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LaSalle's Invariance Principle

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1 Definition of Lyapunov function

Consider a DE

$$\dot{x} = F(x) \tag{1}$$

where $F : \bar{U} \to \mathbb{R}$ is C^1 is an open set containing the bounded set $\bar{U}$. For simplicity we assume:

(A1) $\bar{U}$ has (piecewise) smooth boundary and a trapping region, so that

$$\langle \nabla N_x, F(x) \rangle \geq 0 \quad \text{on } \partial U. \tag{2}$$

(For instance, let $\bar{U}$ be the restriction to a sublevel set of L.) By Theorem, we see that for any $b \in U$, the corresponding trajectory $\varphi(t, b)$ exists for all $t \geq 0$ and stays in $\bar{U}$.

Definition 1. We say that $L : U \to \mathbb{R}$ is a *Lyapunov function* (of the DE (1) in U) if

$$\langle \nabla L(x), F(x) \rangle \leq 0 \quad \text{in } U$$

Definition 2. Define

$$\mathcal{M}_0 = \left\{ x_0 \in \bar{U} : \ \langle \nabla L(x_0), F(x_0) \rangle = 0 \right\}.$$

Definition 3. Define $\mathcal{M}$ to be the *maximal invariant subset* of $\mathcal{M}_0$, i.e.

$$\mathcal{M} = \{ x_0 \in \mathcal{M}_0 : \ \varphi(t, x_0) \text{ exists and belongs to } \mathcal{M}_0 \text{ for all } t \in \mathbb{R} \}.$$

Definition 4. For each $b \in U$, we already know $\varphi(t, b)$ exists for all $t \geq 0$ and is bounded uniformly in $t \geq 0$, define the *omega limit set* $\omega(b)$ to be the collection of all subsequential limits of $\varphi(t, b)$ as $t = t_n \to \infty$, i,e,

$$\omega(b) = \{ \bar{x} \in \bar{U} : \ \exists \{t_n\}_{n=1}^{\infty}, \ t_n \to \infty, \ \bar{x} = \lim_{n\to\infty} \varphi(t_n, b) \}.$$

2 LaSalle's Invariance Principle from the book

Theorem 2.1 (Thm 6.5.3)**.** *Consider the DE* (1) *and assume* **(A1)***. Suppose that*

(A2) *If $\tilde{b} \in \mathcal{M}_0 \setminus \{b^*\}$, then there exists $T > 0$ such that $\varphi(t, b) \notin \mathcal{M}_0$.*

Then b^* is asymptotically stable.

Proof of LaSalle's principle assuming (A2). By (A1), if a trajectory $\mathbf{x}(t)$ starts in U, then it exists for all positive time and stays within $\bar{U}$. Suppose such a trajectory does not converge to $\mathbf{b}_*$. Then there exists a sequence $\{t_n\}$ tending to infinity such that $\{\mathbf{x}(t_n)\}$ is bounded away from $\mathbf{b}_*$. By invoking compactness and passing to a subsequence if necessary, we may assume without loss of generality that there exist $\tilde{b} \neq b$ such that

$$\varphi(t_n, b) \to \tilde{b} \quad \text{as } n \to \infty. \tag{3}$$

Since L is continuous,

$$\lim_{n\to\infty} L(\varphi(t_n, b)) = L(\tilde{b}) = \lim_{t\to\infty} L(\varphi(t, b)), \tag{4}$$

the latter equality because $t \mapsto L(\varphi(t, b))$ is a decreasing function (for any $b \in U$).

Claim 2.1. *There exists $t > 0$ such that*

$$L(\varphi(t, \tilde{b})) < \varphi(\tilde{b}). \tag{5}$$

On the one hand, if $\tilde{b} \notin \mathcal{M}_0$, then

$$\frac{d}{dt} L(\varphi(t, \tilde{b}))|_{t=0} = \langle \nabla L(\tilde{b}), F(\tilde{b}) \rangle < 0,$$

so (5) holds for each $t > 0$.

On the other hand, if $\tilde{b} \in \mathcal{M}_0$, then by (A2), it follows that $\frac{d}{dt} L(\varphi(t, \tilde{b})) < 0$ for some $t > 0$, so that again (5) holds for some $t > 0$. This proves the claim.

By continuity of φ and by semigroup property, we have

$$\varphi(s, \tilde{b}) = \lim_{n\to\infty} \varphi(s, \varphi(t_n, b)) = \lim_{n\to\infty} \varphi(s + t_n, b).$$

By continuous dependence on parameter, we have

$$L(\varphi(s, \tilde{b})) = \lim_{n\to\infty} L(\varphi(t_n + s, b)) = L(\tilde{b}), \tag{6}$$

the latter equality by (4). But (6) contradicts (5), which proves the theorem. □

A stronger version we discussed in class:

Theorem 2.2 (Thm 6.5.3'). *Consider the DE* (1) *and assume* **(A1)**. *Suppose that one of the following holds:*

(A2') *If* $\tilde{b} \in \mathcal{M}_0$ *has maximal interval of existence* $\tilde{I} = (-\infty, \infty)$ *and* $\varphi(t, \tilde{b}) \in \mathcal{M}_0$ *for all* $t \in \mathbb{R}$*, then* $\tilde{b} = b^*$.

Then b^* *is asymptotically stable.*

Proof of LaSalle's principle assuming (A2'). Suppose there exists $b \in U$ such that $\varphi(t, b) \not\to b^*$, then we may argue as before to obtain $\{t_n\} \to \infty$ and $\tilde{b} \neq b$ such that (3) and (4) hold.

Claim 2.2. *We claim that* $\tilde{b} \in \mathcal{M}_0$ *has maximal interval of existence* $\tilde{I} = (-\infty, \infty)$ *and* $\varphi(t, \tilde{b}) \in \mathcal{M}_0$ *for all* $t \in \mathbb{R}$.

Now, by a diagonal trick, we may pass to a subsequence $\{t'_n\}$ such that for each $m \in \mathbb{N}$, there exists $\tilde{b}_{-m}$ such that

$$\lim_{n\to\infty} \varphi(t'_n - m, b) = \tilde{b}_{-m} \qquad \text{for each } m \in \mathbb{N}.$$

Next, define $\gamma : \mathbb{R} \to \bar{U}$ by

$$\gamma(t) = \varphi(t + m, b_{-m}) \qquad \text{where } m \geq |t| + 1. \tag{7}$$

The definition of $\gamma(t)$ is consistent (i.e. independent of choice of m) because for $m_2 > m_1 \geq |t| + 1$

$$\begin{aligned}
\varphi(t + m_2, b_{-m_2}) &= \lim_{n\to\infty} \varphi(t + m_2, \varphi(t'_n - m_2, b)) \\
&= \lim_{n\to\infty} \varphi(t + m_1, \varphi(m_2 - m_1, \varphi(t'_n - m_2, b))) \\
&= \lim_{n\to\infty} \varphi(t + m_1, \varphi(t'_n - m_1, b)) = \varphi(t + m_1, b_{-m_1}).
\end{aligned}$$

Moreover, it is easy to see that for $-\infty < t_1 < t_2 < +\infty$, we have

$$\varphi(t_2 - t_2, \gamma(t_1)) = \gamma(t_2)$$

This shows that $\varphi(t, \tilde{b}) = \gamma(t)$ is an entire trajectory for $t \in \mathbb{R}$. Moreover, observe that (7) implies that

$$\gamma(t) = \varphi(t + m, b_{-m}) = \lim_{n\to\infty} \varphi(t'_n + t, b) \qquad \text{for each } t \in \mathbb{R}$$

so that (4) implies that

$$L(\gamma(t)) \equiv L(\tilde{b}) \qquad \text{for each } t \in \mathbb{R},$$

so that

$$0 = \frac{d}{dt} L(\gamma(t)) = \langle \nabla L(\gamma(t)), F(\gamma(t)) \rangle \quad \text{for all } t \in \mathbb{R}.$$

This shows that $\gamma(t) = \varphi(t, \tilde{b}) \in \mathcal{M}_0$ for all $t \in \mathbb{R}$. This proves Claim 2.2.

Now, by (A2'), we must have $\tilde{b} = \gamma(0) = b^*$. But this is in contradiction with $\tilde{b} \neq b^*$. $\square$

3 General LaSalle's Invariance Principle

Theorem 3.1. *Let $F : \bar{U} \to \mathbb{R}$ be C^1 and assume (A1). Suppose L is a Lyapunov function of* (1) *in U, then*

(a) *$\mathcal{M}$ is nonempty.*

(b) *$\omega(b) \subset \mathcal{M}$ for each $b \in U$.*

$$i.e. \quad \lim_{t \to \infty} \operatorname{dist}(\varphi(t, b), \mathcal{M}) = 0 \quad \textit{for each } b \in U.$$

where $\operatorname{dist}(x, S) = \inf_{y \in S} \|x - y\|$.

Example. For the Duffing's equation

$$\dot{x} = y, \quad \dot{y} = -\beta y + x - x^3$$

with $L(x, y) = \frac{y^2}{2} - \frac{x^2}{2} - \frac{x^4}{4}$, let $\alpha > -\frac{1}{2}$ and take

$$\bar{U} = K_\alpha = \{(x, y) \in \mathbb{R}^2 : \; L(x, y) < \alpha\}$$

. Then L is a Lyapunov function in U and

$$\mathcal{M}_0 = \{(x, y) \in U : \; y = 0\},$$

and

$$\mathcal{M} = \begin{cases} \{(1, 0), (-1, 0)\} & \text{when } -\frac{1}{2} < \alpha < 0 \\ \{(0, 0), (1, 0), (-1, 0)\} & \text{when } \alpha \geq 0 \end{cases}$$