Problem 1 (8 points)**.** Match the following equations to their names: (A) Linear system; (B) Gradient system; (C) Hamiltonian system; (D) Duffing's equation; (E) van der Pol's system; (F) Torqued pendulum equation; (G) Chemostat system; (H) Lotka-Volterra system; (I) Activator-Inhibitor system

(B)(v) $\begin{cases} \dot{x} = D_x\Phi(x,y) \\ \dot{y} = D_y\Phi(x,y) \end{cases}$

(E)(i) $\begin{cases} \dot{x} = y \\ \dot{y} = -\beta(x^2-1)y - x \end{cases}$

(I)$(viii)$ $\begin{cases} \dot{x} = \sigma \frac{1}{1+y}\frac{x^2}{1+x^2/\kappa^2} - x, \\ \dot{y} = \rho\left[\frac{x^2}{1+x^2/\kappa^2} - y\right], \end{cases}$

(D)(ii) $\begin{cases} x'' + \beta x' - x + x^3 = 0 \end{cases}$

(G)(vi) $\begin{cases} \dot{x} = \frac{y}{y+1}x - \rho x, \\ \dot{y} = -\frac{y}{y+1}x - \rho(y-\sigma). \end{cases}$

(F)(iii) $\begin{cases} x'' = -\sin x - \beta x' + \mu \end{cases}$

(C)(iv) $\begin{cases} \dot{x} = D_y\Psi(x,y) \\ \dot{y} = -D_x\Psi(x,y) \end{cases}$

(H)(vii) $\begin{cases} \dot{x} = axy - bx \\ \dot{y} = cy - dxy \end{cases}$

(A)(ix) $\begin{cases} \dot{x} = ax + by + g(t); \\ \dot{y} = -cx - dy + f(t); \end{cases}$

Problem 2. (6 points) Let $F : \mathbb{R} \to \mathbb{R}$ and consider the initial value problem

$$(*) \qquad \dot{x} = F(x) \quad \text{with} \quad x(0) = b.$$

State, without proof, an example of $F \in C(\mathbb{R})$, $b \in \mathbb{R}$ such that the initial value problem $(*)$ has two solutions, and write down two different explicit solutions.

Solution.

Let $F(x) = \frac{3}{2}x^{1/3}$, then $F : \mathbb{R} \to \mathbb{R}$ is continuous and $\dot{x} = F(x)$ has at least two solutions,

$$x_1(t) = 0 \text{ for } t \geq 0, \qquad \text{and} \qquad x_2(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \sqrt{2}, \\ (t-\sqrt{2})^{3/2} & \text{for } t > \sqrt{2}. \end{cases}$$

Problem 3. (10 points) Let $F : \mathbb{R}^d \to \mathbb{R}^d$ and let $b^* \in \mathbb{R}^d$ be a hyperbolic equilibrium of $\dot{x} = F(x)$.
(i) Give the definition of "$\mathcal{M}$ is a 1-dimensional differentiable manifold", assuming $1 < d$.
(ii) Give the definition of "$\mathcal{M}_u$ is a local unstable *manifold* of b^*".

Solution.

(i) A subset $\mathcal{M} \subset \mathbb{R}^d$ is called a k-dimensional manifold if for each $x_0 \in \mathcal{M}$, there exists a neighborhood $\mathcal{N}$ of x_0 in $\mathbb{R}^d$ such that (after suitable orthogonal transformation)
(1)
$$\mathcal{M} \cap \mathcal{N} = \{(x_1, ..., x_d) \in \mathcal{N} : \ (x_{k+1}, ..., x_d) = g(x_1, ..., x_k)\} \quad \text{for some differentiable function } g.$$

(ii) A subset $\mathcal{M}_u \subset \mathbb{R}^d$ is called a local unstable *manifold* of b^* if there exists a neighborhood $\mathcal{N}$ of b^* such that

$$\mathcal{M}_u \cap \mathcal{N} = \Big\{ b \in \mathcal{N} \ : \ \varphi(t,b) \text{ is defined for all } t \leq 0, \quad \varphi((-\infty, 0], b) \subset \mathcal{N}, \quad \varphi(t,b) \to b^* \text{ as } t \to -\infty \Big\}$$

(In addition, since b^* is hyperbolic, the local unstable manifold is tangent to the linear subspace spanned by the unstable eigenvectors. i.e. if we use the orthogonal transformation so that $DF(b^*)$ is block diagnonal as in the textbook, the subspace of unstable eigenvalues is $\{(x_1, ..., x_k, 0, ..., 0)\}$, then (1) holds with $Dg(0) = 0$.)

Problem 4. (8 points) Show that $b^* = 1$ is a locally asymptotically stable equilibrium for

$$\dot{x} = x(x-0.1)(x-0.7)(1-x) \quad \text{in } \mathbb{R}$$

by finding and verifying a strict Lyapunov function. (Do not use linearized stability principle.) Then state without proof the global stable and unstable manifolds for the other equilibrium $b^* = 0$.

Solution.

Let $L(x) = \frac{1}{4}(x-1)^4$, then

$$L'(x)F(x) = (x-1)^3 x(x-0.1)(x-0.7)(1-x) = \begin{cases} < 0 & \text{for } x \in (0.7,\infty) \setminus \{1\}, \\ = 0 & \text{for } x = 1. \end{cases}$$

The equilibrium $b^* = 0$ is unstable since $F'(0) = (-0.1) \times (-0.7) \times 1 > 0$. The unstable manifold is $(-\infty, 0) \cup (0, 0.1)$ and the stable manifold is $\{0\}$.

Problem 5. (8 points) Suppose (i) K is a compact subset of an open set U; (ii) K is a trapping region of $\dot{x} = F(x)$ with smooth boundary, and with $F : U \to \mathbb{R}^d$ being C^1.

(a) Deduce that $\varphi(t, b) \in K$ for all $b \in \partial K$ for $t \in [0, \sup I_b)$.

(b) Show that such solutions exist globally forward in time, i.e. in $[0, \infty)$.

[Hint: You may use the theorems, provided on next page, without proof.] Here $\varphi(t, b)$ denotes the solution of $\dot{x} = F(x)$ with initial data $x(0) = b$ for t in the maximal interval of existence I_b.

Solution.

(a): Let $b_0 \in \partial K$ and let $\{b_n\} \subset \operatorname{Int} K$ be a sequence converging to b_0. By the standard existence uniqueness result, there exists an open interval $I_{b_n} = (-\alpha_n, \beta_n)$ where $\alpha_n, \beta_n \in [0, \infty]$ such that the solution map $\varphi(t, b_n)$ is well defined for $t \in I_{b_n}$.

Moreover, note that by Theorem 4.2.3,

$$\varphi([0, b_n), b_n) \subset K \quad \text{for all } n \geq 1. \tag{2}$$

Next, we claim that for each $t \in [0, \beta_0)$, we have $\varphi(t, b_0) \subset K$.

Indeed, by Theorem 4.,5.1, then it follows that $[0, t] \subset I_{b_n}$ for all $n \gg 1$, and that

$$\varphi(t, b_0) = \lim_{n \to \infty} \varphi(t, b_n).$$

Since $\varphi(t, b_n) \in K$ (thanks to (2)) and K is compact, it follows that the limit $\varphi(t, b_0) \in K$. □

(b): Since $\varphi([0, \beta_0), b_0) \subset K$ and K is a compact set of U, Theorem 4.1.2 says that $\beta_0 = +\infty$. □

Problem 6. (15 points) Suppose $\overline{B} = \{x \in \mathbb{R}^2 : \ \|x\| \leq 1\}$ is a trapping region of $\dot{x} = F(x)$, and where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1.

- **(a)** (11 points) Use Brouwer's fixed point theorem (recalled below) to prove that $\dot{x} = F(x)$ has at least one equilibrium point x^* in $\overline{B}$.
- **(b)** (4 points) Define a $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the above assumptions but the ODE has no equilibrium located in the open set $B = \{x \in \mathbb{R}^2 : \ \|x\| < 1\}$. (Hint: Try a contraction mapping.)

Solution.

(a): By Problem 5, $K = \bar{B}$ is a compact and forward-invariant set, i.e. for any $b \in \bar{B}$, solutions exists globally forward in time, and that

$$\varphi([0,\infty), b) \subset \bar{B}.$$

For $k \in \mathbb{N}$ and $x \in \bar{B}$, define $f_k(x) = \varphi(2^{-k}, b)$. Then (i) $f_k : \bar{B} \to \bar{B}$ (since $\bar{B}$ is a forward-invariant set thanks to Problem 5); (ii) f_k is continuous in $\bar{B}$ (thanks to Theorem 4.5.1). We can then apply Brouwer's fixed point theorem to conclude that, for each k, there exists $x_k \in \bar{B}$ such that

$$\varphi(2^{-k}, x_k) = f_k(x_k) = x_k.$$

By semigroup property, we have

$$\varphi(t, x_k) = x_k \text{ for all } t \in S_k.$$

where

$$S_k = \left\{ \frac{m}{2^k} : \ m \in \mathbb{N} \right\}.$$

Next, by compactness of $\bar{B}$, we pass to a strictly increasing sequence of integers $\{k_n\} \subset \mathbb{N}$ so that

$$\lim_{n\to\infty} x_{k_n} \to x^* \quad \text{ for some } x^* \in \bar{B}.$$

We will show that x^* is an equilibrium for the continuous-time dynamical system.

For any natural number m such that $m \leq k_n$, we have $S_m \subset S_{k_n}$ so that

$$\varphi(t, x_{k_n}) = x_{k_n} \quad \text{ for all } t \in S_m.$$

By Theorem 4.5.1, we can send $k_n \to \infty$ (while fixing t) to deduce that

$$\varphi(t, x^*) = x^* \quad \text{ for all } t \in S_m.$$

Since this holds for any m, it follows that

$$\varphi(t, x^*) = x^* \quad \text{ for all } t \in S' = \left\{ \frac{n'}{2^m} : \ n', m \in \mathbb{N} \right\}.$$

Since S' is dense in $[0,\infty)$ and $\varphi(t, x^*)$ is continuous in t, it follows that

$$\varphi(t, x^*) = x^* \quad \text{ for all } t \geq 0.$$

i.e. x^* is an equilibrium. □

(b): Fix an arbitrary point $x^* \in \partial B$, and Set $F(x) = -(x - x^*)$, then it is clear that $\dot{x} = F(x)$ has a unique equilibrium $x^* \in \bar{B}$. that occurs on the boundary but not in the interior of B.

It remains to show that $\bar{B}$ is a trapping region. Indeed, for any $x \in \partial B$, we have

$$\langle N_x, F(x) \rangle = \langle -x, -(x - x^*) \rangle = \|x\|^2 - \langle x, x^* \rangle \geq 0,$$

where we used Cauchy-Schwartz's inequality and, of course, $\|x\| = \|x^*\| = 1$. □

Apendix.

Theorem. [Brouwer's Fixed Point Theorem] *Let* $\overline{B} = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ *be the closed unit ball in* $\mathbb{R}^d$. *If* $f : \overline{B} \to \overline{B}$ *is continuous, then there exists a point* $x^* \in \overline{B}$ *such that*

$$f(x^*) = x^*.$$

Theorem 4.1.2. *Suppose, regarding the maximal solution of* $\dot{\mathbf{x}} = F(\mathbf{x})$,

$$\mathbf{x}_* : (-\alpha_*, \beta_*) \to \mathbb{R}^d,$$

that $\beta_* < \infty$. *Then for every compact set* $K \subset U$, *there is an* $\varepsilon > 0$ *such that*

$$\mathbf{x}_*(t) \notin K \quad \text{for } \beta_* - \varepsilon < t < \beta_*.$$

Theorem 4.2.3 Suppose that $\mathbf{F} : U \to \mathbb{R}^d$ is C^1 and that K is a compact trapping region for

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}).$$

If K has smooth boundary, then for any $b \in \operatorname{Int} K$, solution exists for all positive time, and that

$$\varphi(t, b) \in K \quad \text{ for all } t \geq 0, \text{ and all } b \in \operatorname{Int} K.$$

Theorem 4.5.1. Suppose $F : U \to \mathbb{R}^d$ is locally Lipschitz, and let $t \mapsto x_0(t) = \varphi(t, b_0)$, $0 \leq t < \beta_0$ (with $\beta_0 \in (0, \infty]$) be a solution forward in time satisfying

$$\frac{d}{dt} x_0(t) = F(x_0) \quad \text{ with initial data } x_0(0) = b_0.$$

Then for each fixed number $T \in (0, \beta_0)$, there exists $C_T > 0$ and $\delta > 0$ containing b_0 such that

$$\sup_{t \in [0,T]} \|\varphi(t, b) - \varphi(t, b_0)\| \leq C_T \|b - b_0\| \quad \text{ for all } b \text{ such that } \|b - b_0\| < \delta.$$
