3) can be established in the same spirit as in statement (2) since when $b = 0$, the downstream end has the same no-flux conditions as the upstream end. We omit the details here. $\square$

By the above lemma and the definition in (2.8), one immediately obtains the boundary behaviors of $T$ and $S$ as follows.

**Corollary 2.1** *Assume that (H) holds. Then the following statements are valid:*

(1) *$T$ (resp. $S$) will firstly decrease strictly from the value $\frac{\alpha_1}{d_1}$ (resp. $\frac{\alpha_2}{d_2}$) at $x = 0$;*

(2) *If, in addition, $b = 0$, then $T$ (resp. $S$) will finally increase strictly to the value $\frac{\alpha_1}{d_1}$ (resp. $\frac{\alpha_2}{d_2}$) at $x = L$.*

## 2.2 Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1. For the sake of clarity, we divide the proof into two parts; see below.

**Proof of Theorem 1.1 (1)**: Suppose to the contrary that there is a positive solution of system (1.6), denoted by $(u, v)$. Recall $T$ and $S$ defined in (2.1). In what follows, we aim to analyze the potential behaviors of $T$ and $S$ and try to deduce a contradiction.

We first claim that $S - \frac{\alpha_2}{d_2}$ must change sign in $[0, L]$.

Otherwise, in view of the function behaviors described in Corollary 2.1 and the condition $\alpha_2 \geq \frac{d_2}{d_1}\alpha_1$, we have

$$S \leq, \not\equiv \frac{\alpha_2}{d_2} \leq \frac{\alpha_2 - \alpha_1}{d_2 - d_1} \quad \text{in} \quad [0, L]. \tag{2.9}$$

If $d_1 \neq d_2$, by restricting identity (2.3) at $(y_1, y_2) = (0, L)$, we see

$$\begin{aligned} &\Big[d_1 - d_2\Big] \cdot \int_0^L \Big[S - \frac{\alpha_2 - \alpha_1}{d_2 - d_1}\Big] \cdot \Big[T - \frac{\alpha_1}{d_1}\Big] \cdot e^{-\frac{\alpha_1}{d_1}x} \cdot u \cdot v \mathrm{d}x \\ &= \; b \cdot \Big[\alpha_2 - \alpha_1\Big] \cdot e^{-\frac{\alpha_1}{d_1}L} \cdot u(L) \cdot v(L) \\ &\geq \; 0, \end{aligned}$$

where the last inequality holds due to $\alpha_2 \geq \frac{d_2}{d_1}\alpha_1 > \alpha_1$, and so

$$\int_0^L \Big[S - \frac{\alpha_2 - \alpha_1}{d_2 - d_1}\Big] \cdot \Big[T - \frac{\alpha_1}{d_1}\Big] \cdot e^{-\frac{\alpha_1}{d_1}x} \cdot u \cdot v \mathrm{d}x \leq 0, \tag{2.10}$$

which, by the virtue of (2.9) and Corollary 2.1, implies that $T - \frac{\alpha_1}{d_1}$ must change sign in $[0, L]$. This fact allows us to find a point $x_1 \in (0, L)$ such that

$$T(0) = T(x_1) = \frac{\alpha_1}{d_1} \qquad \text{and} \qquad T(x) < \frac{\alpha_1}{d_1} \text{ in } (0, x_1),$$

where Corollary 2.1 is used once again. Now, let us use identity (2.4) and evaluate it at $(y_1, y_2) = (0, x_1)$, then we observe

$$\begin{aligned} 0 &< \Big[d_2 - d_1\Big] \int_0^{x_1} \Big[T - \frac{\alpha_2 - \alpha_1}{d_2 - d_1}\Big] \cdot \Big[S - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \mathrm{d}x \\ &= \Big\{d_2 \cdot \big(S - \frac{\alpha_2}{d_2}\big) \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x_1} \\ &\leq 0. \end{aligned}$$

This contradiction confirms the above claim under the assumption $d_1 \neq d_2$.

If $d_1 = d_2 := d$, we confine identity (2.6) at $(y_1, y_2) = (0, L)$ to obtain

$$0 > \int_0^L \left[S - \frac{\alpha_2}{d}\right] \cdot e^{-\frac{\alpha_2}{d}x} \cdot u \cdot v \mathrm{d}x = \left\{b \cdot e^{-\frac{\alpha_2}{d}x} \cdot u \cdot v\right\}\Big|_{x=L} \geq 0,$$

where the first inequality used (2.9). Again, we obtain a contradiction. So, the above claim is proved.

Based on the above claim and $S(L) = (1-b)\frac{\alpha_2}{d_2} \leq \frac{\alpha_2}{d_2}$, we see that $S$ must have at least one local maximum in $(0, L)$ with value bigger than $\frac{\alpha_2}{d_2}$, which enables us to define

$$x_2 := \sup\left\{y \in [0, L] : \ S(y) > \frac{\alpha_2}{d_2}, \ S'(y) = 0 \text{ and } S''(y) \leq 0\right\}.$$

Clearly, $x_2 \in (0, L)$ due to the boundary conditions. Now, we have two cases:

$$\text{case 1}: \ S(x_2) = \frac{\alpha_2}{d_2} \qquad \text{and} \qquad \text{case 2}: \ S(x_2) > \frac{\alpha_2}{d_2}.$$

We first discuss case 1, i.e., $S(x_2) = \frac{\alpha_2}{d_2}$ holds.

*Claim 1.1.* $S'(x_2) = S''(x_2) = 0$.

By the definition of $x_2$, we have $S'(x_2) = 0$ and $S''(x_2) \leq 0$. Suppose for contradiction that $S''(x_2) < 0$, then there exists small $\delta > 0$ such that

$$S(x) < \frac{\alpha_2}{d_2} \qquad \text{for} \qquad x \in (x_2 - \delta, x_2).$$

Using the definition of $x_2$ again, the above inequality yields $x_2 \leq x_2 - \delta$, a contradiction. Thus claim 1.1 is true.

*Claim 1.2.* $T(x_2) < 0$.

Evaluating the equation of $S$ (see the second equation of (2.2)) at $x = x_2$ and using claim 1.1, one easily sees

$$T(x_2) = \frac{r'(x_2) - v(x_2)S(x_2)}{u(x_2)} < 0,$$

as desired.

*Claim 1.3.* $S(x) \leq \frac{\alpha_2}{d_2}$ for $x \in [x_2, L]$.

Note $S(x_2) = \frac{\alpha_2}{d_2}$. If $b > 0$, $S(L) < \frac{\alpha_2}{d_2}$; if $b = 0$, by Corollary 2.1, $S$ will increase to $\frac{\alpha_2}{d_2}$ at $x = L$. So, for any $b \geq 0$, if this claim is not true, then $S$ must have one local maximum in $(x_2, L)$ with value larger than $\frac{\alpha_2}{d_2}$, contradicting the definition of $x_2$. The proof of this claim is complete.

*Claim 1.4.* $T - \frac{\alpha_1}{d_1}$ must change sign in $(x_2, L)$.

If not, $T \leq \frac{\alpha_1}{d_1}$ $(\leq \frac{\alpha_2 - \alpha_1}{d_2 - d_1})$ in $[x_2, L]$ since either $T(L) < \frac{\alpha_1}{d_1}$ $(b > 0)$ or $T(x) < \frac{\alpha_1}{d_1}$ for $x$ close to $L$ (Corollary 2.1).

If $d_1 \neq d_2$, confining identity (2.4) at $(x_2, L)$, we attain

$$
\begin{aligned}
0 &< \Big[d_2 - d_1\Big] \int_{x_2}^{L} \Big[T - \frac{\alpha_2 - \alpha_1}{d_2 - d_1}\Big] \cdot \Big[S - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \mathrm{d}x \\
&= \Big\{b \cdot \Big[\alpha_1 - \alpha_2\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=L} + \Big\{d_1 \cdot \Big[T - \frac{\alpha_1}{d_1}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=x_2} \\
&\leq 0,
\end{aligned}
$$

a contradiction.

If $d_1 = d_2$, we restrict identity (2.6) at $(x_2, L)$ to get

$$
\begin{aligned}
0 &< \Big[\alpha_1 - \alpha_2\Big] \int_{x_2}^{L} \Big[S - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \mathrm{d}x \\
&= \Big\{b \cdot \Big[\alpha_1 - \alpha_2\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=L} + \Big\{d_1 \cdot \Big[T - \frac{\alpha_1}{d_1}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=x_2} \\
&\leq 0,
\end{aligned}
$$

again, a contradiction.

Hence, claim 1.4 is established.

We are now able to deduce a contradiction for case 1.

Resting on claims 1.2 and 1.4, we can select a point $x_3 \in (x_2, L)$ such that

$$
T(x) < \frac{\alpha_1}{d_1} \ (\leq \frac{\alpha_2 - \alpha_1}{d_2 - d_1}) \ \text{ in } \ [x_2, x_3) \quad \text{ and } \quad T(x_3) = \frac{\alpha_1}{d_1}. \tag{2.11}
$$

If $d_1 \neq d_2$, we evaluate identity (2.4) at $(x_2, x_3)$ to obtain the following contradiction

$$
\begin{aligned}
0 &< \Big[d_2 - d_1\Big] \int_{x_2}^{x_3} \Big[T - \frac{\alpha_2 - \alpha_1}{d_2 - d_1}\Big] \cdot \Big[S - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \mathrm{d}x \\
&= \ \Big\{\Big[d_2\big(S - \frac{\alpha_2}{d_2}\big) - d_1\big(T - \frac{\alpha_1}{d_1}\big)\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x_2}^{x_3} \\
&= \ \Big\{d_2 \cdot (S - \frac{\alpha_2}{d_2}) \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=x_3} + \Big\{d_1 \cdot (T - \frac{\alpha_1}{d_1}) \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=x_2} \\
&< 0,
\end{aligned}
$$

where claim 1.3 and (2.11) are used. While if $d_1 = d_2$, we apply (2.6) at $(x_2, x_3)$ to derive another contradiction

$$
\begin{aligned}
0 &< \Big[\alpha_1 - \alpha_2\Big] \int_{x_2}^{x_3} \Big[S - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \mathrm{d}x \\
&= \ \Big\{d_2 \cdot (S - \frac{\alpha_2}{d_2}) \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=x_3} + \Big\{d_1 \cdot (T - \frac{\alpha_1}{d_1}) \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=x_2} \\
&< 0,
\end{aligned}
$$

where claims 1.2 and 1.3 are used. In a word, we can derive a contradiction for case 1.

Next, we deal with case 2, i.e., $S(x_2) > \frac{\alpha_2}{d_2}$ holds.

*Claim 2.1.* There exists $x_4 \in (x_2, L)$ such that $S'(x) \le 0$ in $[x_2, x_4]$, $S(x_4) = \frac{\alpha_2}{d_2}$ and $S \le \frac{\alpha_2}{d_2}$ in $[x_4, L]$.

Recall the behaviors of $S$ near $x = L$ in claim 1.3. Since now $S(x_2) > \frac{\alpha_2}{d_2}$, $S$ will touch the value $\frac{\alpha_2}{d_2}$ at some location in $(x_2, L)$. Denote the one that is closest to $x_2$ by $x_4$, and clearly, $S(x_4) = \frac{\alpha_2}{d_2}$. Furthermore, we must have

$$S'(x) \le 0 \text{ in } [x_2, x_4] \qquad \text{and} \qquad S \le \frac{\alpha_2}{d_2} \text{ in } [x_4, L]$$

since otherwise $S$ will achieve a local maximum in $(x_2, L)$ with value greater than $\frac{\alpha_2}{d_2}$, contradicting the definition of $x_2$. Therefore, claim 2.1 is verified.

*Claim 2.2.* $T - \frac{\alpha_1}{d_1}$ must change sign in $(x_4, L)$.

The proof of this claim is similar to that of claim 1.4.

*Claim 2.3.* $T(x_4) < \frac{\alpha_1}{d_1}$.

Suppose for contradiction that $T(x_4) \ge \frac{\alpha_1}{d_1}$. By the definition of $x_2$, one can derive from the equation of $S$ that $T(x_2) < 0$. Now we assert that $T'(x_4) \ge 0$. Otherwise, $T$ will achieve a positive local maximum in $(x_2, x_4)$, in which, $S$ is positive, contradicting Lemma 2.2. Restricting identity (2.7) at $x = x_4$ and using the inequalities $T(x_4) \ge \frac{\alpha_1}{d_1}$ and $T'(x_4) \ge 0$ as well as claim 2.1, we find

$$\begin{aligned} 0 \ge & -d_1 T'(x_4) + d_1 \cdot T(x_4) \cdot \left[\frac{\alpha_1}{d_1} - T(x_4)\right] \\ = & -d_2 S'(x_4) + d_2 \cdot S(x_4) \cdot \left[\frac{\alpha_2}{d_2} - S(x_4)\right] \\ = & -d_2 S'(x_4) \\ \ge & \ 0, \end{aligned}$$

which implies

$$T'(x_4) = S'(x_4) = T(x_4) - \frac{\alpha_1}{d_1} = 0.$$

This identity allows us to further deduce from the equation of $S$ in (2.2) that

$$S''(x_4) = \frac{u(x_4)T(x_4) + v(x_4)S(x_4) - r'(x_4)}{d_2} > 0,$$

which, together with $S'(x_4) = 0$, leads to a contradiction with the behaviors of $S$ nearby $x = x_4$ depicted in claim 2.1. Hence, claim 2.3 is proved.

We now derive a contradiction for case 2.

In view of claims 2.2 and 2.3, we can find a point $x_5 \in (x_4, L)$ such that

$$T(x) < \frac{\alpha_1}{d_1} \ (\le \frac{\alpha_2 - \alpha_1}{d_2 - d_1}) \ \text{ in } \ [x_4, x_5) \quad \text{ and } \quad T(x_5) = \frac{\alpha_1}{d_1}.$$

Then we can perform a similar analysis as in the end of case 1 on the interval $(x_4, x_5)$ to get a contradiction.

In summary, based on the assumption at the beginning, logically we demonstrate that either case 1 or case 2 happens. But no matter which case happens, we can always deduce a contradiction, which shows that there is no positive solution. $\square$

**Remark 2.1** *We include here a discussion about the possibility of case 1 and case 2. First, the occurring of case 2 is easy to understand, e.g., if $S$ has finitely many local maximum points, then case 2 must happen. The interesting thing is that case 1 may also happen for general smooth functions. We give a specific example below:*

$$S(x) = \begin{cases} e^{\frac{1}{x-x_2}} \sin \frac{1}{x-x_2} + \frac{\alpha_2}{d_2}, & x \in (x_2 - \delta, x_2), \\ \frac{\alpha_2}{d_2}, & x \in [x_2, x_2 + \delta), \end{cases}$$

*where $\delta$ is a small positive number. It is easy to see that $S$ constructed above is a smooth function in $(x_2 - \delta, x_2 + \delta)$, and that there is a sequence of local maximum points in $(x_2 - \delta, x_2)$ with value bigger than $\frac{\alpha_2}{d_2}$ converging to $x_2$. For this example, we have $S(x_2) = \frac{\alpha_2}{d_2}$. So, in general, logically we cannot exclude case 1. Such details were ignored in our recent work [37, Theorem 1.1], and we point out here that by using the same arguments as above, one can make up for this gap. We also note that case 1 will not happen if the environment is spatially homogeneous since in that case one can apply the Cauchy-Kowalevski theory [20] to prove the analyticity of the solution.*

**Proof of Theorem 1.1 (2)**: Again, we use the contradiction argument. Suppose that there is a positive solution of system (1.6), denoted by $(u, v)$.

By a transformation $u = e^{\frac{\alpha_1}{2d_1}x} w$, we can derive from the equation of $u$ that

$$\begin{cases} d_1 w_{xx} + \left[ r - u - v - \frac{\alpha_1^2}{4d_1} \right] w = 0, & 0 < x < L, \\ d_1 w_x = \frac{1}{2}\alpha_1 w, & x = 0, \\ d_1 w_x = (\frac{1}{2} - b)\alpha_1 w, & x = L. \end{cases} \tag{2.12}$$

Since $r'(x) \leq, \not\equiv 0$ in $[0, L]$ and $\alpha_1 \geq 2\sqrt{r(0)d_1}$, we see from the equation of $w$ that $w_{xx} > 0$ in $(0, L)$, which together with the boundary condition $w_x(0) = \frac{1}{2}\frac{\alpha_1}{d_1} w(0) > 0$, yields $w_x > 0$ in $[0, L]$. If $b \in [\frac{1}{2}, \infty)$, then we immediately obtain a contradiction by looking at $w_x(L) = (\frac{1}{2} - b)\frac{\alpha_1}{d_1} w(L) \leq 0$.

Next, we consider the case $b \in [0, \frac{1}{2})$.

Indeed, from the above analysis, we can further deduce that

$$u_x = e^{\frac{\alpha_1}{2d_1}x} \left[ w_x + \frac{\alpha_1}{2d_1} w \right] > 0 \quad \text{in} \quad [0, L]. \tag{2.13}$$

Now, let us recall $T$ and $S$ defined in (2.1). By evaluating identity (2.4) at $(0, L)$, we find

$$\begin{aligned}&\Big[d_2 - d_1\Big]\int_0^L \Big[T + \frac{\alpha_1 - \alpha_2}{d_2 - d_1}\Big] \cdot \Big[S - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v \mathrm{d}x \\ &= \Big\{b \cdot \Big[\alpha_1 - \alpha_2\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot u \cdot v\Big\}\Big|_{x=L} \\ &\geq 0,\end{aligned}$$

where the last inequality used the condition $\alpha_1 \geq \alpha_2$. By (2.13) and the assumptions $0 < d_1 < d_2$ and $\alpha_1 \geq \alpha_2$, we see that the term $T + \frac{\alpha_1 - \alpha_2}{d_2 - d_1}$ is non-negative and not identically zero. Thus, the above inequality implies that $S \leq \frac{\alpha_2}{d_2}$ in $[0, L]$ cannot happen, and so $S$ must be greater than $\frac{\alpha_2}{d_2}$ somewhere in $[0, L]$. Recall the boundary behaviors of $S$ in Corollary 2.1 and the fact that $S(L) < \frac{\alpha_2}{d_2}$ if $b > 0$, we then can conclude that $S$ must attain a positive local maximum in $(0, L)$, say at $x = x^*$. By Lemma 2.2, $T(x^*) = \Big[\frac{u_x}{u}\Big]\Big|_{x=x^*} < 0$, i.e., $u_x(x^*) < 0$. This leads to a contradiction with (2.13). $\square$

# 3 An auxiliary problem

In this section, we primarily focus on the following special case of setting (1.2):

$$\frac{d_2}{d_1} = \frac{\alpha_2}{\alpha_1} := k \ (\neq 1), \tag{3.1}$$

which indicates that the dispersal strategies of two competitors are proportional. For this special case, we can reduce system (1.1) to

$$\begin{cases} u_t = d_1 u_{xx} - \alpha_1 u_x + u\Big[r(x) - u - v\Big], & 0 < x < L, t > 0, \\ v_t = k\Big[d_1 v_{xx} - \alpha_1 v_x\Big] + v\Big[r(x) - u - v\Big], & 0 < x < L, t > 0, \\ d_1 u_x(x,t) - \alpha_1 u(x,t) = 0, & x = 0, L, t > 0, \\ d_1 v_x(x,t) - \alpha_1 v(x,t) = 0, & x = 0, L, t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & 0 < x < L, \\ v(x,0) = v_0(x) \geq, \not\equiv 0, & 0 < x < L, \end{cases} \tag{3.2}$$

where one sees that the difference between two species is measured by the proportional constant $k$.

It turns out that the population dynamics of the above auxiliary problem (3.2) will be very useful in the establishment of our main results: Theorems 1.2 and 1.3 (see Sections 4 and 5). So, we study it here separately.

Our main goal is to investigate how the proportional constant $k$ influences the global dynamics of system (3.2) and the conclusion can be stated as follows.

**Theorem 3.1** *Given $d_1, \alpha_1 > 0$ and $0 < k \neq 1$. Assume that $r(x)$ is not in the form of $r(x) = C_0 \cdot e^{\frac{\alpha_1}{d_1}x}$ for some positive constant $C_0$. Then the following statements are valid:*

(1) *If $k > 1$, then $(\widetilde{u}, 0)$ is g.a.s;*

(2) *If $0 < k < 1$, then $(0, \widetilde{v})$ is g.a.s.*

We note here that if $r(x) = C_0 \cdot e^{\frac{\alpha_1}{d_1}x}$ for some positive constant $C_0$, then there will be the so-called ideal free distribution and the system is degenerate in the sense that there is a continuum of positive steady states $\big(k_1 \cdot e^{\frac{\alpha_1}{d_1}x}, k_2 \cdot e^{\frac{\alpha_1}{d_1}x}\big)$ with $k_1 + k_2 = C_0$, which consists of the global attractor. See [3, 4, 5] and references therein.

Our main approach to prove the above theorem is to use the concavity of the principal eigenvalue with respect to $k$. So, we go first to consider the following two linear eigenvalue problems:

$$\begin{cases} -k\Big[d_1\zeta_{xx} - \alpha_1\zeta_x\Big] - h(x)\zeta = \mu\zeta, & 0 < x < L, \\ d_1\zeta_x(0) - \alpha_1\zeta(0) = d_1\zeta_x(L) - \alpha_1\zeta(L) = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} -\Big[d_1\vartheta_{xx} - \alpha_1\vartheta_x\Big] - \tau h(x)\vartheta = \nu\vartheta, & 0 < x < L, \\ d_1\vartheta_x(0) - \alpha_1\vartheta(0) = d_1\vartheta_x(L) - \alpha_1\vartheta(L) = 0, \end{cases} \tag{3.4}$$

where $k, \tau > 0$ and $h(x) \in C([0, L])$. Denote by $\mu_1(k)$ and $\nu_1(\tau)$, respectively, the principal eigenvalue of problems (3.3) and (3.4). Then we have

**Lemma 3.1** *Assume that $h(x)$ is nonconstant. Then $\nu_1(\tau)$ is strictly concave in $\tau$. Moreover, if $\nu_1(1) = 0$, then*

$$\nu_1(\tau) \begin{cases} > 0, & \textit{for} \quad 0 < \tau < 1, \\ < 0, & \textit{for} \quad \tau > 1. \end{cases} \tag{3.5}$$

*As a consequence, $\mu_1(k)$ is also concave in $k$ and*

$$\mu_1(k) \begin{cases} > 0, & \textit{for} \quad k > 1, \\ < 0, & \textit{for} \quad 0 < k < 1. \end{cases} \tag{3.6}$$

**Proof:** The concavity of $\nu_1$ in $\tau$ can be proved by standard arguments, see, e.g., [31]. The inequalities in (3.5) follow from the concavity and the fact $\nu_1(0) = \nu_1(1) = 0$. The concavity of $\mu_1$ in $k$ holds due to $\mu_1(k) = k \cdot \nu_1(\frac{1}{k})$ and $\mu_1''(k) = \frac{1}{k^3} \cdot \nu_1''(\frac{1}{k})$. The inequalities in (3.6) follow from (3.5) and $\mu_1(k) = k \cdot \nu_1(\frac{1}{k})$. $\square$

The above result is quite useful to determine the linear stability of semi-trivial steady states.

**Lemma 3.2** *Given $d_1, \alpha_1 > 0$ and $0 < k \neq 1$. Assume further that $r(x)$ is not in the form of $r(x) = C_0 e^{\frac{\alpha_1}{d_1}x}$ for some positive constant $C_0$. Then the following statements are true:*

(1) *If $k > 1$, then $(\widetilde{u}, 0)$ is linearly stable and $(0, \widetilde{v})$ is linearly unstable;*

(2) *If $0 < k < 1$, then $(\widetilde{u}, 0)$ is linearly unstable and $(0, \widetilde{v})$ is linearly stable.*

**Proof:** From the equation of $\widetilde{u}$, we see $\nu_1(1) = 0$, where $h(x)$ in (3.4) is chosen to be $r(x) - \widetilde{u}(x)$, which is nonconstant due to the assumption on $r(x)$. The desired results would then follow by an application of (3.6) in Lemma 3.1. $\square$

We can also employ the concavity of the principal eigenvalue to derive the following non-existence result.

**Lemma 3.3** *Given $d_1, \alpha_1 > 0$ and $0 < k \neq 1$. Assume further that $r(x)$ is not in the form of $r(x) = C_0 e^{\frac{\alpha_1}{d_1}x}$ for some positive constant $C_0$. Then system (3.2) has no co-existence steady state.*

**Proof:** Arguing indirectly, we suppose that there is a co-existence steady state denoted by $(u, v)$ which satisfies $u, v > 0$ in $[0, L]$ and

$$\begin{cases} 0 = d_1 u_{xx} - \alpha_1 u_x + u\Big[r(x) - u - v\Big], & 0 < x < L, \\ 0 = k\Big[d_1 v_{xx} - \alpha_1 v_x\Big] + v\Big[r(x) - u - v\Big], & 0 < x < L, \\ d_1 u_x(x) - \alpha_1 u(x) = 0, & x = 0, L, \\ d_1 v_x(x) - \alpha_1 v(x) = 0, & x = 0, L. \end{cases} \tag{3.7}$$

If $r(x) - u - v$ is nonconstant, then the first equation in (3.7) implies $\nu_1(1) = 0$, where $h(x)$ in (3.4) now is chosen as $r(x) - u - v$. By Lemma 3.1, $\mu_1(k) \neq 0$ for any given $0 < k \neq 1$, which causes a contradiction with the second equation of (3.7). Thus $r(x) - u - v \equiv C_1$ in $[0, L]$ for some constant $C_1$. A direct integration of the first equation in (3.7) over $(0, L)$ yields $C_1 = 0$, that is, $r(x) - u - v \equiv 0$ in $[0, L]$. Substituting this fact into (3.7) and using the boundary conditions, we find

$$\begin{cases} d_1 u_x - \alpha_1 u \equiv 0, & 0 < x < L, \\ d_1 v_x - \alpha_1 v \equiv 0, & 0 < x < L, \end{cases}$$

i.e.,

$$\begin{cases} u(x) = u(0) \cdot e^{\frac{\alpha_1}{d_1}x}, & 0 < x < L, \\ v(x) = v(0) \cdot e^{\frac{\alpha_1}{d_1}x}, & 0 < x < L. \end{cases}$$

Hence, $r(x) = u + v = \Big[u(0) + v(0)\Big] \cdot e^{\frac{\alpha_1}{d_1}x}$, contradicting our assumption. Therefore, there is no co-existence steady state. $\square$

Finally, we present the proof of Theorem 3.1 as follows.

**Proof of Theorem 3.1**: In view of the theory of abstract competitive systems developed in [18, 19], this theorem follows directly from Lemmas 3.2 and 3.3. □

# 4 Stability of semi-trivial steady states

This section is devoted to the proof of Theorem 1.2, which will be displayed in subsections 4.1 and 4.2 (two parts). Also, we exhibit some other sufficient conditions for the local stability in subsection 4.3 that will be used in Section 5.

Throughout this section, we always assume that $(H)$ holds and $0 < d_1 < d_2$ even it is not mentioned explicitly.

To analyze the linear stability of two semi-trivial steady states $(\widetilde{u}, 0)$ and $(0, \widetilde{v})$ (the existence of such solutions is due to the positivity of $r(x)$, see [37, Corollary 2.1]), we introduce the following auxiliary linear eigenvalue problem:

$$\begin{cases} \Big[d\varphi_x - \alpha\varphi\Big]_x + \eta(x)\varphi + \lambda\varphi = 0, & 0 < x < L, \\ d\varphi_x(0) - \alpha\varphi(0) = d\varphi_x(L) - \alpha\varphi(L) = 0, & \end{cases} \tag{4.1}$$

where $d, \alpha > 0$ and $\eta(x) \in C([0, L])$. It is well-known (see, e.g., [21, 32]) that problem (4.1) admits a principal eigenvalue, denoted by $\lambda_1 = \lambda_1(d, \alpha, \eta)$, which is simple, and the corresponding eigenfunction, denoted by $\varphi_1$, can be chosen strictly positive in $[0, L]$. Moreover, the linear stabilities of $(\widetilde{u}, 0)$ and $(0, \widetilde{v})$ are determined, respectively, by the sign of $\lambda_1(d_2, \alpha_2, r - \widetilde{u})$ and $\lambda_1(d_1, \alpha_1, r - \widetilde{v})$. Specifically, $(\widetilde{u}, 0)$ is linearly stable (resp. linearly unstable) if $\lambda_1(d_2, \alpha_2, r - \widetilde{u}) > 0$ (resp. $< 0$); $(0, \widetilde{v})$ is linearly stable (resp. linearly unstable) if $\lambda_1(d_1, \alpha_1, r - \widetilde{v}) > 0$ (resp. $< 0$).

## 4.1 Proof of Theorem 1.2 (1)

We divide the proof of Theorem 1.2 (1) into two steps. Step 1: the stability of $(\widetilde{u}, 0)$ (see Lemma 4.6), which is based on Lemmas 3.1-3.5. Step 2: estimate of $\alpha_2^{**}$ (see Lemma 4.13), which is based on Lemmas 3.7-3.12.

**Lemma 4.1** *If* $\alpha_1 \geq 2\sqrt{r(0)d_1}$, *then* $\widetilde{u}_x > 0$ *in* $[0, L]$.

**Proof:** The proof is similar to that of the claim in Theorem 1.1 (2).

Set $\widetilde{u} = e^{\frac{\alpha_1}{2d_1}x} w$. Then we find

$$\begin{cases} d_1 w_{xx} + \Big[r - \widetilde{u} - \frac{\alpha_1^2}{4d_1}\Big] w = 0, & 0 < x < L, \\ d_1 w_x = \frac{1}{2}\alpha_1 w, & x = 0, L. \end{cases} \tag{4.2}$$

By assumption (H) and $\alpha_1 \geq 2\sqrt{r(0)d_1}$, $w_{xx} > 0$ in $(0, L)$. Note $w_x(0) = \frac{1}{2}\frac{\alpha_1}{d_1}w(0) > 0$, so $w_x > 0$ in $[0, L]$, that is, $\widetilde{u}_x = e^{\frac{\alpha_1}{2d_1}x}\Big[w_x + \frac{\alpha_1}{2d_1}w\Big] > 0$ in $[0, L]$. □

**Lemma 4.2** *If $\alpha_1 \geq 2\sqrt{r(0)d_1}$, then $\int_0^L \widetilde{u}(x)dx < \int_0^L r(x)dx$.*

**Proof:** Dividing the equation of $\widetilde{u}$ by $\widetilde{u}$ and integrating the resulting equation over $(0, L)$, we arrive at

$$\int_0^L \widetilde{u}(x)\mathrm{d}x - \int_0^L r(x)\mathrm{d}x = d_1 \int_0^L \left[\frac{\widetilde{u}_x}{\widetilde{u}} - \frac{\alpha_1}{d_1}\right] \cdot \frac{\widetilde{u}_x}{u}\mathrm{d}x. \tag{4.3}$$

Set $p := \frac{\widetilde{u}_x}{\widetilde{u}}$. Then by some straightforward computations, $p$ satisfies

$$\begin{cases} -d_1 p_{xx} + \Big[\alpha_1 - 2d_1 p\Big] p_x + \widetilde{u}p = r'(x), & 0 < x < L, \\ p(0) = p(L) = \frac{\alpha_1}{d_1} > 0. \end{cases}$$

By assumption (H) and the maximum principle,

$$p := \frac{\widetilde{u}_x}{\widetilde{u}} < \frac{\alpha_1}{d} \quad \text{in} \quad (0, L), \tag{4.4}$$

which, together with Lemma 4.1, implies that the right side of (4.3) is strictly negative, and so the desired result follows. □

**Lemma 4.3** *If $\alpha_1 \geq 2\sqrt{r(0)d_1}$, then $\frac{\partial \lambda_1}{\partial \alpha_2} > 0$ for $\alpha_2 > 0$, where $\lambda_1 = \lambda_1(d_2, \alpha_2, r - \widetilde{u})$.*

**Proof:** Recall the eigenfunction $\varphi_1$ corresponding to $\lambda_1$ and introduce the transformation $\psi_1 = e^{-\frac{\alpha_2}{d_2}x}\varphi_1$. Then $\psi_1$ satisfies

$$\begin{cases} d_2 \psi_{1xx} + \alpha_2 \psi_{1x} + \big(r - \widetilde{u}\big)\psi_1 + \lambda_1 \psi_1 = 0, & 0 < x < L, \\ \psi_{1x} = 0, & x = 0, L. \end{cases} \tag{4.5}$$

We claim that $\psi_{1x} < 0$ in $(0, L)$.

Multiplying (4.5) by $e^{\frac{\alpha_2}{d_2}x}$ and integrating the result over $(0, L)$, we obtain

$$\int_0^L e^{\frac{\alpha_2}{d_2}x} \psi_1 \Big[r - \widetilde{u} + \lambda_1\Big] \mathrm{d}x = 0. \tag{4.6}$$

Since $r_x \leq, \not\equiv 0$ and $\widetilde{u}_x > 0$ (Lemma 4.1) in $[0, L]$, we see that $r - \widetilde{u} + \lambda_1$ changes sign exactly once, say, at $x = x^* \in (0, L)$, and

$$r - \widetilde{u} + \lambda_1 \begin{cases} > 0, & \text{for} \quad x \in [0, x^*), \\ < 0, & \text{for} \quad x \in (x^*, L). \end{cases}$$

that is,

$$\Big[e^{\frac{\alpha_2}{d_2}x} \psi_{1x}\Big]_x \begin{cases} < 0, & \text{for} \quad x \in [0, x^*), \\ > 0, & \text{for} \quad x \in (x^*, L), \end{cases}$$

As ${\psi_1}_x = 0$ at $x = 0, L$, we have $e^{\frac{\alpha_2}{d_2}x}{\psi_1}_x < 0$ in $(0, L)$, i.e., ${\psi_1}_x < 0$ in $(0, L)$. Hence, the above claim is verified.

Now, differentiating equation (4.5) with respect to $\alpha_2$, we get

$$\begin{cases} d_2{\psi_1}'_{xx} + \alpha_2{\psi_1}'_x + {\psi_1}_x + (r - \widetilde{u}){\psi_1}' + \lambda_1{\psi_1}' + \lambda_1'\psi_1 = 0, & 0 < x < L, \\ {\psi_1}'_x = 0, & x = 0, L, \end{cases} \tag{4.7}$$

where the prime notation denotes differentiating with respect to $\alpha_2$. Multiplying (4.7) by $e^{\frac{\alpha_2}{d_2}x}\psi_1$ and (4.5) by $e^{\frac{\alpha_2}{d_2}x}{\psi_1}'$, subtracting the results and integrating in $(0, L)$, we obtain

$$\lambda_1' \;=\; -\frac{\int_0^L e^{\frac{\alpha_2}{d_2}x}\psi_1{\psi_1}_x \mathrm{d}x}{\int_0^L e^{\frac{\alpha_2}{d_2}x}{\psi_1}^2 \mathrm{d}x} > 0, \tag{4.8}$$

where the above claim is used. This completes the proof. $\square$

**Lemma 4.4** *If $\alpha_1 \ge 2\sqrt{r(0)d_1}$, then $\lambda_1 = \lambda_1(d_2, \alpha_2, r - \widetilde{u})|_{\alpha_2 = 0} < 0$, i.e., $(\widetilde{u}, 0)$ is linearly unstable.*

**Proof:** When $\alpha_2 = 0$, $\lambda_1 = \lambda_1(d_2, \alpha_2, r - \widetilde{u})$ satisfies

$$\begin{cases} d_2{\varphi_1}_{xx} + (r - \widetilde{u})\varphi_1 + \lambda_1\varphi_1 = 0, & 0 < x < L, \\ {\varphi_1}_x(0) = {\varphi_1}_x(L) = 0. \end{cases} \tag{4.9}$$

Dividing equation (4.9) by $\varphi_1$ and integrating the result over $(0, L)$, we find

$$\lambda_1 = -\frac{d_2}{L}\int_0^L \frac{{\varphi_1}_x^2}{{\varphi_1}^2}\mathrm{d}x - \frac{1}{L}\int_0^L \big(r - \widetilde{u}\big)\mathrm{d}x < 0,$$

due to Lemma 4.2. $\square$

**Lemma 4.5** *Given $d_1, d_2, \alpha_1 > 0$. Then for $\alpha_2 > 0$ large, $(\widetilde{u}, 0)$ is linearly stable.*

**Proof:** We establish this result under a weaker condition than (H): $\frac{r_x}{r} < \frac{\alpha_1}{d_1}$ in $[0, L]$. The essential idea is to analyze the behavior of $\widetilde{u}$ at the boundary $x = L$.

We first claim that if $\frac{r_x}{r} < \frac{\alpha_1}{d_1}$ in $[0, L]$, then $\widetilde{u}(L) > r(L)$.

Suppose for contradiction that $\widetilde{u}(L) \le r(L)$. Let $w = e^{-\frac{\alpha_1}{d_1}x}\widetilde{u}$. Then $w$ satisfies

$$\begin{cases} d_1 w_{xx} + \alpha_1 w_x + e^{\frac{\alpha_1}{d_1}x} w\Big[e^{-\frac{\alpha_1}{d_1}x} r(x) - w\Big] = 0, & 0 < x < L, \\ w_x(0) = w_x(L) = 0. \end{cases}$$

Define

$$h(x) := w(x) - e^{-\frac{\alpha_1}{d_1}x} r(x) = e^{-\frac{\alpha_1}{d_1}x}\Big[\widetilde{u}(x) - r(x)\Big] \quad \text{for} \quad x \in [0, L].$$

Clearly, $h(L) \leq 0$. Since $\frac{r_x}{r} < \frac{\alpha_1}{d_1}$ in $[0, L]$, we have

$$h'(L) = w'(L) - e^{-\frac{\alpha_1}{d_1}L}\Big[r'(L) - \frac{\alpha_1}{d_1}r(L)\Big] > 0,$$

which implies

$$h(x) < 0 \ \text{ in } \ (L-\delta, L) \ \text{ for some } \ \delta > 0.$$

This leads us to deduce from the equation of $w$ that

$$d_1 w_{xx} + \alpha_1 w_x = d_1 e^{-\frac{\alpha_1}{d_1}x}\Big[e^{\frac{\alpha_1}{d_1}x} w_x\Big]_x < 0 \ \text{ in } \ (L-\delta, L),$$

which, together with $w_x(L) = 0$, further yields

$$w_x(x) > 0 \ \text{ in } \ (L-\delta, L).$$

Since $w_x(0) = 0$, we can choose $x_0 \in [0, L)$ such that

$$w_x(x_0) = w_x(L) = 0, \ \ w_x(x) > 0 \ \text{ in } \ (x_0, L),$$

and consequently,

$$w_{xx}(x_0) \geq 0 \ \text{ and } \ h_x(x) = w_x(x) - e^{-\frac{\alpha_1}{d_1}x}\Big[r_x(x) - \frac{\alpha_1}{d_1}r(x)\Big] > 0 \ \text{ in } \ [x_0, L],$$

which, in view of $h(L) \leq 0$, implies $h(x_0) < 0$. Evaluating the equation of $w$ at the position $x = x_0$, we observe

$$0 \leq \Big[d_1 w_{xx}\Big]\Big|_{x=x_0} = -\Big\{e^{\frac{\alpha_1}{d_1}x} w\Big[e^{-\frac{\alpha_1}{d_1}x} r(x) - w\Big]\Big\}\Big|_{x=x_0} < 0.$$

This contradiction confirms the above claim.

On the other hand, it follows from [6] that

$$\lambda_1(d_2, \alpha_2, r - \widetilde{u}) \to \widetilde{u}(L) - r(L) \quad \text{as} \quad \alpha_2 \to \infty.$$

By the virtue of the above claim, the desired result holds. □

**Lemma 4.6** *If $\alpha_1 \geq 2\sqrt{r(0)d_1}$, then there is a positive number $\alpha_2^{**}$ such that $(\widetilde{u}, 0)$ is linearly stable for $\alpha_2 > \alpha_2^{**}$ and linearly unstable for $\alpha_2 < \alpha_2^{**}$.*

**Proof:** The existence of $\alpha_2^{**}$ directly follows from Lemmas 4.3, 4.4 and 4.5. □

In what follows, we make some preparations for determining the asymptotic behaviors of $\alpha_2^{**}$ in the sense of $\alpha_1 \to \infty$. The proof of the following result adopts some idea from [7, Lemma 3.2].

**Lemma 4.7** *If* $\alpha_1 \geq 2\sqrt{r(0)d_1}$, *then*

$$e^{\left(\frac{\alpha_1}{d_1}+\frac{\widetilde{u}(L)}{\alpha_1}\right)\left(x-L\right)} \;\leq\; \frac{\widetilde{u}(x)}{\widetilde{u}(L)} \;\leq\; e^{\left(\frac{\alpha_1}{d_1}-\frac{2r(0)}{\alpha_1}\right)\left(x-L\right)} \quad \textit{for} \;\; x\in[0,L]. \tag{4.10}$$

**Proof:** Set $\overline{u}(x) := \widetilde{u}(L)e^{\left(\frac{\alpha_1}{d_1}-\frac{2r(0)}{\alpha_1}\right)\left(x-L\right)}$. By direct calculations,

$$d_1\overline{u}_x - \alpha_1\overline{u} = \widetilde{u}(L)\cdot\frac{-2r(0)d_1}{\alpha_1}\cdot e^{\left(\frac{\alpha_1}{d_1}-\frac{2r(0)}{\alpha_1}\right)\left(x-L\right)},$$

which particularly implies $\Big[d_1\overline{u}_x - \alpha_1\overline{u}\Big]\Big|_{x=0} \leq 0$. Moreover,

$$\begin{aligned}
\Big[d_1\overline{u}_x - \alpha_1\overline{u}\Big]_x + \overline{u}\Big(r-\widetilde{u}\Big) &= \widetilde{u}(L)e^{\left(\frac{\alpha_1}{d_1}-\frac{2r(0)}{\alpha_1}\right)\left(x-L\right)}\Big[r(x)-2r(0)+\frac{4r^2(0)d_1}{\alpha_1^2}-\widetilde{u}\Big]\\
&< \widetilde{u}(L)e^{\left(\frac{\alpha_1}{d_1}-\frac{2r(0)}{\alpha_1}\right)\left(x-L\right)}\Big[-r(0)+\frac{4r^2(0)d_1}{\alpha_1^2}\Big] \;\; (\text{as } r(x)\leq r(0))\\
&\leq 0 \quad (\text{as } \alpha_1 \geq 2\sqrt{r(0)d_1}).
\end{aligned}$$

In summary, $\overline{u}(x)$ satisfies

$$\begin{cases} d_1\overline{u}_{xx} - \alpha_1\overline{u}_x + \overline{u}\Big[r-\widetilde{u}\Big] \leq 0, & 0<x<L,\\ d_1\overline{u}_x(0) - \alpha_1\overline{u}(0) \leq 0, \quad \overline{u}(L)=\widetilde{u}(L). \end{cases}$$

Now, let us set $w = \overline{u}(x) - \widetilde{u}(x)$, which satisfies

$$\begin{cases} d_1w_{xx} - \alpha_1w_x + w\Big[r-\widetilde{u}\Big] \leq 0, & 0<x<L,\\ d_1w_x(0) - \alpha_1w(0) \leq 0, \quad w(L)=0. \end{cases}$$

By a further transformation $w = e^{\frac{\alpha_1}{2d_1}x}z$, we find

$$\begin{cases} d_1z_{xx} + \Big[r-\widetilde{u}-\frac{\alpha_1^2}{4d_1}\Big]z \leq 0, & 0<x<L,\\ d_1z_x(0) - \frac{\alpha_1}{2}z(0) \leq 0, \quad z(L)=0. \end{cases} \tag{4.11}$$

Since $r-\widetilde{u}-\frac{\alpha_1^2}{4d_1} < r(0)-\frac{\alpha_1^2}{4d_1} \leq 0$, by the maximum principle,

$$z \geqslant \min\{z(0), z(L), 0\} = \min\{z(0), 0\} \quad \text{in} \quad [0,L]. \tag{4.12}$$

We claim that $z(0) \geqslant 0$. Otherwise, if $z(0)<0$, then $z$ attains the minimum at $x=0$, which implies that $z_x(0)\geq 0$. However, it follows from the boundary condition at $x=0$ that $d_1z_x(0) \leq (\alpha_1/2)z(0) < 0$, which is a contradiction. Thus, our assertion holds. Combining this claim with (4.12), we have

$$z \geq 0 \quad \text{in} \quad [0,L],$$

that is,

$$\overline{u}(x) \geq \widetilde{u}(x) \quad \text{in} \quad [0, L].$$

This proves the second inequality in (4.10).

Next, set $\underline{u} := \widetilde{u}(L)e^{\left(\frac{\alpha_1}{d_1}+\frac{\widetilde{u}(L)}{\alpha_1}\right)\left(x-L\right)}$. Then, for any $x \in [0, L]$, we have

$$d_1\underline{u}_x - \alpha_1\underline{u} = \underline{u} \cdot \frac{d_1\widetilde{u}(L)}{\alpha_1},$$

and

$$\begin{aligned}\Big[d_1\underline{u}_x - \alpha_1\underline{u}\Big]_x + \underline{u}\Big(r - \widetilde{u}\Big) &= \underline{u}\Big[\widetilde{u}(L) + \frac{d_1\widetilde{u}^2(L)}{\alpha_1^2} + r(x) - \widetilde{u}(x)\Big] \\ &> 0 \quad (\text{as } \widetilde{u}(L) \geq \widetilde{u}(x)),\end{aligned}$$

which allow us to use the same arguments as above to prove $\widetilde{u}(x) - \underline{u}(x) \geq 0$ in $[0, L]$, that is, the first inequality in (4.10) holds. $\square$

**Lemma 4.8** *If $\alpha_1 \geq 2\sqrt{r(0)d_1}$, then as $\alpha_1 \to \infty$, $\widetilde{u}(L) \to 2r(L)$.*

**Proof:** We first illustrate that $\widetilde{u}(L)$ is uniformly bounded for $\alpha_1 \gg 1$. A direction integration of the equation of $\widetilde{u}$ gives

$$\int_0^L \widetilde{u}^2(x)\mathrm{d}x \ = \ \int_0^L \widetilde{u}(x)r(x)\mathrm{d}x. \tag{4.13}$$

By Lemma 4.7,

$$\widetilde{u}(L)\int_0^L e^{2\left(\frac{\alpha_1}{d_1}+\frac{\widetilde{u}(L)}{\alpha_1}\right)\left(x-L\right)}\mathrm{d}x \ \leq \ \int_0^L r(x)e^{\left(\frac{\alpha_1}{d_1}-\frac{2r(0)}{\alpha_1}\right)\left(x-L\right)}\mathrm{d}x,$$

which, in view of $r(x) \leq r(0)$ for all $x \in [0, L]$, implies

$$\widetilde{u}(L)\frac{1 - e^{2\left(\frac{\alpha_1}{d_1}+\frac{\widetilde{u}(L)}{\alpha_1}\right)\left(-L\right)}}{2\left(\frac{\alpha_1}{d_1} + \frac{\widetilde{u}(L)}{\alpha_1}\right)} \ \leq \ \frac{r(0)}{\frac{\alpha_1}{d_1} - \frac{2r(0)}{\alpha_1}}.$$

Clearly, for sufficiently large $\alpha_1$, we have $1 - e^{2\left(\frac{\alpha_1}{d_1}+\frac{\widetilde{u}(L)}{\alpha_1}\right)\left(-L\right)} \geq \frac{1}{2}$. Hence, we can further deduce from the above inequality that for sufficiently large $\alpha_1$,

$$\begin{aligned}\widetilde{u}(L) &\leq 4r(0) \cdot \frac{\frac{\alpha_1}{d_1} + \frac{\widetilde{u}(L)}{\alpha_1}}{\frac{\alpha_1}{d_1} - \frac{2r(0)}{\alpha_1}} \\ &\leq 8r(0) \cdot \Big[1 + \frac{d_1}{\alpha_1^2}\widetilde{u}(L)\Big] \quad (\text{as } \frac{2r(0)}{\alpha_1} \leq \frac{1}{2}\frac{\alpha_1}{d_1}).\end{aligned}$$

Therefore,

$$\widetilde{u}(L) \le \frac{8r(0)}{1-8r(0)\frac{d_1}{\alpha_1^2}} \le 16r(0), \quad \text{for } \alpha_1 \gg 1.$$

Next, we show the desired limit. By introducing a new variable $y = \frac{\alpha_1}{d_1}(L-x)$, we observe from (4.13) that

$$\widetilde{u}(L)\int_0^{\frac{\alpha_1}{d_1}L}\Big[\frac{\widetilde{u}(L-\frac{d_1}{\alpha_1}y)}{\widetilde{u}(L)}\Big]^2 \mathrm{d}y = \int_0^{\frac{\alpha_1}{d_1}L} r(L-\frac{d_1}{\alpha_1}y)\frac{\widetilde{u}(L-\frac{d_1}{\alpha_1}y)}{\widetilde{u}(L)}\mathrm{d}y. \tag{4.14}$$

Also, the inequality in Lemma 4.7 can be written as

$$e^{-y\Big[1+\frac{\widetilde{u}(L)d_1}{\alpha_1^2}\Big]} \;\le\; \frac{\widetilde{u}(L-\frac{d_1}{\alpha_1}y)}{\widetilde{u}(L)} \;\le\; e^{-y\Big[1-\frac{2r(0)d_1}{\alpha_1^2}\Big]}, \quad y\in[0,\frac{\alpha_1}{d_1}L], \tag{4.15}$$

which, together with the above proved fact, implies that

$$\frac{\widetilde{u}(L-\frac{d_1}{\alpha_1}y)}{\widetilde{u}(L)} \to e^{-y} \quad \text{pointwisely as} \quad \alpha_1\to\infty.$$

Passing to the limit in (4.14), we have

$$\begin{aligned}\lim_{\alpha_1\to\infty}\widetilde{u}(L) &= \frac{\int_0^\infty r(L)e^{-y}\mathrm{d}y}{\int_0^\infty e^{-2y}\mathrm{d}y}\\ &= 2r(L).\end{aligned}$$

We note here that thanks to the estimate in (4.15), the function $\frac{\widetilde{u}(L-\frac{d_1}{\alpha_1}y)}{\widetilde{u}(L)}$ is integrable in $(0,\infty)$ uniformly for large $\alpha_1$. This together with the uniform boundedness of $r(x)$ allows us to directly take the above limit. □

Now, we give the upper bound for $\alpha_2^{**}$.

**Lemma 4.9** *For $\alpha_1 \ge 2\sqrt{r(0)d_1}$, we always have*

$$\alpha_2^{**} \leqslant \frac{d_2}{d_1}\alpha_1. \tag{4.16}$$

**Proof:** Suppose for contradiction that $\alpha_2^{**} > \frac{d_2}{d_1}\alpha_1$. Then for $\alpha_2 = \frac{d_2}{d_1}\alpha_1$, by Lemma 4.6, $(\widetilde{u},0)$ is linearly unstable, but by Theorem 3.1 (1) (with $k$ there be $\frac{d_2}{d_1}>1$), $(\widetilde{u},0)$ is g.a.s, which, clearly, is impossible. So the inequality (4.16) holds. □

To establish the lower bound of $\alpha_2^{**}$, we adopt some idea from [8] and define an auxiliary function

$$F(\tau) = \int_0^L e^{\tau x}\Big[r(x)-\widetilde{u}(x)\Big]\mathrm{d}x, \quad \text{for any} \quad \tau\ge 0.$$

**Lemma 4.10** *If $\alpha_1 \geq 2\sqrt{r(0)d_1}$, then $F(\tau)$ has a unique positive root, denoted by $\tau^*$, and we have*

$$F(\tau)\begin{cases} > 0, & for \quad \tau \in [0, \tau^*), \\ < 0, & for \quad x \in (\tau^*, \infty). \end{cases}$$

**Proof:** By Lemma 4.1, $\widetilde{u}_x > 0$ in $[0, L]$, so $r - \widetilde{u}$ is strictly decreasing in $[0, L]$. Note that (4.13) can be written as $\int_0^L \widetilde{u}[r - \widetilde{u}]\mathrm{d}x = 0$, which implies that $r - \widetilde{u}$ changes sign exactly once, say, at $x = x^* \in (0, L)$. Hence, $r - \widetilde{u} > 0$ in $[0, x^*)$ and $r - \widetilde{u} < 0$ in $(x^*, L]$. This leads us to further deduce the following inequality

$$\begin{aligned} \frac{d}{d\tau}\Big(e^{-\tau x^*}F(\tau)\Big) &= \frac{d}{d\tau}\left\{\int_0^L e^{\tau(x-x^*)}\Big[r - \widetilde{u}\Big]\mathrm{d}x\right\} \\ &= \int_0^L e^{\tau(x-x^*)}\Big[r - \widetilde{u}\Big]\Big(x - x^*\Big)\mathrm{d}x \\ &< 0. \end{aligned}$$

So $e^{-\tau x^*}F(\tau)$ has at most one positive root, i.e., $F(\tau)$ has at most one positive root. Since $F(0) = \int_0^L \Big[r(x) - \widetilde{u}(x)\Big]\mathrm{d}x > 0$ (Lemma 4.2), it remains to show that $F(\tau) < 0$ for sufficiently large $\tau$. Notice that $\widetilde{u}(L) > r(L)$ always holds (due to their monotonicity and the fact that $r - \widetilde{u}$ changes sign), we then have

$$\begin{aligned} \lim_{\tau\to\infty} \tau e^{-\tau L}F(\tau) &= \lim_{\tau\to\infty} \tau \int_0^L e^{\tau(x-L)}\Big[r - \widetilde{u}\Big]\mathrm{d}x \\ &= r(L) - \widetilde{u}(L) \\ &< 0, \end{aligned}$$

where the second equality used integration by parts and some straightforward computations. Hence, $F(\tau) < 0$ for $\tau \gg 1$, as desired. □

**Lemma 4.11** *If $\alpha_1 \geq 2\sqrt{r(0)d_1}$, then $\alpha_2^{**} \geq \tau^* d_2$.*

**Proof:** Recall $\alpha_2^{**} > 0$ is uniquely determined in Lemma 4.6 and it satisfies

$$\begin{cases} d_2\varphi_{xx} - \alpha_2^{**}\varphi_x + (r - \widetilde{u})\varphi = 0, & 0 < x < L, \\ d_2\varphi_x - \alpha_2^{**}\varphi = 0, & x = 0, L, \end{cases}$$

for some positive function $\varphi$. Rewrite the above equation as

$$d_2\Big[e^{\frac{\alpha_2^{**}}{d_2}x}\big(e^{-\frac{\alpha_2^{**}}{d_2}x}\varphi\big)_x\Big]_x + (r - \widetilde{u})\varphi = 0, \quad 0 < x < L. \tag{4.17}$$

Dividing equation (4.17) by $e^{-\frac{\alpha_2^{**}}{d_2}x}\varphi$ and integrating the result over $(0, L)$, we see

$$d_2\int_0^L \frac{e^{\frac{\alpha_2^{**}}{d_2}x}\Big[\big(e^{-\frac{\alpha_2^{**}}{d_2}x}\varphi\big)_x\Big]^2}{\Big(e^{-\frac{\alpha_2^{**}}{d_2}x}\varphi\Big)^2}\mathrm{d}x + \int_0^L e^{\frac{\alpha_2^{**}}{d_2}x}\Big(r - \widetilde{u}\Big)\mathrm{d}x = 0,$$

which implies $\int_0^L e^{\frac{\alpha_2^{**}}{d_2}x}(r-\widetilde{u})\mathrm{d}x<0$, that is, $F(\frac{\alpha_2^{**}}{d_2})<0$. By Lemma 4.10, we must have $\frac{\alpha_2^{**}}{d_2}>\tau^*$. □

**Lemma 4.12** *The following estimate holds:*

$$\liminf_{\alpha_1\to\infty}\frac{\tau^*}{\alpha_1}\ \geq\ \frac{1}{d_1}.$$

**Proof:** We show that for any $\epsilon>0$,

$$F\big(\frac{\alpha_1}{d_1}(1-\epsilon)\big)>0,\quad \text{for}\quad \alpha_1\gg 1. \tag{4.18}$$

If this is true, by Lemma 4.10, $\tau^*>\frac{\alpha_1}{d_1}(1-\epsilon)$, i.e.,

$$\liminf_{\alpha_1\to\infty}\frac{\tau^*}{\alpha_1}\geq\frac{1}{d_1}(1-\epsilon).$$

The desired result would then follow due to the arbitrariness of $\epsilon$.

We next prove (4.18). Direct computations yield

$$\begin{aligned}
&\frac{\alpha_1}{d_1}\cdot F\Big(\frac{\alpha_1}{d_1}(1-\epsilon)\Big)\cdot e^{-\frac{\alpha_1}{d_1}\big(1-\epsilon\big)L}\\
=\ &\frac{\alpha_1}{d_1}\int_0^L e^{\frac{\alpha_1}{d_1}\big(1-\epsilon\big)\big(x-L\big)}\Big[r(x)-\widetilde{u}(x)\Big]\mathrm{d}x\\
\geq\ &\frac{\alpha_1}{d_1}\int_0^L e^{\frac{\alpha_1}{d_1}\big(1-\epsilon\big)\big(x-L\big)}\Big[r(x)-\widetilde{u}(L)e^{\big(\frac{\alpha_1}{d_1}-\frac{2r(0)}{\alpha_1}\big)\big(x-L\big)}\Big]\mathrm{d}x\\
=\ &\int_0^{\frac{\alpha_1}{d_1}L} e^{-\big(1-\epsilon\big)y}\Big[r\big(L-\frac{d_1}{\alpha_1}y\big)-\widetilde{u}(L)e^{-\big(1-\frac{2r(0)d_1}{\alpha_1^2}\big)y}\Big]\mathrm{d}y\quad \big(y=\frac{\alpha_1}{d_1}(L-x)\big)\\
\to\ &\frac{r(L)\epsilon}{(1-\epsilon)(2-\epsilon)}>0\quad(\text{as }\alpha_1\to\infty),
\end{aligned}$$

where the first inequality used Lemma 4.7 and the last step used Lemma 4.8. Hence, (4.18) holds true. □

**Lemma 4.13** *The following limits hold:*

$$\lim_{\alpha_1\to\infty}\frac{\alpha_2^{**}}{\alpha_1}=\frac{d_2}{d_1}. \tag{4.19}$$

**Proof:** By Lemmas 4.9 and 4.11,

$$\frac{\tau^* d_2}{\alpha_1}\leq\frac{\alpha_2^{**}}{\alpha_1}\leqslant\frac{d_2}{d_1},$$

which, in view of Lemma 4.12, immediately implies (4.19). □

**Proof of Theorem 1.2 (1)**: This statement follows directly from Lemmas 4.6 and 4.13. □

## 4.2 Proof of Theorem 1.2 (2)

Similar to the previous subsection, here we first discuss the stability of $(0,\widetilde{v})$ (see Lemma 4.14), then make proper estimates on $\alpha_1^{**}$ (see Lemma 4.18), and finally prove Theorem 1.2 (2).

Arguing in the same manner as in Lemma 4.6, one can obtain the following result:

**Lemma 4.14** *If $\alpha_2 \geq 2\sqrt{r(0)d_2}$, then there is a positive number $\alpha_1^{**}$ such that $(0,\widetilde{v})$ is linearly stable for $\alpha_1 > \alpha_1^{**}$ and linearly unstable for $\alpha_1 < \alpha_1^{**}$.*

Next, we go to estimate $\alpha_1^{**}$. We first display a lower bound for $\alpha_1^{**}$.

**Lemma 4.15** *For any $\alpha_2 \geq 2\sqrt{r(0)d_2}$, we have $\alpha_1^{**} \geq \frac{d_1}{d_2}\cdot\alpha_2$.*

**Proof:** Arguing indirectly, we suppose $\alpha_1^{**} < \frac{d_1}{d_2}\cdot\alpha_2$ for $\alpha_2 \geq 2\sqrt{r(0)d_2}$. Then for $\alpha_1 = \frac{d_1}{d_2}\cdot\alpha_2$, by Lemma 4.14, $(0,\widetilde{v})$ is linearly stable, but according to Theorem 3.1 (1), $(\widetilde{u},0)$ is g.a.s, which causes a contradiction. □

Note that $\alpha_1^{**}$ determined in Lemma 4.14 satisfies the following equation

$$\begin{cases} d_1\varphi_{xx} - \alpha_1^{**}\varphi_x + \varphi(r-\widetilde{v}) = 0, & 0<x<L, \\ d_1\varphi_x - \alpha_1^{**}\varphi = 0,, & x = 0, L, \end{cases} \tag{4.20}$$

where $\varphi > 0$ in $[0,L]$ and is normalized by $\varphi(L) = 1$. We then have the following estimate for $\varphi$.

**Lemma 4.16** *For sufficiently large $\alpha_2$, we have*

$$e^{\left(\frac{\alpha_1^{**}}{d_1}+\frac{\widetilde{v}(L)}{\alpha_1^{**}}\right)\left(x-L\right)} \;\leq\; \varphi(x) \;\leq\; e^{\left(\frac{\alpha_1^{**}}{d_1}-\frac{2r(0)}{\alpha_1^{**}}\right)\left(x-L\right)} \quad \textit{for } x\in[0,L]. \tag{4.21}$$

**Proof:** This lemma can be justified by using the same arguments as in Lemma 4.7. We note here that when one goes to check the upper and lower bounds given in (4.21) are, respectively, super- and sub-solution of system (4.20), keep in mind two things: (1) $\alpha_1^{**}\to\infty$ if $\alpha_2\to\infty$ (Lemma 4.15); (2) for $\alpha_2 \geq 2\sqrt{r(0)d_2}$, one can prove as before that $\widetilde{v}$ is strictly increasing in $(0,L)$ and so $\widetilde{v}(L)$ is the global maximum. □

**Lemma 4.17** $\forall\ \epsilon > 0$, *for sufficiently large $\alpha_2$,*

$$\frac{\alpha_1^{**}}{\alpha_2} \;\leq\; \frac{d_1}{d_2}(1+\epsilon).$$

**Proof:** We use the contradiction argument. Suppose that there is $\epsilon_0 > 0$ such that $\frac{\alpha_1^{**}}{\alpha_2} > \frac{d_1}{d_2}(1+\epsilon_0)$. A direction integration of (4.20) in $(0,L)$ produces $\int_0^L \varphi(r-\widetilde{v})\mathrm{d}x = 0$. By a variable transformation $\frac{\alpha_1^{**}}{d_1}(L-x) = y$, we have

$$\int_0^{\frac{\alpha_1^{**}}{d_1}L} \varphi\big(L-\frac{d_1}{\alpha_1^{**}}y\big)r\big(L-\frac{d_1}{\alpha_1^{**}}y\big)\mathrm{d}y = \int_0^{\frac{\alpha_1^{**}}{d_1}L} \widetilde{v}\big(L-\frac{d_1}{\alpha_1^{**}}y\big)\varphi\big(L-\frac{d_1}{\alpha_1^{**}}y\big)\mathrm{d}y. \tag{4.22}$$

Also, inequality (4.21) in Lemma 4.16 can be written as

$$e^{-\left[1+\frac{\widetilde{v}(L)d_1}{(\alpha_1^{**})^2}\right]y} \le \varphi(L-\frac{d_1}{\alpha_1^{**}}y) \le e^{-\left[1-\frac{2r(0)d_1}{(\alpha_1^{**})^2}\right]y}. \tag{4.23}$$

In addition, we can apply the same idea as in Lemma 4.7 to prove the following inequality for $\widetilde{v}$

$$e^{\left(\frac{\alpha_2}{d_2}+\frac{\widetilde{v}(L)}{\alpha_2}\right)\left(x-L\right)} \le \frac{\widetilde{v}(x)}{\widetilde{v}(L)} \le e^{\left(\frac{\alpha_2}{d_2}-\frac{2r(0)}{\alpha_2}\right)\left(x-L\right)} \quad \text{for } x\in[0,L],$$

which, in term of $y$-variable, becomes

$$e^{-y\left[\frac{\alpha_2 d_1}{d_2\alpha_1^{**}}+\frac{\widetilde{v}(L)d_1}{\alpha_2\alpha_1^{**}}\right]} \le \frac{\widetilde{v}(L-\frac{d_1}{\alpha_1^{**}}y)}{\widetilde{v}(L)} \le e^{-y\left[\frac{\alpha_2 d_1}{d_2\alpha_1^{**}}-\frac{2r(0)d_1}{\alpha_2\alpha_1^{**}}\right]}, \tag{4.24}$$

for $y\in[0,\frac{\alpha_1^{**}}{d_1}L]$. By our assumption $\frac{\alpha_1^{**}}{\alpha_2} > \frac{d_1}{d_2}(1+\epsilon_0)$, passing to a subsequence if necessary, we may assume that

$$\text{as} \quad \alpha_2\to\infty, \qquad \frac{\alpha_2 d_1}{\alpha_1^{**}d_2} \quad\to\quad \delta\in[0,\frac{1}{1+\epsilon_0}]. \tag{4.25}$$

Moreover, based on the above estimates in (4.23) and (4.24), we can take a limit in (4.22) to obtain

$$\int_0^\infty r(L)e^{-y}\mathrm{d}y = \int_0^\infty 2r(L)e^{-(1+\delta)y}\mathrm{d}y, \tag{4.26}$$

where the fact $\lim_{\alpha_2\to\infty}\widetilde{v}(L) = 2r(L)$ (similar to Lemma 4.8) is used. By a direct computation, we can derive from (4.26) that $\delta = 1$, which contradicts (4.25). □

**Lemma 4.18** *As* $\alpha_2\to\infty$, $\frac{\alpha_1^{**}}{\alpha_2}\to\frac{d_1}{d_2}$.

**Proof:** By Lemmas 4.15 and 4.17, we have for any $\epsilon>0$,

$$\frac{d_1}{d_2} \le \frac{\alpha_1^{**}}{\alpha_2} \le \frac{d_1}{d_2}(1+\epsilon), \quad \forall\alpha_2\gg1,$$

which implies the desired result due to the arbitrariness of $\epsilon$. □

**Proof of Theorem 1.2 (2)**: This statement follows directly from Lemmas 4.14 and 4.18. □

### 4.3 Other sufficient conditions for stability

In this subsection, we aim to give several types of sufficient conditions for the stability of semi-trivial steady states, which are useful in the proof of Theorem 1.3.

We first present, as a complement of Lemma 4.5, the linear stability of $(0, \widetilde{v})$ under the effect of strong advection.

**Lemma 4.19** *Given $d_1, d_2, \alpha_1 > 0$. Then for $\alpha_2 > 0$ large, $(0, \widetilde{v})$ is always linearly unstable.*

**Proof:** Clearly, we need to determine the sign of $\lambda_1 = \lambda_1(d_1, \alpha_1, r - \widetilde{v})$. Let $\psi_1 = e^{-\frac{\alpha_1}{d_1}x}\varphi_1$. Then $\psi_1$ satisfies

$$\begin{cases} d_1\Big[e^{\frac{\alpha_1}{d_1}x}\psi_{1x}\Big]_x + \Big[r - \widetilde{v}\Big]e^{\frac{\alpha_1}{d_1}x}\psi_1 + \lambda_1 e^{\frac{\alpha_1}{d_1}x}\psi_1 = 0, \quad 0 < x < L, \\ \psi_{1x}(0) = \psi_{1x}(L) = 0. \end{cases}$$

Dividing the above equation by $\psi_1$ and then integrating over $(0, L)$, we finally attain

$$d_1 \int_0^L \frac{e^{\frac{\alpha_1}{d_1}x}\psi_{1x}^2}{\psi_1^2}\mathrm{d}x + \int_0^L \Big[r(x) - \widetilde{v}\Big]e^{\frac{\alpha_1}{d_1}x}\mathrm{d}x + \lambda_1 \int_0^L e^{\frac{\alpha_1}{d_1}x}\mathrm{d}x = 0,$$

which implies

$$\lambda_1 \int_0^L e^{\frac{\alpha_1}{d_1}x}\mathrm{d}x \le -\int_0^L r(x)e^{\frac{\alpha_1}{d_1}x}\mathrm{d}x + e^{\frac{\alpha_1}{d_1}L}\int_0^L \widetilde{v}\mathrm{d}x.$$

By using the fact that $\int_0^L \widetilde{v}\mathrm{d}x \to 0$ as $\alpha_2 \to \infty$ (see, e.g., [23, Lemma 2.5]), we see that $\lambda_1 < 0$ provided $\alpha_2$ is large. Hence, the desired result follows. $\square$

A symmetric version of the above lemma can be stated as follows.

**Lemma 4.20** *Given $d_1, d_2, \alpha_2 > 0$. Then for $\alpha_1 > 0$ large, $(\widetilde{u}, 0)$ is always linearly unstable.*

**Remark 4.1** *In the above two lemmas, the condition that we need on $r(x)$ is the positivity, more general than the assumption (H). Moreover, by an easy inspection, the above two lemmas hold also for the special case $d_1 = d_2 := d$.*

At the end of this subsection, we verify a very useful local stability result which plays an important role in determining the global dynamics stated in Theorem 1.3 (2).

**Lemma 4.21** *Assume that assumption* (H) *holds, $0 < d_1 < d_2$, $\alpha_1 \ge \alpha_2$, and $\alpha_1 \ge 2\sqrt{r(0)d_1}$. Then $(\widetilde{u}, 0)$ is always linearly unstable.*

**Proof:** Recall the equation of $\widetilde{u}$

$$\begin{cases} d_1\widetilde{u}_{xx} - \alpha_1\widetilde{u}_x + \Big[r(x) - \widetilde{u}\Big]\widetilde{u} = 0, & 0 < x < L, \\ d_1\widetilde{u}_x(x) - \alpha_1\widetilde{u}(x) = 0, & x = 0, L, \end{cases}$$

and rewrite it as

$$\begin{cases} d_2\widetilde{u}_{xx} - \alpha_2\widetilde{u}_x + \Big[r(x) - \widetilde{u}\Big]\widetilde{u} = \Big[d_2 - d_1\Big]\widetilde{u}_{xx} + \Big[\alpha_1 - \alpha_2\Big]\widetilde{u}_x, & 0 < x < L, \\ d_1\widetilde{u}_x(x) - \alpha_1\widetilde{u}(x) = 0, & x = 0, L. \end{cases} \tag{4.27}$$

Also, recall the linearized problem (4.1) at $(\widetilde{u}, 0)$

$$\begin{cases} d_2\varphi_{1xx} - \alpha_2\varphi_{1x} + \Big[r(x) - \widetilde{u}\Big]\varphi_1 + \lambda_1\varphi_1 = 0, & 0 < x < L, \\ d_2\varphi_1(x) - \alpha_2\varphi_1(x) = 0, & x = 0, L. \end{cases} \tag{4.28}$$

Multiplying the first equation of (4.27) by $e^{-\frac{\alpha_2}{d_2}x}\varphi_1$ and integrating the result over $(0, L)$, we see

$$\begin{aligned} &\left\{\Big[d_2\widetilde{u}_x - \alpha_2\widetilde{u}\Big]\cdot e^{-\frac{\alpha_2}{d_2}x}\cdot\varphi_1\right\}\Big|_0^L - d_2\int_0^L e^{\frac{\alpha_2}{d_2}x}\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\widetilde{u}\Big]_x\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\varphi_1\Big]_x\mathrm{d}x \\ &\qquad + \int_0^L\Big[r(x) - \widetilde{u}\Big]\cdot e^{-\frac{\alpha_2}{d_2}x}\cdot\widetilde{u}\cdot\varphi_1\mathrm{d}x \\ &= \left\{\Big[(d_2 - d_1)\widetilde{u}_x + (\alpha_1 - \alpha_2)\widetilde{u}\Big]\cdot e^{-\frac{\alpha_2}{d_2}x}\cdot\varphi_1\right\}\Big|_0^L \\ &\qquad - \int_0^L\Big[(d_2 - d_1)\widetilde{u}_x + (\alpha_1 - \alpha_2)\widetilde{u}\Big]\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\varphi_1\Big]_x\mathrm{d}x, \end{aligned}$$

from which we can deduce

$$\begin{aligned} &-d_2\int_0^L e^{\frac{\alpha_2}{d_2}x}\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\widetilde{u}\Big]_x\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\varphi_1\Big]_x\mathrm{d}x + \int_0^L\Big[r(x) - \widetilde{u}\Big]\cdot e^{-\frac{\alpha_2}{d_2}x}\cdot\widetilde{u}\cdot\varphi_1\mathrm{d}x \\ &= -\int_0^L\Big[(d_2 - d_1)\widetilde{u}_x + (\alpha_1 - \alpha_2)\widetilde{u}\Big]\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\varphi_1\Big]_x\mathrm{d}x. \end{aligned} \tag{4.29}$$

On the other hand, multiply the first equation of (4.28) by $e^{-\frac{\alpha_2}{d_2}x}\widetilde{u}$ and integrate the result over $(0, L)$, we then obtain

$$\begin{aligned} -d_2\int_0^L e^{\frac{\alpha_2}{d_2}x}\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\widetilde{u}\Big]_x\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\varphi_1\Big]_x\mathrm{d}x + \int_0^L\Big[r(x) - \widetilde{u}\Big]\cdot e^{-\frac{\alpha_2}{d_2}x}\cdot\widetilde{u}\cdot\varphi_1\mathrm{d}x& \\ +\lambda_1\int_0^L e^{\frac{\alpha_2}{d_2}x}\cdot\widetilde{u}\cdot\varphi_1\mathrm{d}x = 0.& \end{aligned} \tag{4.30}$$

Subtracting (4.29) from (4.30), we find

$$\lambda_1 = \frac{\int_0^L\Big[(d_2 - d_1)\widetilde{u}_x + (\alpha_1 - \alpha_2)\widetilde{u}\Big]\cdot\Big[e^{-\frac{\alpha_2}{d_2}x}\varphi_1\Big]_x\mathrm{d}x}{\int_0^L e^{-\frac{\alpha_1}{d}x}\cdot\widetilde{u}\cdot\varphi_1\mathrm{d}x}. \tag{4.31}$$

Based on assumptions, we see from Lemma 4.1 that $\widetilde{u}_x > 0$ in $(0, L)$ and from Lemma 4.3 that $\left[e^{-\frac{\alpha_2}{d_2}x}\varphi_1\right]_x < 0$ in $(0, L)$. Then, putting these facts into (4.31), one finds $\lambda_1 < 0$, that is, $(\widetilde{u}, 0)$ is linearly unstable. $\square$

# 5 Global dynamics

In this section, we aim to study the global dynamics of system (1.1) under the assumption (H). We first discuss the general case (1.2) in subsection 5.1, where Theorem 1.3 is proved. Then in subsection 5.2, we deal with the case (1.3) and establish Theorem 1.4.

## 5.1 General case (1.2)

In this subsection, we aim to prove Theorem 1.3. The whole proof is divided into the following three parts.

**Proof of Theorem 1.3 (1)**: We first establish the existence of $\alpha_2^*$ by the following three steps.

*Step 1.* $(\widetilde{u}, 0)$ is g.a.s for any $\alpha_2 \in (\frac{d_2}{d_1}\alpha_1, \infty)$.

We prove this step by further developing the arguments introduced in [37].

Let $\Lambda = (\frac{d_2}{d_1}\alpha_1, \infty)$ and define the following three sets

$$S_{\alpha_2} := \Big\{(u_0, v_0) : 0 \le u_0 \le \widetilde{u},\ 0 \le v_0 \le \widetilde{v},\ u_0, v_0 \not\equiv 0\Big\},$$

$$A := \Big\{\alpha_2 \in \Lambda : (\widetilde{u}, 0) \text{ is g.a.s for any initial condition } (u_0, v_0) \text{ in } S_{\alpha_2}\Big\},$$

$$B := \Big\{\alpha_2 \in \Lambda : (0, \widetilde{v}) \text{ is g.a.s for any initial condition } (u_0, v_0) \text{ in } S_{\alpha_2}\Big\}.$$

Clearly, $A \cap B = \emptyset$. Combining the non-existence result Theorem 1.1 and the general result for abstract competitive systems [19, Theorem B] together, we see that $\Lambda = A \cup B$. Moreover, it can be proved by using the same arguments as in [37, Theorem 1.2] that both $A$ and $B$ are open.

We now illustrate that $A$ is not empty. By Lemma 4.19, $(0, \widetilde{v})$ is linearly unstable when $\alpha_2$ is large. Recall the non-existence result Theorem 1.1 (1). Then we can apply [19, Theorem B] to conclude that all large $\alpha_2$ belong to $A$, and thus $A$ is not empty.

We next claim that $B$ must be empty. Otherwise, one observes that an open interval $\Lambda$ (which is connected) equals a union of two disjoint open sets, which clearly is impossible.

The above analysis tells us that $A = \Lambda = (\frac{d_2}{d_1}\alpha_1, \infty)$. Using the non-existence result Theorem 1.1 again, we see that for any $\alpha_2 \in (\frac{d_2}{d_1}\alpha_1, \infty)$, $(0, \widetilde{v})$ is either linearly unstable or neutrally stable. Next, no matter which case occurs, one can apply the same arguments as in [37, Theorem 1.2] to further prove that $(\widetilde{u},0)$ is always g.a.s. Thus, step 1 is established.

*Step 2.* $(\widetilde{u},0)$ is g.a.s for $\alpha_2 = \frac{d_2}{d_1}\alpha_1$.

This step follows directly from Theorem 3.1.

*Step 3.* $(\widetilde{u},0)$ is g.a.s for $\alpha_2 \in (\frac{d_2}{d_1}\alpha_1 - \epsilon, \frac{d_2}{d_1}\alpha_1)$ for some small $\epsilon > 0$.

For $\alpha_2 = \frac{d_2}{d_1}\alpha_1$, in view of Lemmas 3.2 and 3.3, $(\widetilde{u},0)$ is linearly stable, $(0,\widetilde{v})$ is linearly unstable, and there is no co-existence steady state. By using the perturbation arguments (see, e.g., [33]), step 3 follows immediately.

Now, with all parameters fixed except $\alpha_2$, let us define

$$\alpha_2^* := \inf\Big\{\gamma > 0 : (\widetilde{u},0) \text{ is g.a.s for any } \alpha_2 \in [\gamma,\infty)\Big\}.$$

Then we must have $\alpha_2^* \in [0, \frac{d_2}{d_1}\alpha_1)$.

Next, we prove two properties of $\alpha_2^*$ descried in (1.9).

For the first one, it suffices to prove that there exists some $\epsilon = \epsilon(d_1, d_2, r) > 0$ small such that if $\alpha_1 \in (0,\epsilon)$, then for any $\alpha_2 > 0$, $(\widetilde{u},0)$ is g.a.s. If $\alpha_1 = \alpha_2 = 0$, it is well known that $(\tilde{u},0)$ is g.a.s (see [9]). By perturbation argument, there exists some small positive constant $\epsilon_1 = \epsilon_1(d_1,d_2,r)$ such that for $\alpha_1, \alpha_2 \in (0,\epsilon_1)$, $(\tilde{u},0)$ is g.a.s. Set $\epsilon = (d_1/d_2)\epsilon_1$. We consider two cases:

$$(i)\ 0 < \alpha_1 < \epsilon \text{ and } 0 < \alpha_2 < \epsilon_1 \quad \text{ and } \quad (ii)\ 0 < \alpha_1 < \epsilon \text{ and } \alpha_2 \geq \epsilon_1.$$

For case $(i)$, since $d_1 < d_2$, we have $\epsilon < \epsilon_1$, and thus $(\tilde{u},0)$ is g.a.s. For case $(ii)$, by the definition of $\epsilon$, we have $\alpha_2 \geq \epsilon_1 = (d_2/d_1)\epsilon > (d_2/d_1)\alpha_1$, which, in view of step 1 above, also gives that $(\tilde{u},0)$ is g.a.s. Lastly, we define $\epsilon_0$ as the supremum of such $\epsilon$.

For the second one, by the definitions of $\alpha_2^*$ and $\alpha_2^{**}$, we have

$$\alpha_2^{**} \leq \alpha_2^* < \frac{d_2}{d_1}\alpha_1, \quad \text{for any } \alpha_1 \geq 2\sqrt{r(0)d_1},$$

which, in view of $\lim_{\alpha_1\to\infty} \frac{\alpha_2^{**}}{\alpha_1} = \frac{d_2}{d_1}$ (Theorem 1.2), implies the desired result. $\square$

**Proof of Theorem 1.3 (2)**: Based on Theorem 1.1 (2) and Lemma 4.21, we see that under the conditions $0 < d_1 < d_2$, $\alpha_1 \geq \alpha_2$, $\alpha_1 \geq 2\sqrt{r(0)d_1}$ and (H), there is no co-existence steady state and $(\widetilde{u},0)$ is always linearly unstable. In view of the general result for abstract competitive systems [19, Theorem B], we can conclude that $(0,\widetilde{v})$ is g.a.s under these conditions. In particular, for any $\alpha_2 > 0$, there exists $\widetilde{\alpha}_1 = \widetilde{\alpha}_1(\alpha_2) := \max\{2\sqrt{r(0)d_1}, \alpha_2\} > 0$ such that $(0,\widetilde{v})$ is g.a.s for all $\alpha_1 \geq \widetilde{\alpha}_1$.

We claim that the above result continues to hold if $\widetilde{\alpha}_1$ is replaced by $\widetilde{\alpha}_1 - \epsilon$ for some small $\epsilon > 0$. In view of Lemma 4.21, $(\widetilde{u},0)$ is still linearly unstable under such small perturbations. So, to establish this claim, we just need to prove that for any given $\alpha_2 > 0$, there is no co-existence steady state for $\alpha_1 \in (\widetilde{\alpha}_1 - \epsilon, \widetilde{\alpha}_1)$. Suppose to the contrary that there is a sequence $\{{\alpha_1}^n\}_{n=1}^{\infty}$ satisfying ${\alpha_1}^n < \widetilde{\alpha}_1$ for all $n$ and ${\alpha_1}^n \to \widetilde{\alpha}_1$ as $n \to \infty$ such that system (1.1) with $\alpha_1 = {\alpha_1}^n$ has a co-existence steady state denoted by $(u_n, v_n)$. By elliptic regularity [10], we may assume, passing to a subsequence if necessary, that as $n \to \infty$, $(u_n, v_n) \to (u^*, v^*)$ in $C^2([0, L])$, where $u^*, v^* \geq 0$ and satisfy

$$\begin{cases} d_1 u^*_{xx} - \widetilde{\alpha}_1 u^*_x + u^*\Big[r - u^* - v^*\Big] = 0, & 0 < x < L, \\ d_2 v^*_{xx} - \alpha_2 v^*_x + v^*\Big[r - u^* - v^*\Big] = 0, & 0 < x < L, \\ d_1 u^*_x(0) - \widetilde{\alpha}_1 u^*(0) = d_1 u^*_x(L) - \widetilde{\alpha}_1 u^*(L) = 0, & \\ d_2 v^*_x(0) - \alpha_2 v^*(0) = d_2 v^*_x(L) - \alpha_2 v^*(L) = 0. & \end{cases}$$

By Theorem 1.1 (2), $u^*, v^* > 0$ in $[0, L]$ cannot happen. Therefore,

$$\text{either} \quad u^* = v^* = 0 \quad \text{or} \quad u^* > 0 = v^* \quad \text{or} \quad v^* > 0 = u^*. \tag{5.1}$$

If $u^* = v^* = 0$, let $\widehat{u}_n = \frac{u_n}{\|u_n\|_{L^\infty}}$ and $\widehat{v}_n = \frac{v_n}{\|v_n\|_{L^\infty}}$. Then, using the elliptic regularity again, we may assume that $(\widehat{u}_n, \widehat{v}_n) \to (\widehat{u}, \widehat{v})$ in $C^2([0, L])$ as $n \to \infty$, where $\widehat{u}, \widehat{v} > 0$ in $[0, L]$ (due to $\|\widehat{u}\|_{L^\infty} = \|\widehat{v}\|_{L^\infty} = 1$ and the maximum principle) and satisfy

$$\begin{cases} d_1 \widehat{u}_{xx} - \widetilde{\alpha}_1 \widehat{u}_x + \widehat{u} r = 0, & 0 < x < L, \\ d_2 \widehat{v}_{xx} - \alpha_2 \widehat{v}_x + \widehat{v} r = 0, & 0 < x < L, \\ \text{no flux boundary conditions.} & \end{cases}$$

A direct integration of the above equation over $(0, L)$ yields $\int_0^L \widehat{u} r \mathrm{d}x = 0$, which, clearly, is impossible. If $u^* > 0 = v^*$, using the same notations as above, we see

$$\begin{cases} d_1 u^*_{xx} - \widetilde{\alpha}_1 u^*_x + u^*\Big[r - u^*\Big] = 0, & 0 < x < L, \\ d_2 \widehat{v}_{xx} - \alpha_2 \widehat{v}_x + \widehat{v}\Big[r - u^*\Big] = 0, & 0 < x < L, \\ \text{no flux boundary conditions,} & \end{cases} \tag{5.2}$$

which can be rearranged as

$$\begin{cases} d_2 u^*_{xx} - \alpha_2 u^*_x + u^*\Big[r - u^*\Big] = \Big[d_2 - d_1\Big] u^*_{xx} + \Big[\widetilde{\alpha}_1 - \alpha_2\Big] u^*_x, & 0 < x < L, \\ d_2 \widehat{v}_{xx} - \alpha_2 \widehat{v}_x + \widehat{v}\Big[r - u^*\Big] = 0, & 0 < x < L. \end{cases} \tag{5.3}$$

Multiplying the first equation in (5.3) by $e^{-\frac{\alpha_2}{d_2}x}\widehat{v}$ and the second one by $e^{-\frac{\alpha_2}{d_2}x}u^*$, subtracting the resulting equations and then integrating over $[0, L]$, one finally obtains

$$\int_0^L \Big[(d_2 - d_1) u^*_x + (\widetilde{\alpha}_1 - \alpha_2) u^*\Big] \cdot \Big[\frac{\widehat{v}_x}{\widehat{v}} - \frac{\alpha_2}{d_2}\Big] \cdot e^{-\frac{\alpha_2}{d_2}x} \cdot \widehat{v} \mathrm{d}x = 0. \tag{5.4}$$

Returning to the first equation in (5.2) and noting the definition of $\widetilde{\alpha}_1$, one can firstly prove similarly as Lemma 4.1 that $u_x^* > 0$ in $(0, L)$. Then applying this fact together with the arguments in the proof of the estimate (4.4), one can further derive from the second equation in (5.2) that $\frac{\widehat{v}_x}{\widehat{v}} - \frac{\alpha_2}{d_2} < 0$ in $(0, L)$. Finally, putting these estimates into (5.4), one finds that the left side is negative, a contradiction. If $u^* = 0 < v^*$, then we have

$$\begin{cases} d_1 \widehat{u}_{xx} - \widetilde{\alpha}_1 \widehat{u}_x + \widehat{u}\Big[r - v^*\Big] = 0, & 0 < x < L, \\ d_2 v^*_{xx} - \alpha_2 v^*_x + v^*\Big[r - v^*\Big] = 0, & 0 < x < L, \\ \text{no flux boundary conditions.} \end{cases}$$

Again, by using the definition of $\widetilde{\alpha}_1$, one can justify $\widehat{u}_x > 0$ in $(0, L)$. Then by using the same arguments as in the proof of (4.4), one can directly verify $\frac{v_x^*}{v^*} - \frac{\alpha_2}{d_2} < 0$ in $(0, L)$. Clearly, we can deduce a similar identity to (5.4) with $u^*$ and $\widehat{v}$ there replaced by $\widehat{u}$ and $v^*$, respectively, and then a similar contradiction can be derived. This contradiction argument confirms the above claim.

With all parameters fixed except $\alpha_1$, now we can define

$$\alpha_1^* := \inf\Big\{\gamma > 0 : (0, \widetilde{v}) \text{ is g.a.s for any } \alpha_1 \in [\gamma, \infty)\Big\}.$$

Based on the above claim, $\alpha_1^* < \widetilde{\alpha}_1$. Moreover, by Theorem 3.1, $\alpha_1^* > \frac{d_1}{d_2}\alpha_2$ for each $\alpha_2 > 0$. Hence, (1.10) is confirmed. □

**Proof of Theorem 1.3 (3)**: We continue to use some arguments in Theorem 1.3 (1) to prove this statement.

We first demonstrate that for any $\alpha_1 > 0$ with $\alpha_2^*(\alpha_1) > 0$, there must be a co-existence steady state for some $\alpha_2 \in (0, \alpha_2^*(\alpha_1)]$. Define $\Gamma_1 := (0, \frac{d_2}{d_1}\alpha_1)$ and recall $A$ and $B$ defined in the proof of Theorem 1.3 (1). Suppose for contradiction that for any $\alpha_2 \in \Gamma_1 := (0, \frac{d_2}{d_1}\alpha_1)$, there is no co-existence steady state. Then in view of Theorem 1.3 (1), $(\frac{d_2}{d_1}\alpha_1 - \epsilon, \frac{d_2}{d_1}\alpha_1) \subset A$ for some small $\epsilon > 0$, so $A$ is non-empty, and so $B$ must be empty, otherwise one sees that an open interval $(0, \frac{d_2}{d_1}\alpha_1)$ equals a union of two disjoint open sets, which is impossible. In other words, $A = (0, \frac{d_2}{d_1}\alpha_1)$. Arguing in the same spirit as in Theorem 1.3 (1), we can conclude that $(\widetilde{u}, 0)$ is g.a.s for $\alpha_2 \in (0, \frac{d_2}{d_1}\alpha_1)$, and consequently for all $\alpha_2 \in (0, \infty)$, that is, $\alpha_2^*(\alpha_1) = 0$, contradicting our assumption. Hence, there must be a co-existence steady state for some $\alpha_2 \in (0, \frac{d_2}{d_1}\alpha_1)$. Indeed, by the definition of $\alpha_2^*$, such $\alpha_2$ must appear in $(0, \alpha_2^*]$.

Similarly, by defining $\Gamma_2 := (0, \widetilde{\alpha}_1(\alpha_2))$ (note $\widetilde{\alpha}_1(\alpha_2)$ is defined in the proof of Theorem 1.3 (2)), one can use the claim established in Theorem 1.3 (2) and the same idea as above to verify the second part of this statement. □

## 5.2 Special case (1.3)

We now treat the special case (1.3). For simplicity, let $d_1 = d_2 := d > 0$, and then write system (1.1) as

$$\begin{cases} u_t = du_{xx} - \alpha_1 u_x + u\Big[r(x) - u - v\Big], & 0 < x < L, t > 0, \\ v_t = dv_{xx} - \alpha_2 v_x + v\Big[r(x) - u - v\Big], & 0 < x < L, t > 0, \\ du_x(x,t) - \alpha_1 u(x,t) = 0, & x = 0, L, t > 0, \\ dv_x(x,t) - \alpha_2 v(x,t) = 0, & x = 0, L, t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & 0 < x < L, \\ v(x,0) = v_0(x) \geq, \not\equiv 0, & 0 < x < L. \end{cases} \tag{5.5}$$

Note that Theorem 1.1 (1) implies that system (5.5) above has no co-existence steady state. So, for this case, the global dynamics can be simply determined by the local stability of semi-trivial steady states.

We present the proof of Theorem 1.4 as follows.

**Proof of Theorem 1.4**: We first analyze the linear stability of $(\widetilde{u}, 0)$.

Recall the equation of $\widetilde{u}$

$$\begin{cases} d\widetilde{u}_{xx} - \alpha_1 \widetilde{u}_x + \Big[r(x) - \widetilde{u}\Big]\widetilde{u} = 0, & 0 < x < L, \\ d\widetilde{u}_x(x) - \alpha_1 \widetilde{u}(x) = 0, & x = 0, L, \end{cases} \tag{5.6}$$

and rewrite the linearized problem (4.1) at $(\widetilde{u}, 0)$ as

$$\begin{cases} d\varphi_{1xx} - \alpha_1 \varphi_{1x} + \Big[r(x) - \widetilde{u}\Big]\varphi_1 + \lambda_1 \varphi_1 = \Big[\alpha_2 - \alpha_1\Big]\varphi_{1x}, & 0 < x < L, \\ d\varphi_1(x) - \alpha_1 \varphi_1(x) = \Big[\alpha_2 - \alpha_1\Big]\varphi_1(x), & x = 0, L. \end{cases} \tag{5.7}$$

Multiply the first equation of (5.6) by $e^{-\frac{\alpha_1}{d}x}\varphi_1$ and integrate over $(0, L)$, we then see

$$\begin{aligned} -d\int_0^L e^{\frac{\alpha_1}{d}x} \cdot \Big[e^{-\frac{\alpha_1}{d}x}\widetilde{u}\Big]_x \cdot \Big[e^{-\frac{\alpha_1}{d}x}\varphi_1\Big]_x \mathrm{d}x + \int_0^L \Big[r(x) - \widetilde{u}\Big] \cdot e^{-\frac{\alpha_1}{d}x} \cdot \widetilde{u} \cdot \varphi_1 \mathrm{d}x \\ = 0, \end{aligned} \tag{5.8}$$

where the no-flux boundary conditions are used. Similarly, multiplying the first equation in (5.7) by $e^{-\frac{\alpha_1}{d}x}\widetilde{u}$ and then integrating over $(0, L)$, we obtain

$$\begin{aligned} \Big[(\alpha_2 - \alpha_1) \cdot e^{-\frac{\alpha_1}{d}x}\widetilde{u} \cdot \varphi_1\Big]\Big|_0^L - d\int_0^L e^{\frac{\alpha_1}{d}x} \cdot \Big[e^{-\frac{\alpha_1}{d}x}\widetilde{u}\Big]_x \cdot \Big[e^{-\frac{\alpha_1}{d}x}\varphi_1\Big]_x \mathrm{d}x \\ + \int_0^L \Big[r(x) - \widetilde{u}\Big] \cdot e^{-\frac{\alpha_1}{d}x} \cdot \widetilde{u} \cdot \varphi_1 \mathrm{d}x + \lambda_1 \int_0^L e^{-\frac{\alpha_1}{d}x}\widetilde{u} \cdot \varphi_1 \mathrm{d}x \\ = \Big[\alpha_2 - \alpha_1\Big] \int_0^L e^{-\frac{\alpha_1}{d}x}\widetilde{u} \cdot \varphi_{1x} \mathrm{d}x \\ = \Big[(\alpha_2 - \alpha_1) \cdot e^{-\frac{\alpha_1}{d}x}\widetilde{u} \cdot \varphi_1\Big]\Big|_0^L - \Big[\alpha_2 - \alpha_1\Big] \int_0^L \Big[e^{-\frac{\alpha_1}{d}x}\widetilde{u}\Big]_x \cdot \varphi_1 \mathrm{d}x, \end{aligned}$$

which can be reduced to

$$\begin{aligned}
&-d\int_0^L e^{\frac{\alpha_1}{d}x}\cdot\Big[e^{-\frac{\alpha_1}{d}x}\widetilde{u}\Big]_x\cdot\Big[e^{-\frac{\alpha_1}{d}x}\varphi_1\Big]_x\mathrm{d}x+\int_0^L\Big[r(x)-\widetilde{u}\Big]\cdot e^{-\frac{\alpha_1}{d}x}\cdot\widetilde{u}\cdot\varphi_1\mathrm{d}x\\
&=-\lambda_1\int_0^L e^{-\frac{\alpha_1}{d}x}\widetilde{u}\cdot\varphi_1\mathrm{d}x-\Big[\alpha_2-\alpha_1\Big]\int_0^L\Big[e^{-\frac{\alpha_1}{d}x}\widetilde{u}\Big]_x\cdot\varphi_1\mathrm{d}x.
\end{aligned}\tag{5.9}$$

Combining (5.8) and (5.9) together, we find

$$\lambda_1\int_0^L e^{-\frac{\alpha_1}{d}x}\widetilde{u}\cdot\varphi_1\mathrm{d}x=\Big[\alpha_1-\alpha_2\Big]\int_0^L\Big[e^{-\frac{\alpha_1}{d}x}\widetilde{u}\Big]_x\cdot\varphi_1\mathrm{d}x.\tag{5.10}$$

On the other hand, following the proof of (4.4), we can conclude that

$$p:=\frac{\widetilde{u}_x}{\widetilde{u}}<\frac{\alpha_1}{d}\quad\text{in}\quad(0,L),$$

which implies that the right side of (5.10) is positive. Hence, $\lambda_1>0$, that is, $(\widetilde{u},0)$ is linearly stable.

Following the above approach, one can prove that $(0,\widetilde{v})$ is linearly unstable.

In view of the non-existence result of co-existence steady state (Theorem 1.1) and the theory of abstract competitive systems [18, 19], we can conclude that $(\widetilde{u},0)$ is g.a.s. $\square$

# 6 Discussion

In this paper, we studied a Lotka-Volterra type reaction-diffusion-advection system, which can be applied to describe the competition between two aquatic species living in a river/stream with unidirectional water flow or in a vertical water column. It is assumed that two species are competing for the same resources that are distributed decreasingly across space and undergoing different dispersal strategies as reflected by their diffusion and/or advection rates. Under this assumption, we explore the joint impact of movement strategy and environmental heterogeneity on the outcome of competition. It turns out that the competitive exclusion principle holds in most situations and which species has more competitive advantages depends on the size relation between their advection speeds $\alpha_1$ and $\alpha_2$ with $0<d_1<d_2$ fixed; more precisely, we find two critical values $\alpha_2^*(\alpha_1)$ (for given $\alpha_1>0$) and $\alpha_1^*(\alpha_2)$ (for given $\alpha_2>0$) to describe when species $u$ or $v$ is a superior one (Theorem 1.3); see also Figure 1 for a geometric description of these results. In particular, we achieve a thorough understanding when two population only differ in their advection speeds (Theorem 1.4) and when the dispersal strategies of two competitors are proportional (Theorem 3.1).

Compared with the existing work [38], the new ingredient of our model lies in the involvement of environmental heterogeneity, as described by the resource function. Such a consideration, biologically, is more reasonable and meaningful since it reflects a more realistic situation where the resources may vary from point to point, and mathematically, is much more difficult to deal with as many arguments used before can not be equally applied. We need to develop new arguments to study the properties of both semi-trivial steady states and co-existence steady states. Furthermore, by an easy inspection, one immediately finds that dramatic changes in the dynamical behaviors may happen when the spatial variations are involved. Recall two observations obtained in [38], as mentioned in our introduction section. Then the first observation now may become invalid in some cases due to $\alpha_2^* \equiv 0$ for all small $\alpha_1 > 0$ (Theorem 1.3).

We next discuss the other related work [37] ($\alpha_1 = 0 < \alpha_2$). Obviously, the current work deals with a much more general situation than [37]. If we return to Figure 1, then [37] analyzed the population dynamics on the special line $\alpha_1 = 0$, while now we consider a more wider region: the first quadrant. Mathematically, our main contribution lies in the generalization of the previous non-existence result to the current version Theorem 1.1 (1) with very general boundary conditions, in which we introduce new ingredients to overcome the difficulty caused by the non-analyticity of stationary solutions, and also in the finding of a new observation Theorem 1.1 (2). Moreover, by using a wide range of arguments, we provide a good understanding on the global dynamics for this general situation.

We end this section by proposing several research problems. The first one refers to the linear stability of semi-trivial steady states. Although now we obtain a clear understanding on this problem under certain condition on $\alpha_1$ or $\alpha_2$ (see Theorem 1.2), it is not yet completely solved. We suspect that the principal eigenvalue of the current diffusion-advection type operator is no longer monotonic in the advection rate if the conditions given in Theorem 1.2 are removed. The other tough issue is about the properties of co-existence steady state, say, existence/non-existence, local stability, and uniqueness. Currently, we have made progress on the non-existence property (Theorem 1.1), but for the latter two, so far there seems no effective ways or techniques. Motivated by a recent work of He and Ni [14] (for reaction-diffusion systems without advection), we may propose a conjecture for system (1.1): every co-existence steady state, if exists, is linearly stable. If this conjecture holds true, then we can say more about the region between $\alpha_2^*$ and $\alpha_1^*$. We leave these challenging problems for future investigations.

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# References

[1] I. Averill, K.-Y. Lam and Y. Lou, The role of advection in a two-species competition model: A Bifurcation Approach. *Memoirs of AMS*, **245**, #1161, 2017.

[2] R.S. Cantrell, C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*. Series in Mathematical and Computational Biology, John Wiley and Sons, Chichester, UK, 2003.

[3] R.S. Cantrell, C. Cosner, and Y. Lou, Evolution of dispersal and ideal free distribution, *Math Bios.*, Eng., **7** (2010) 17-36.

[4] R.S. Cantrell, C. Cosner, and Y. Lou, Evolutionary stability of ideal dispersal strategies in patchy environments, *J. Math. Biol.*, **65** (2012) 943-965.

[5] R.S. Cantrell, C. Cosner, Y. Lou and S. Schreiber, Evolution of natal dispersal in spatially heterogeneous environments, *Math Biosciences*, **283** (2017) 136-144.

[6] X.F. Chen, Y. Lou, Effects of diffusion and advection on the smallest eigenvalue of an elliptic operator and their applications, *Indiana Univ. Math. J.*, **61** (2012) 45-80.

[7] R.H. Cui, K.-Y. Lam and Y. Lou, Dynamics and asymptotic profiles of steady states to an epidemic model in advective environments, *J. Differential Equations*, **263** (2017), 2343-2373.

[8] R.H. Cui, Y. Lou, Spatial SIS epidemic models in advective environments, *J. Differential Equations*, **261** (2016), 3305-3343.

[9] J. Dockery, V. Hutson, K. Mischaikow, and M. Pernarowski, The evolution of slow dispersal rates: a reaction-diffusion model, *J. Math. Biol.*, **37** (1998) 61-83.

[10] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 2001.

[11] A. Hastings, Can spatial variation alone lead to selection for dispersal? *Theor. Popul. Biol.*, **24** (1983), 244-251.

[12] X. He, W.-M. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system I: Heterogeneity vs. homogeneity, *J. Differential Equations*, **254** (2013), 528-546.

[13] X. He, W.-M. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system II: The general case, *J. Differential Equations*, **254** (2013), 4088-4108.

[14] X. He, W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system: Diffusion and spatial heterogeneity I, *Comm. Pure. Appl. Math.*, **69** (2016), 981-1014.

[15] X. He, W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system with equal amount of total resources, II, *Calc. Var. Partial Differential Equations*, **55** (2016), Art. 25, 20 pp.

[16] X. He, W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system with equal amount of total resources, III, *Calc. Var. Partial Differential Equations*, **56** (2017), Art. 132, 26 pp.

[17] P. Hess, *Periodic-parabolic boundary value problems and positivity.* Pitman Research Notes in Mathematics Series, 247. Longman, Harlow, UK; Wiley, New York, 1991.

[18] P. Hess, A.C. Lazer, On an abstract competition model and applications, *Nonlinear Analysis T.M.A.*, **16** (1991) 917-940.

[19] S.-B. Hsu, H. Smith, and P. Waltman, Competitive exclusion and coexistence for competitive systems on ordered Banach spaces, *Trans. Amer. Math. Soc.*, **348** (1996) 4083-4094.

[20] S. Kowalevski, Zur theorie der partiellen differentialgleichung, *J. Reine Angew. Math.*, **80** (1875) 1-32.

[21] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, *Uspekhi Mat. Nauk (N. S.)*, **3** (1948), 3-95.

[22] K.-Y. Lam and D. Munther, A remark on the global dynamics of competitive systems on ordered Banach spaces, *Proc. Amer. Math. Soc.*, **144** (2016), 1153-1159.

[23] K.-Y. Lam, Y. Lou, and F. Lutscher, Evolution of dispersal in closed advective environments, *J. Biol. Dyn.*, **9** (2015), 188-212.

[24] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Differential Equations*, **223** (2006), 400-426.

[25] Y. Lou, Some challenging mathematical problems in evolution of dispersal and population dynamics, *Tutorials in mathematical biosciences. IV*, 171-205, *Lecture Notes in Math. 1922, Math. Biosci. Subser.*, Springer, Berlin, 2008.

[26] Y. Lou, F. Lutscher, Evolution of dispersal in open advective environments, *J. Math. Biol.*, **69** (2014), 1319-1342.

[27] Y. Lou, D.M. Xiao, and P. Zhou, Qualitative analysis for a Lotka-Volterra competition system in advective homogeneous environment, *Discrete Contin. Dyn. Syst. A*, **36** (2016) 953-969.

[28] Y. Lou, P. Zhou, Evolution of dispersal in advective homogeneous environment: the effect of boundary conditions, *J. Differential Equations*, **259** (2015) 141-171.

[29] F. Lutscher, M. A. Lewis, and E. McCauley, Effects of heterogeneity on spread and persistence in rivers, *Bull. Math. Biol.*, **68** (2006), 2129-2160.

[30] F. Lutscher, E. Pachepsky, and M.A. Lewis, The effect of dispersal patterns on stream populations, *SIAM Rev.*, **47** (2005), 749-772.

[31] W.-M. Ni, *The Mathematics of Diffusion*, CBMS-NSF Regional Conference Series in Applied Mathematics, **82**, SIAM, Philedelphia, (2011).

[32] H. L. Smith, *Monotone Dynamical System. An Introduction to the Theory of Competitive and Cooperative Systems*, Math. Surveys Monogr., **41**, Amer. Math. Soc., Providence, RI, 1995.

[33] H.L. Smith, X.-Q. Zhao, Dynamics of a periodically pulsed bio-reactor model, *J. Differential Equations*, **155** (1999), 368-404.

[34] D.C. Speirs, W.S. C. Gurney, Population persistence in rivers and estuaries, *Ecology*, **82** (2001), 1219-1237.

[35] A.M. Trimbee, G.P. Harris, Use of time-series analysis to demonstrate advection rates of different variables in a small lake, *J. Plankton Res.*, **5** (1983), 819-833.

[36] O. Vasilyeva, F. Lutscher, Population dynamics in rivers: analysis of steady states, *Can. Appl. Math. Quart.*, **18** (2011), 439-469.

[37] X.-Q. Zhao, P. Zhou, On a Lotka-Volterra competition model: the effects of advection and spatial variation, *Calc. Var. Partial Differential Equations*, **55** (2016), Art. 73, 25 pp.

[38] P. Zhou, On a Lotka-Volterra competition system: diffusion vs advection, *Calc. Var. Partial Differential Equations*, **55** (2016), Art. 137, 29 pp.