Note on omega limit sets

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1 Property of limit sets

Consider $F : U \to \mathbb{R}^d$, where U is an open set in $\mathbb{R}^d$, and consider the dynamical system $\varphi(t, z)$ defined on

$$\Omega := \{(t, b) : \ b \in U, \quad I_b = (-\alpha(b), \beta(b))\}$$

with I_z being the maximal interval of existence of the ODE $\dot{x} = F(x)$ with initial data $x(0) = z$.

Definition of omega limit set. We define the *omega limit set* $\omega(b)$ to be the collection of all subsequential limits of $\varphi(t, b)$ as $t = t_n \to \infty$, i,e,

$$\omega(b) = \{\bar{x} \in \bar{U} : \ \exists \{t_n\}_{n=1}^{\infty}, \ t_n \to \infty, \ \bar{x} = \lim_{n\to\infty} \varphi(t_n, b)\}.$$

It can also equivalently be defined as

$$\omega(b) = \begin{cases} \emptyset & \text{if } \sup\, I_b < +\infty, \\ \bigcap\limits_{k\in\mathbb{N}} \overline{\varphi([k, \infty), b)} & \text{if } \sup\, I_b = +\infty. \end{cases}$$

I want to discuss the following proposition in Schaeffer and Cain.

Proposition 7.2.4. *Every ω-limit set $\omega(\mathbf{b})$ is a closed invariant subset of $\mathcal{U}$.*

1.1 The proposition as it is stated is wrong

I think the above proposition is not true as it is stated: Counter example is the following:

$$U = \{(r, \theta) : 0 < r < \frac{1}{|\cos\theta|}\} = \{(x, y) \in \mathbb{R}^2 : \ |x| < 1\} \setminus \{(0, 0)\}.$$

The dynamical system is:

$$\dot{\theta} = 1, \quad \dot{r} = r(1 - r).$$

Then

- $I_{(r,\theta)} = (-\infty, \infty)$ if and only if $0 < r < 1$.
- Fix arbitrary initial data in the open unit disk (i.e. $0 < r < 1$, $\theta \in S^1$), we have $\omega((r, \theta)) = S^1 \cap U$. (Remember the dynamical system is not defined outside of U.)
- Due to the first bullet point, $S^1 \cap U$ is not invariant, since we have $r = 1$.

If you want an example of $U = \mathbb{R}^2$, simply consider

$$y(t) = \Phi(x(t)) \quad \text{where} \;\; \Phi(x_1, x_2) = (\tan\left(\frac{2}{\pi}x_1\right), x_2)$$

which satisfies the ODE

$$\dot{y}(t) = D\Phi(x(t))\dot{x}(t) = D\Phi\left(\Phi^{-1}(y(t))\right) F\left(\Phi^{-1}(y(t))\right) := \tilde{F}(y(t)) \tag{1}$$

The orbits of $x(t)$ in U and that of $y(t)$ in $\mathbb{R}^2$ are in one-one correspondence.

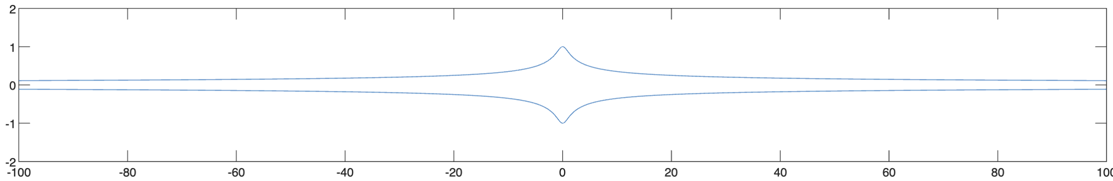

Figure 1: The omega limit set of (1) starting at $(0, 0.5)$. It consists of two disconnected pieces: $\omega^+ = \{(y_1, y_2) : \ \Phi^{-1}(y) = e^{i\theta}$ for some $0 < \theta < \pi\}$ and $\omega^- = \{(y_1, y_2) : \ \Phi^{-1}(y) = e^{i\theta}$ for some $-\pi < \theta < 0\}$.

1.2 Where in the proof has a gap?

The gap in the proof of Proposition 7.2.4 in the textbook appears to be:

$$\varphi(t_*, z_n) = \varphi(t_*, z_0) \quad \text{where} \quad z_n = \varphi(t_n, b) \to z_0.$$

The application of "continuous dependence on initial data" (Theorem 4.5.1 in Schaeffer and Cains) requires that $\varphi(t_*, z_0)$ is well-defined, i.e. $[0, t_*] \subset I_{z_0}$, but that is what is needed to be proved! (To prove $\omega(b)$ is invariant, we need to show that $I_{z_0} = (-\infty, \infty)$ for every $z_0 \in \omega(b)$.)

1.3 How to make it right?

One way is to assume that

(A) $\varphi([t_0, \infty), b)$ is precompact in U. (i.e. its closure is compact set and is contained in the open set U.)

Contrast the following properties with the counter example in Figure 1. The following theorem is applicable in particular if one consider the dynamical system within a trapping region (a forward-invariant region), such as the region bounded by a limit cycle.

1.4 Useful results/properties of omega limit sets under assumption (A)

Theorem 1.1. *Let $\dot{x} = F(x)$, $F : U \to \mathbb{R}^d$ be locally Lipschitz in the open set U and let $b \in U$ be fixed. Suppose* **(A)** *holds, then*

(a) *$\omega(b)$ is a compact set in U.*

(b) *$\omega(b)$ is connected.*

(c) *$\omega(b)$ is invariant; i.e. for each $z_0 \in \omega(b)$, $I_{z_0} = (-\infty, \infty)$ and $\varphi(\mathbb{R}, z_0) \subseteq \omega(b)$.*

(d) *$\omega(b)$ is chain-recurrent. (Think of a homoclinic orbit. We won't be proving it. See Figure 2 for the definition.)*

Remark. For the definition of *chain-recurrence*, see Figure 2, which is from P. 6 in my note for complete Lyapunov function.

Proof of compactness. Here trajectory initiating at b exists for all time according to **(A)**. We first prove the equivalent condition claimed earlier in the note:

$$\omega(b) = \tilde{\omega}(b), \qquad \text{where} \quad \tilde{\omega}(b) = \bigcap_{k \in \mathbb{N}} \overline{\varphi([k, \infty), b)}. \tag{2}$$

Let $z_0 \in \omega(b)$ then there exists a sequence $t_n \to \infty$ such that $\varphi(t_n, b) \to z_0$. Thus z_0 belongs to the closure of $\varphi([k,\infty), b)$ for each k. Thus $z_0 \in \tilde{\omega}(b)$. This proves one direction of inclusion.

For the opposite direction, let $z_0 \in \tilde{\omega}(b)$, then $z_0 \in \overline{\varphi([k,\infty), b)}$ for each k. Hence, for each k, we can choose $t_k \in [k, \infty)$ such that $\|\varphi(t_k, b) - z_0\| < \frac{1}{k}$. This proves that $z_0 \in \omega(b)$.

Having proved (2), it is clear that $\omega(b)$ is the intersection of decreasing sequence of compact (closed and bounded) sets $\overline{\varphi[k,\infty), b)}$, so it is closed and bounded itself and hence it is compact. ☐

Proof of connectedness. Suppose $\omega(b)$ is not connected, i.e. there exists open sets A, B in $\mathbb{R}^d$ such that

$$\begin{cases} A \cap B = \emptyset \\ \omega(b) \subset A \cap B \\ A \cap \omega(b) \neq \emptyset, \quad \text{and} \quad B \cap \omega(b) \neq \emptyset \end{cases}$$

Take $z \in A \cap \omega(b)$ and $z' \in B \cap \omega(b)$. Then there exists two sequences $t_n, t'_n \to \infty$ such that

$$\varphi(t_n, b) \to z \quad \text{and} \quad \varphi(t'_n, b) \to z'.$$

Without loss of generality, we may assume

$$\begin{cases} t_n \leq t'_n < t_{n+1} \quad \text{for all } n \geq 1, \\ \varphi(t_n, b) \in A \quad \text{and} \quad \varphi(t'_n, b) \in B \quad \text{for all } n \geq 1. \end{cases}$$

Now, $\varphi([t_n, t'_n], b)$, being the image of a continuous mapping from an interval, is connected and has nonempty intersection with A and B, hence it cannot possibly be contained in $A \cup B$, so there exists $s_n \in (t_n, t'_n)$ that $\varphi(s_n, b) \in \mathbb{R}^d \setminus (A \cup B)$. Note that $s_n \to \infty$ since $t_n, t'_n \to \infty$.

By **(A)**, we can pass to a subsequence s'_n such that $z'' = \lim_{n\to\infty} \varphi(s'_n, b)$ exists, and belongs to $\omega(b)$. However, it also belongs to the closed set $(A \cup B)^c$. This is a contradiction to the disconnectedness of $\omega(b)$. ☐

Proof of invariance of $\omega(b)$; i.e. Proposition 7.2.4 holds assuming (A)

Suppose **(A)** holds.

Let $z_0 \in \omega(b)$ be given. Then there exists a sequence $t_n \to \infty$ such that

$$\varphi(t_n, b) \to z_0.$$

Step #1. Solvability forward in time. i.e. $I_{z_0} \supseteq [0, \infty)$

To see this, suppose

$$\sup I_{z_0} = t_*.$$

Then for each $t \in [0, t_*)$, we have by continuous dependence

$$\varphi(t, z_0) = \lim_{n\to\infty} \varphi(t, \varphi(t_n, b)) = \lim_{n\to\infty} \varphi(t + t_n, b) \in \omega(b) \quad \text{for any } t > 0 \text{ such that } t \in I_{z_0}.$$

Hence the trajectory initiating at z_0 never leave the compact subset $\omega(b)$. (**Here we used (A)**, otherwise see the counter example with $\omega(b)$ being noncompact and disconnected; see Figure 2.) This proves $t_* = +\infty$ and implies the claim.

Step #2. Solvability backward in time. i.e. $I_{z_0} \supseteq [-m, 0]$ for all $m \geq 1$.

Claim 1.1. *We claim that z_0 has maximal interval of existence $I_{z_0} = (-\infty, \infty)$ and $\varphi(t, z_0) \in \omega(b)$ for all $t \in \mathbb{R}$.*

Now, by a diagonal trick, we may pass to a subsequence $\{t_n'\} \subset \{t_n\}$ such that for each $m \in \mathbb{N}$, there exists z_{-m} such that

$$\lim_{n\to\infty} \varphi(t_n' - m, b) = z_{-m} \quad \text{for each } m \in \mathbb{N}.$$

This part is due to (A).

It follows as above that for each $m \in \mathbb{N}$,

$$\begin{cases} z_{-m} \in \omega(b), \\ I_{z_{-m}} \supseteq [0,\infty), \\ \varphi([0,\infty), z_{-m}) \subset \omega(b). \end{cases}$$

Having shown that solutions initiating at z_{-m} exists forward in time, we can now apply continuous dependence to obtain

$$\varphi(m, z_{-m}) = \lim_{n\to\infty} \varphi(m, \varphi(t_n' - m, b)) = \lim_{n\to\infty} \varphi(t_n', b) = z_0. \tag{3}$$

This implies z_{-m} and z_0 lies on the same orbit, i.e.

$$I_{z_0} = I_{z_{-m}} - m \supseteq [-m,\infty) \quad \text{for each } m \in \mathbb{N}.$$

and

$$\varphi([-m,\infty), z_0) = \varphi([0,\infty), z_{-m}) \subset \omega(b) \quad \text{for each } m \in \mathbb{N}.$$

Since $m \in \mathbb{N}$ is arbitrary, this shows that

$$I_{z_0} = (-\infty,\infty) \quad \text{and that} \quad \varphi(\mathbb{R}, z_0) \subset \omega(b).$$

This completes the proof. □

Recall that a subset A of S is said to be *internally chain transitive* with respect to the semiflow φ if, for two points $u_0, v_0 \in A$, and any $\delta > 0$, $T > 0$, there is a finite sequence

$$C_{\delta,T} = \{u^{(1)} = u_0, u^{(2)}, \ldots, u^{(m)} = v_0; t_1, \ldots, t_{m-1}\}$$

with $u^{(j)} \in A$ and $t_j \geq T$, such that $\|\varphi(t_j, u^{(j)}) - u^{(j+1)}\| < \delta$ for all $1 \leq i \leq m-1$. The sequence $C_{\delta,T}$ is called a (δ, T)-chain connecting u_0 and v_0. Define the *chain recurrent set* $R(S)$ to be the set of all $u_0 \in S$ such that for any $T \gg 1$, and $\delta \ll 1$ there exists a (δ, T)-chain connecting u_0 to itself.

Figure 2: Definition of chain-recurrence; taken rom P. 6 in my note for complete Lyapunov function.

2 Existence of equilibrium for simply-connected trapping region

Theorem 2.1 (Brouwer's Fixed Point Theorem). *Let $D^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^d$. If $f : D^d \to D^d$ is continuous (a continuous mapping from the closed ball and whose image is contained in the closed ball), then there exists a point $x^* \in D^d$ such that*

$$f(x^*) = x^*.$$

Exercise. Suppose $\overline{U} \subset \mathbb{R}^d$ is a trapping region of $\dot{x} = F(x)$ with C^1 boundary, and with $F : \bar{U} \to \mathbb{R}^d$ being C^1. Suppose there exists a diffeomorphism $\Phi : \overline{U} \to D^d$. Use Brouwer's fixed point theorem to prove that $\dot{x} = F(x)$ has at least one equilibrium point x^* in $\overline{U}$.

[Hint: Step 1. Use the procedure in Section 1.1 to assume, WLOG, that $\overline{U} = D^d$; Step 2. For each n, let $f_n : \bar{U} \to \bar{U}$ be given by $f_n(x) = \varphi(2^{-n}, x)$, then apply BFPT to get a fixed point x_n. Then $\varphi(t, x_n) = x_n$ if $t = m2^{-n}$ for any $m \geq 1$; Step 3. Pass to the limit $x_n \to x^*$ and show that x^* is an equilibrium point.]