### 8.5.4 Exchange of Stability

Let's state the main result straightaway and explore context later. The principle of exchange of stability deduces stability information for a higher-dimensional bifurcation problem

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}, \mu), \tag{8.50}$$

based on information about the reduced equation. For this theorem, hypothesis (8.25) must be strengthened to require that the $d-1$ nonzero eigenvalues of $\mathbf{DF}_*$ lie in the left half-plane, i.e.,

$$\lambda_1(\mathbf{DF}_*) = 0, \qquad \Re\lambda_j(\mathbf{DF}_*) < 0, \; j = 2, \ldots, d. \tag{8.51}$$

For the proof, which expands on the less-than-inspiring calculations used to prove Lemma 8.5.2, we refer you to Section 1.4 of [31].

**Theorem 8.5.3.** *Let $g(x, \mu)$ be the reduced function from a standard Lyapunov–Schmidt reduction of (8.50) at a bifurcation point where (8.51) is satisfied. Then in an appropriately small neighborhood of $(\mathbf{x}_*, \mu_*)$, an equilibrium $(\mathbf{x}_0, \mu_0) \in \mathbb{R}^d \times \mathbb{R}$ of (8.50) is stable or unstable if at the corresponding equilibrium $(x_0, \mu_0) \in \mathbb{R} \times \mathbb{R}$ of the reduced problem, $\partial_x g(x_0, \mu_0)$ is negative or positive, respectively.*

1. (a) For the bifurcations of (8.5), (8.8), and (8.16), if you haven't already done so, verify that the predictions of Theorem 8.5.3 regarding exchange of stability are consistent with the relevant bifurcation diagrams in the text.

$$
\begin{array}{llll}
(a) & x' & = & \sigma(y-x), \\
(b) & y' & = & \rho x - y - xz, \\
(c) & z' & = & -\beta z + xy,
\end{array}
\quad = G(\overset{(x_1,x_2,x_3)}{\overset{\|}{x}}, \rho) \qquad (8.5)
$$

$$
\begin{array}{ll}
\text{(a)} & \mathbf{e}_1 \text{ is a null eigenvector of } \mathbf{DF}_*, \\
\text{(b)} & \text{the range of } \mathbf{DF}_* \text{ is spanned by } \mathbf{e}_2, \ldots, \mathbf{e}_d,
\end{array}
\qquad (8.26)
$$

5. Write the Lorenz system (8.5) in vector notation $\mathbf{x}' = \mathbf{G}(\mathbf{x}, \rho)$, where $\mathbf{x} = (x, y, z)$. Given a square matrix $S$, consider a transformed unknown $\overline{\mathbf{x}} = S^{-1}\mathbf{x}$, which satisfies the ODE $\overline{\mathbf{x}}' = \mathbf{F}(\overline{\mathbf{x}}, \rho)$ with $\mathbf{F}(\overline{\mathbf{x}}, \rho) = S^{-1}\mathbf{G}(S\overline{\mathbf{x}}, \rho)$. Determine $S$ so that at the bifurcation point, $\mathbf{DF}_*$ satisfies (8.26).

Trivial branch is $(x, y, z) = (0, 0, 0)$

$\rho_* = 1$ (see p. 331)

At the present coordinate,

$$
DG_* = DG(0,0,0)\Big|_{\rho=1} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

- $\ker DG_* = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.
- $\operatorname{Range} DG_* = \operatorname{span}\left\{ \begin{pmatrix} -\sigma \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\beta \end{pmatrix} \right\}$.

$$
= \operatorname{span}\left\{ \begin{pmatrix} -\sigma \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.
$$

| $\bar{x} = S^{-1}x$ | $x$ |
| --- | --- |
| $e_1$ | $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ |
| $e_2$ | $\begin{pmatrix} -\sigma \\ 1 \\ 0 \end{pmatrix}$ |
| $e_3$ | $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ |

$x = S\bar{x}$.

$$
X = \underbrace{\begin{pmatrix} 1 & -\sigma & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S} \overline{X}
$$

$$S = \begin{pmatrix} 1 & -\sigma & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1} = \frac{1}{1+\sigma}\begin{bmatrix} 1 & -1 & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & 1+\sigma \end{bmatrix}^{T}$$

---

In the new coordinate $\bar{x} = S^{-1}x$

$$\frac{d}{dt}\bar{x} = S^{-1}G(S\bar{x}, \rho) := \underline{F(\bar{x}, \rho)}.$$

$$\left( \text{Trivial branch} \quad \bar{x}_{eq}(\rho) = S^{-1}x_{eq}(\rho) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \rho^{*} = 1, \right)$$

$$DF_{*} = D_{\bar{x}}F(\bar{x}, \rho)\Big|_{\substack{\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \rho = 1}}$$

$$= D_{\bar{x}} S^{-1} G(S\bar{x}, \rho)\Big|_{\bar{x}=0,\ \rho=1}.$$

$$= S^{-1} D_{\bar{x}} \underline{\underline{G}}(\underline{S\bar{x}}, \underline{\rho})\Big|_{\bar{x}=0, \rho=1}$$

$$= S^{-1}\,\underline{D_{x}G\left(S\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, 1\right)} \cdot S$$

$$= S^{-1} \cdot DG_{*} \cdot S$$

$$= \frac{1}{1+\sigma}\begin{pmatrix} 1 & -1 & \\ \sigma & 1 & \\ & & 1 \end{pmatrix}^T \begin{pmatrix} -\sigma & \sigma & \\ 1 & -1 & \\ & & -\beta \end{pmatrix} \begin{pmatrix} 1 & -\sigma & \\ 1 & 1 & \\ & & 1 \end{pmatrix}$$

$$= \frac{1}{1+\sigma}\begin{pmatrix} 1 & \sigma & \\ -1 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma(1+\sigma) & 0 \\ 0 & -(\sigma+1) & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

$$= \frac{1}{1+\sigma}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma^2-2\sigma-1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1-\sigma & 0 \\ 0 & 0 & -\frac{\beta}{1+\sigma} \end{pmatrix}}_{\underline{\underline{DF_*}}}$$

6. (a) Draw the graph, $y$ vs. $\sigma$, defined by solving the reduced equation (8.30) for the chemostat, $g(y, \sigma) = 0$, where

$$\dot{y} = g(y, \sigma) = (\sigma - y)\left(\frac{y}{y+1} - \rho\right).$$

(b) Label each branch of your graph according as $\partial_y g$ is positive or negative there.

(c) Along the equilibrium branch $x = 0$, $y = \sigma$ of (8.18) with $\sigma < \rho/(\rho + 1)$, determine whether the equilibrium is stable or unstable.

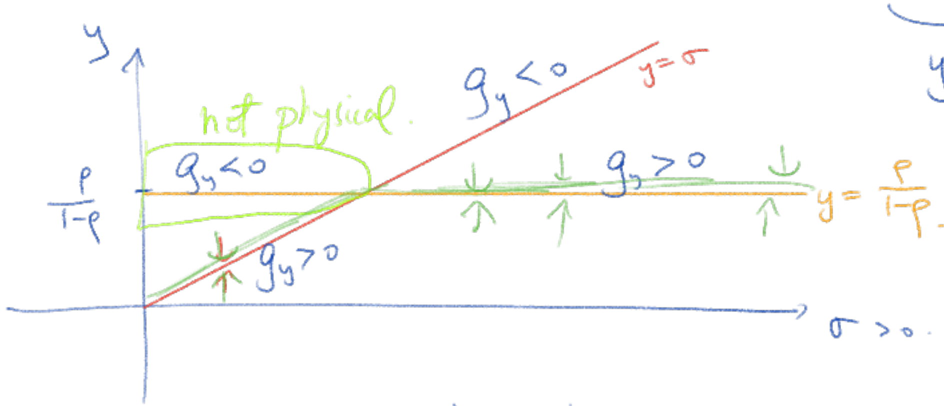
(d) Recalling the addendum to Theorem 8.5.3, complete a bifurcation diagram for the chemostat by identifying stable and unstable branches on your graph.

bif. para.

$$\dot{y} = g(y, \sigma) = (\sigma - y)\left(\frac{y}{y+1} - \rho\right).$$

$\boxed{0 < \rho < 1}$ fixed. $\boxed{\sigma > 0 \quad \text{bif. para.}}$

Two branches: $\{y = \sigma\}$ $\quad \left\{\frac{y}{y+1} = \rho\right\} = \underbrace{\left\{y = \frac{\rho}{1-\rho}\right\}}_{y_{eq}(\sigma) = \frac{\rho}{1-\rho}}$.




We used nonstandard reduction $\quad x + y = \sigma$

In this nonstandard transform $\longrightarrow$ $x = \sigma - y$

the brach $g_y > 0 \Leftrightarrow$ stability $\qquad y = \sigma$ —— no bacteria.

$g_y < 0 \Leftrightarrow$ unstability $\qquad y = \frac{\rho}{1-\rho}$ —— $x = \sigma - \frac{\rho}{1-\rho}$.

$\rho = \frac{y}{y+1}$