# Math 6411; Assignments

Adrian Lam

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Link to textbook: Schaeffer and Cain

## Solution to Ch. 4, #20.

Let $\varphi(t,b) = (x(t), y(t))$ be a solution to the Duffing's equation with the friction sign reversed:

$$\dot{x} = y \qquad \dot{y} = \beta y + x - x^3,$$

where $\beta > 0$. We fix an arbitrary initial data $b \in \mathbb{R}^2$ and let $t_* \in (0, \infty]$ be the maximal time of existence of the solution $t \mapsto \varphi(t,b)$.

Define $E(x,y) : \mathbb{R}^2 \to \mathbb{R}$ by

$$E(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} + 1$$

and define $\bar{E}(t) = E(x(t), y(t))$, **which has the domain** $t \in [0, t_*)$. Then it is easy to show that

$$(\dagger) \quad E(x,y) \geq \frac{y^2}{2} + \frac{x^4}{8} + \min_{z \in \mathbb{R}} \left[ -\frac{z^2}{2} + \frac{z^4}{8} + 1 \right] = \frac{y^2}{2} + \frac{x^4}{8} + \frac{1}{2}$$

and hence

$$(*) \qquad \frac{d}{dt}\bar{E}(t) = \beta y(t)^2 \leq 2\beta \bar{E}(t) \quad \textbf{in the interval } [0, t_*).$$

**I emphasize here that** $(*)$ **holds only as long as the solution** $(x(t), y(t))$ **to the Duffing's equation exists, which is** $[0, t_*)$**.** Multiply the differential inequality on both sides by $e^{-2\beta t}$, we derive that

$$\frac{d}{dt}(e^{-2\beta t}\bar{E}(t)) \leq 0 \quad \textbf{in the interval } [0, t_*).$$

which implies that

$$(e^{-2\beta t}\bar{E}(t)) \leq (e^{-2\beta t}\bar{E}(t))\Big|_{t=0} \quad \Rightarrow \quad \bar{E}(t) \leq \bar{E}(0)e^{2\beta t} \quad \text{ in } [0, t_*).$$

To proceed, suppose to the contrary that $t_* < \infty$. Then define

$$\mathcal{K} = \left\{ (x,y) \in \mathbb{R}^2 : \ E(x,y) \leq E(b)e^{2\beta t_*} \right\} \quad \text{which is obviously a compact set in } \mathbb{R}^2.$$

Then we have $(x(t), y(t)) \in \mathcal{K}$ in $[0, t_*)$, all the way to the maximal time $t_*$. This is in contradiction to Theorem 4.1.2.