Smooth Boundaries and Inward Normal Vectors

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1. C^1 Boundary Manifold in $\mathbb{R}^n$

Let $\mathcal{K} \subset \mathbb{R}^n$ be a domain (an open set). We say that $\mathcal{K}$ has a C^1 **boundary** if for every point $p \in \partial\mathcal{K}$, there exists:

- a neighborhood $U \subset \mathbb{R}^n$ of p,
- a C^1 function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$,

such that, after a suitable rotation of coordinates, we can write

$$\partial\mathcal{K} \cap U = \big\{(x_1, x_2, \ldots, x_n) \in U : x_1 = \psi(x_2, \ldots, x_n)\big\}, \tag{1}$$

and

$$\mathcal{K} \cap U = \big\{(x_1, x_2, \ldots, x_n) \in U : x_1 > \psi(x_2, \ldots, x_n)\big\}. \tag{2}$$

Thus locally, the boundary $\partial\mathcal{K}$ is the graph of a C^1 function, and $\mathcal{K}$ lies on one side of that graph.

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2. Inward-Pointing Normal Vector

At a point

$$p = \big(\psi(x_2, \ldots, x_n),\, x_2, \ldots, x_n\big) \in \partial\mathcal{K},$$

we define a **local defining function**

$$g(x_1, \ldots, x_n) := x_1 - \psi(x_2, \ldots, x_n).$$

Then:

$$\partial\mathcal{K} = \{g(x) = 0\}, \qquad \mathcal{K} = \{g(x) > 0\}.$$

The gradient of g is

$$\nabla g(x) = -\big(-1,\ \partial_{x_2}\psi(x_2,\ldots,x_n),\ \ldots,\ \partial_{x_n}\psi(x_2,\ldots,x_n)\big).$$

Hence the **inward-pointing unit normal vector** at $p \in \partial\mathcal{K}$ is

$$\nu_{\text{in}}(p) = \frac{\nabla g(p)}{\|\nabla g(p)\|} = \frac{\big(-1,\ \partial_{x_2}\psi(p),\ \ldots,\ \partial_{x_n}\psi(p)\big)}{\sqrt{1+\sum_{j=2}^{n}\big(\partial_{x_j}\psi(p)\big)^2}}.$$

The **outward-pointing normal** is simply

$$\nu_{\text{out}}(p) = -\nu_{\text{in}}(p).$$

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Theorem of trapping region: Easy version

Theorem 0.1 (4.2.2)**.** *Suppose that* $\mathbf{F} : U \to \mathbb{R}^d$ *is* C^1 *and that* $\mathcal{K}$ *is a compact trapping region for*

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}).$$

If the initial data $\mathbf{b}$ *lies in the interior of* $\mathcal{K}$*, then the solution* $\mathbf{x}(t)$ *to equation (4.1) exists for all positive time and moreover lies in the interior of* $\mathcal{K}$*.*

If the inequality (4.11) is replaced by strict inequality,

$$\forall \mathbf{x} \in \partial\mathcal{K}, \quad \langle N_{\mathbf{x}}, \mathbf{F}(\mathbf{x})\rangle > 0, \tag{4.12}$$

only a few lines suffice to prove this result:

Proof of Theorem 4.2.2 assuming (4.12). By Theorem 4.1.2, the solution may cease to exist only if it first leaves $\mathcal{K}$. If, coming from inside $\mathcal{K}$, the trajectory $\mathbf{x}(t)$ reaches a point on $\partial\mathcal{K}$, then at that point the (outward) normal velocity must be nonnegative. Precisely, define

$$t_* = \sup\{t :\ x(s) \in \text{Int}\mathcal{K} \ \text{ for all } s \in [0,t]\}.$$

And use the local coordinate (1) and (2) in a neighborhood U containing the point $x_* = x(t_*) \in \partial\mathcal{K}$, then there is a $\delta > 0$ such that $x(t) \in U$ for $t \in (t_* - \delta, t_* + \delta)$. By definition of t_*,

$$g(x(t)) > 0 \text{ for } t \in (t_* - \delta, t_*), \qquad \text{and} \qquad g(x(t_*)) = 0.$$

It follows that $\frac{d}{dt}g(x(t)) \leq 0$ at the time $t = t_*$. Since g is defining function of $\partial\mathcal{K}$, we have $N_x = \frac{\nabla g(x)}{|\nabla g(x)|}$. Hence,

$$0 \geq \frac{d}{dt}g(x(t))\big|_{t=t_*} = \nabla g(x(t_*)) \cdot \dot{x}(t_*) = |\nabla g(x)|\langle N_{x_*}, F(x_*)\rangle.$$

and this contradicts the trapping hypothesis (4.12). $\square$