The Competitiveness of Joint Bidding in Multiple-Units Uniform-Price Auctions.*

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Abstract

Recent literature on joint bidding in a single-unit, Common-Value (CV) auction argues that the inference and precision that emerges from pooling information by joint bidders enhances competition. Such information pooling mitigates adverse selection and results in more aggressive bidding and may even increase revenue despite the reduction in the number of bidders. We analyze here, for the first time, joint bidding in a simple CV, uniform-price auction with multi-unit demand. This introduces an opposite force due to (increased) monopsony power of the joint bidders, called demand reduction (DR). We show that the pro-competitive benefit from joint bidding in single unit auctions does not generalize to a multi-unit environment, even when DR is disallowed. With DR, the scope for improved competition is further eroded.

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**Introduction**

Anti- and pro-competitives arguments regarding joint bidding agreements in auctions made joint bidding a subject for public policy debates and concerns for over 30 years. The main argument against joint bidding, reduction in competition, is articulated by J. Markham (1970), who studied the auction market for offshore petroleum tracts:

... where joint bids are simply substituted for independent solo bids, they reduce the total number of bids. That is, if all the participating firms bid individually on a given tract in any event, joint bids would simply be substituted for a larger number of independent solo bids. If the resulting reduction in total bids is substantial, competition may be adversely affected.

Such concerns often resulted in litigation and/or regulation. A notable example is the 1975 Congress’ Energy Policy and Conservation Act that disallowed joint bidding in the outer continental shelf government lease auctions. Other important examples of restrictions on joint bidding in areas such as state and municipal bonds markets and auctions of the London Tea Broker’s are mentioned and referenced in James Smith (1983). Despite concerns, joint bidding still plays an important role in government auctions, generating roughly 42% of government revenues in the broadband PCS auctions, according to Vijay Krishna and John Morgan (1997, KM hereafter).

Earlier empirical studies claim that joint bidding is not associated with a reduction in the total number of bids in many markets, seemingly a necessary condition for Markham’s critique to apply, thus, not surprisingly, the anti-competitive claim is not supported by the data. These studies also offer factors that are favorable for competition and allocation efficiency. Those factors include, risk diversification, weakening of liquidity or capital constraints, and sharing of private information. No doubt, those supportive arguments help explain the large role that joint bidding plays currently.

In a recent work KM highlights a new pro-competitive argument, an *inference effect*, for joint bidding resulting from information pooling. They consider a simple pure common-value (CV), single-unit, second-price auction. Since a bidder wins only when her signal is the highest, her willingness to pay may be well below the expected value of the item conditional on her own private signal. KM argue that information pooling by joint bidders mitigates such winner’s curse (WC) and leads to higher bidding. The more aggressive bidding is offset by having fewer bids; yet, it can result in higher revenues as well. Larry DeBrock and James Smith (1983) used somewhat a different intuition to make a similar claim:

We show that joint bidding increases the total social value of the lease offering and, in most cases, does not significantly decrease the percentage of social value captured by the government. These results follow from the fact that pooling of information concerning a priory unknown tract value allows for more accurate estimates. The anti-competitive effect of a reduced number of bidders tends to offset by the well known fact that better informed participants bid more aggressively. (p. 395, italics added.)
All the previous analyses of joint bidding restricted attention to single-unit auctions. Here, we maintain the pure CV assumption and study, for the first time, joint bidding in a uniform-price auction, but with *multi-unit demand*. We find that the inference/precision effect intuition which was developed for joint bidding in a single-unit auctions fails in multi-unit, as it does not necessarily induce more aggressive bidding. This intuition ignores strategic considerations of pooling information in joint bidding with multiple-units.

Important auctions often offer identical multiple-units, and bidders demand more than one unit. Similar to an horizontal mergers that increasing the monopoly power held by the merging entity, joint bidding increases the monopsony power of the bidding consortium. For example, even if a individual bidders originally demand one unit, the bidding consortium would typically have a multi-unit demand. A scope for a new strategic consideration, unavailable to single unit bidders, is now available to the group, leading them to lower their bid on the second (and beyond) unit(s) demanded. The group recognizes that although a lower bid may cause them not to (otherwise) win that unit, it also reduces, in such cases, the price paid on the unit(s) won. This additional and opposite economic force, called demand reduction (DR), is absent in auctions with a single unit and may overcome the inference/precision effect.

Lawrence Ausubel and Peter Cramton (1996) showed that, except in “knife edge” cases, DR is part of equilibrium in uniform-price-auction. Experimental and quasi-experimental evidence suggests that indeed bidders exercise DR (see John Kagel and Dan Levin (2001a, 2001b); John List and David Lucking-Reiley (2000)). To separate the inference/precision effect from the DR effect, we create a synthetic, in-between, model with joint bidding that disallows groups to exercise their increased monopsony power. We show that joint bidding may induce less aggressive bidding thus necessarily lowers revenue even in the absence of DR. We also show that with DR revenue may be lower in informational environments where joint bidding indeed leads to higher bidding and revenues in its absence. The auction model employed is the simplest that still maintains all the relevant economics tensions. Being so simple the analysis does not allow a sweeping, general conclusions. The results nevertheless, strongly suggest that modeling multi-unit demand and recognizing DR is critically important for analyzing joint bidding.

We introduce the benchmark model in section 2. In section 3 we introduce joint bidding without DR, characterize equilibrium and rank revenues. Bidding with DR is presented in section 4. We summarize and conclude in section 5.

The Benchmark Model

**Preliminary:** Let \( n \) be the number of bidders in the benchmark model, M1, and let \( m : n/2 \geq 2 \) be the number of pairs of bidders who share information in the other two models, denoted by M2 and M3. We simplify by assuming that the seller supply, \( S \), is either one or two identical units and that her valuation of the (each) item is zero. Denote by \( R^*_M \), seller’s revenue
per unit (price) when supply is $S = \{1, 2\}$, in model $M = \{1, 2, 3\}$. Each one of the $n$ bidders receives an informative signal $X_i$, $i = 1, 2, \ldots, n$. The random variables $X_i$'s are i.i.d. from a distribution function $F \in \mathcal{F}$ (footnote on $\mathcal{F}$ with $F \in \mathcal{F}$; $0$ on $\mathcal{F}$, $1$). Without loss of generality, we order the signals, $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ and assume that the CV, $V$, takes the form:

$$V : >_{\geq 1} c_i Y_i, c_i \geq 0 \quad \text{and} \quad >_{\geq 1} c_i = 1.$$ 

In a frequently used CV model, $V$ is the average of all signals, i.e., $c_i = \frac{1}{n} - i$ (e.g., see KM). In most of this work we use the following special form of $V$:

$$V = \begin{cases} Y_1 & \text{if } Y_1 \geq Y_2, \\ \frac{1}{2} (Y_1 + Y_2) & \text{if } Y_1 < Y_2, \end{cases}$$

$V$ is a weighted average of the two highest signals. We expect a serious WC problem in a single unit second-price auction. The winner in the symmetric Nash equilibrium is the bidder holding $Y_1$. However, since winning is the event that matters, a bidder's (max) willingness to pay is guided by conditioning on that event, namely, $E[B \mid X_i] = Y_i$.$\quad x$, which is typically well below the expectation of $V$ conditional on their own signal, $E[B \mid X_i] = x$. The difference between the two expectations, here $\hat{\alpha} E[B \mid X_i] = x$.$\hat{\alpha}$, is often used to measure the extent of the adverse selection problem.

2.1 The Benchmark Model [M1]: In M1 each of the $n$ bidders receives a signal $X_i = x$, has a single unit demand, and submits one sealed bid. When $S = 1$, the highest bidder is awarded the unit and pays the second highest bid. When $S = 2$, the highest two bids are awarded one unit each and pay the third highest bid, the highest rejected bid. Let the bidding strategy, $F \in \mathcal{F}$, from $\mathcal{R}_+ \setminus \mathcal{R}_+$ represent the bidding functions of bidder $i$ who observes $X_i = x$, under $S = \{1, 2\}$.

2.2 M1 Under $S = 1$: Consider the following bidding function for bidder $i$ who observes $X_i = x$:

$$V = \begin{cases} Y_1 & \text{if } Y_1 \geq Y_2, \\ \frac{1}{2} (Y_1 + Y_2) & \text{if } Y_1 < Y_2. $$

**Proposition:** With one unit supply, $S = 1$: A. The bidding function $F \in \mathcal{F}$ is the unique symmetric equilibrium in M1. B. When $J = 0$, i.e., $V = Y_2$, seller extracts full surplus in M1 (footnote $\mathcal{F}$).

**Proof:** A1. The bidding strategy $F \in \mathcal{F}$: $x$ is a symmetric equilibrium. Consider any bidder $i$ who observes $X_i = x$, and assume all other bidders follow the proposed bidding strategy. If $X_i = Y_1$, bidder $i$ wins the unit, pays $Y_2$ and earns $J Y_1 = Y_2$. Thus lowering the bid and risking losing the unit cannot help, and raising the bid does not matter. If $X_i = Y_2$, bidder $i$ does not win any unit. Here, lowering the bid does not matter, and if by raising it bidder $i$ outbids the holder of $Y_1$ she wins the unit but gains $J Y_1 = Y_2$. This is not possible (strict inequality - J $9$) rather than zero, since the price is $Y_1$. Thus, no one wishes to deviate.
unilaterally from the proposed strategy.

A2. Assume that there is another symmetric equilibrium, \( F^1_i \in R \). This implies the existence of an interval \( a, b \) with \( a \) where \( F^1_i \) is strictly above or strictly below \( F^2_i \). Consider bidder \( i \) who observes \( X_i : x \leq Y \). and assume that \( F^1_i \); \( F^2_i \): \( x \). If \( X_i : x = Y \), lowering the bid from \( F^1_i \) to \( F^2_i \) matters with strictly positive probability. This is, \( 0 < z \), \( a < z \). \( x \). such that if \( y_2 > 5 \), then \( x \). \( \sigma \) bidder \( i \) loses the unit with the proposed deviation. It implies that the price \( i \) would otherwise pay for the unit exceeds \( Y \) resulting in a loss. If \( X_i : x > Y \), then lowering the bid from \( F^1_i \) to \( F^2_i \) no matter what. Thus, bidder \( i \) is better off following such a deviation from \( F^1_i \) for multi-unit auctions. The structure of the proof for the case that \( F^1_i \) \( F^2_i \) \( F^2_i \), \( b \) is similar and is omitted. We conclude that \( F^2_i \) \( F^2_i \) cannot be another symmetric equilibrium.

B. Follows directly from the fact that in this case the winner, the holder of the highest signal, pays \( Y \). \( V \).

2.3 M1 Under \( S \): 2: Consider the following bidding function for bidder \( i \) who observes \( X_i : x \):

\[
\sigma \quad F^1_i : E Y_1 \quad Y_2 : x = Y_3 \quad x, \quad \text{equality} \quad x > 9 \quad 1, \quad \text{if and only if}
\]

\( J \): 0. footnote

**Proposition** With two units supply, \( S \): 2, the bidding function, \( F^2_i \) is the unique SNE for M1.

**proof** The proof that \( F^2_i \) is a symmetric equilibrium is similar to the proof to part A1 in Proposition 1. The proof of uniqueness is proven with the same arguments as in Levin and Harstad (1986) and is omitted.

Under \( S \): 1, a bidder’s max willingness to pay is disciplined by the condition of having is the highest of all \( n \) signals where under \( S \): 2, the conditions is of having (only) the second highest of all \( n \) signals. Thus, the increased supply reduces the WC and results in more aggressive bidding as \( F^2_i \); \( x = F^1_i \); \( J \); 0 and \( x > 9 \). Higher bidding due to an increase in supply helps to interpret our results below for multi-unit auctions. The impact on the price from more aggressive bidding is offset by the fact that with \( S \): 2, the price is set by the holder of \( Y \), rather than by the holder of \( Y_2 \), as it is with \( S \): 1. It follows that:

\[
\sigma \quad R_1 : Y_2, \quad R_2 : F^2_i Y_3 \cdot \sigma
\]

Is it possible that the reduction in WC due to an increased supply can result in an increase in the expected price, \( E \sigma \sigma \) apparently invalidating the law of demand?

**Proposition** There exist \( 0 < 9 \) such that: \( A \): \( 5 \beta, 9 \) \( \beta \); \( E \sigma \sigma \) \( B \).
proof. A. The proof to the first part follows from the fact (Proposition 1 part B) that with
J : 0, the seller extracts full surplus under S : 1, but not under S : 2. Thus, for J : 0,
E[Rj] ; E[Rj] which together with continuity in J, establishes the proof.

B. For any given realization of Y3 : y3 9 1,
where we used integration by part and some tedious but straightforward simplifications.
Since E[Rj ] is strictly increasing in t, strictly negative at t : y3, and strictly positive
at t : 1, we have:
E[Rj ] ? R|Y3 : y3 ; 1 X y3 1 ? P(y)1 P(y)1 dt ? y3 ; Yi ? J P

\[ \hat{X}_{y3} \frac{g(y)p}{1 + F(Y_3)p} dt \] where \( \hat{t} \) solves \( \int_{Y_3}^\infty \frac{g(y)p}{1 + F(Y_3)p} \) 0. Since the last strict
inequality holds for - Y3 : y3 9 1, we conclude that E[Rj ] ? R|j ; 0, for J : 1, and
continuous in J, which establishes the proof.

Thus, if the CV, \( V_i \) is “close enough” to \( Y_1 \), the expected price necessarily rises as we
increase supply! The reason for this anomaly is that the increase in supply “shifts” the demand
as well due to favorable informational impact on bidders’ willingness to pay. Similar such
anomalies are discussed in Bulow I. Jeremy and Paul D. Klemperer, (2000). Note however that
this anomaly, an increase in price as a result of increase in supply, is quite different than the
one that would emerge with social influences and or consumption externalities. See Becker

The Constrained Joint Bidding Model

In the constrained joint bidding model, [M2], we randomly match two bidders to form m
pairs from the original n single unit demand bidders. Denote by \( P_k \), the information set, \( \hat{X}^k : x^k, q^k, a^k, k = 1, 2, ... , m \), of the \( k^{th} \) pair. Assume that, \( x^k \geq q^k \), i.e., \( x^k \) and \( q^k \) are
the max and the min respectively in \( P_k \). Simplify by writing \( P_k : \hat{x}, q, a \), or even just
\( P : \hat{x}, q \), when \( k \) is not needed. Each pair consists of two bidders each with a single unit
demand. Thus, the pair demands two units and submits two bids when \( S : 2 \). However, when
\( S : 1 \), only one unit is available and the pair submits only one bid. Let the bidding strategy
\( \hat{G}_k \hat{Y}_k \) represent the bidding function of the \( k^{th} \) pair who observes \( P_k \) when
\( S : 1 \), and let \( \hat{G}_k \hat{Y}_k \) represent the bidding strategy of the \( k^{th} \) pair who observes \( P_k \) when
\( S : 2 \). Without a loss of generality, assume that
\( \hat{G}_k \hat{Y}_k \) Order the \( X^k \)s so that \( X^1, X^2, ..., X^m \) and denote the phrase “the pair
holding \( X^i \)” by \( PH^i \). The special feature of M2 when \( S : 2 \) (motivated below) is that
\( \hat{G}_k \hat{Y}_k \) i.e., each of the m pairs must submit the same bid for their two units
demand. Ausubel and Cramton (1996) have shown that generically \( \hat{G}_2 \hat{Y}_2 \hat{P} ; \hat{G}_2 \hat{Y}_2 \hat{P} \)
(except in special situations). Thus, the above equality constraint is typically binding.

The inference effect intuition developed for joint bidding in a single unit auction is appealing. Pooling information mitigates the WC and induces higher bidding. To account for the adverse selection in the symmetric equilibrium of a second-price auction, a bidder behaves as if holding the highest signal. Assume that the information set of the group can be summarized by one real number, \( Z \). Winning in the SNE of M2 implies the highest \( Z \). However, having the highest \( Z \) in M2 does not necessarily imply that all other individual signals are lower than the highest \( Z \). This means that there is less “bad news” to account for in winning, which encourages higher bidding. A similar claim for first-price auction environment is made by DeBrock and Smith (1983), who argue that more precise information leads to more aggressive bidding.

M2 allows us to focus on joint bidding with multi-unit demand that mitigates the WC but without the complication of allowing each pair to exploit their increased monopsony power. Thus, we can examine whether the inference (precision) effect intuition that joint bidding leads to higher price generalizes to auctions with multi-unit demand even without DR. We show that it does not. Clearly, such analysis is incomplete. Monopsony power induces bidders in a uniform-price auction format with multi-unit demand to reduce their bids on the second (and beyond) unit(s), since they realize that such lower bidding may favorably reduce the price on units won. This strategic force, absent in the single unit literature, is introduced in the third stage, M3, where we allow DR.

3.1 M2 Under \( S : 1 \): Consider the following bidding function for a pair who observes \( P : \tilde{\lambda} x, q \tilde{\lambda} \):

\[
\forall x \in \tilde{\lambda} P \quad G_i \tilde{\lambda}, q \in \tilde{\lambda} x. 
\]

**Proposition** With only one unit supply, \( S : 1 \), the bidding function, \( G_i \tilde{\lambda}, q \tilde{\lambda} x \), is the unique symmetric equilibrium to M2.

**proof** The proof mimics the proof to Proposition 1.

When \( S : 1 \), joint bidders demand only a single unit. Does information pooling by the group raise bidding in this case? In M1, bidders bid their signal \( X_i \). In M2, pairs summarize their information by \( X_i \), their highest signal, and bid it as in M1. Thus, joint bidding does not raise bidding. Not surprising then is that revenue necessarily falls. The price in M2 is determined by the second highest bid which is \( X^2 \). With a probability of \( \frac{n-2}{n-1} \), the holders of the two highest signals are not matched, and the price is, \( X^2 : Y_2 \), as in M1. However, with a probability of \( \frac{1}{n-1} \), the holders of the two highest signals are matched resulting in a lower price, since in such event, \( X^2 : Y_3 \). Revenue in M2 with \( S : 1 \) is equal to, \( \frac{n^2-2}{n-1} Y'_2 + \frac{1}{n-1} Y'_3 \), 9 \( Y_2 \). We summarize these observations (proof is omitted):
**Proposition** Under $S$: 1. Bidding in $M_2$ is not higher than in $M_1$. 2. $E[R_1^2]; E[R_1^2]$

Our model $M_2$ under $S$: 1 captures the spirit of Markham’s critique cited in the introduction: Joint bidding does not affect bidding, so the only consequence is the possible elimination of a competitive bid. In the above model joint bidders use $X_i$, their highest signal, to summarize their information set. Thus, conditional on winning, $X_i$ is the highest of $n$ signals in $M_2$ as it is in $M_1$. As a result, joint bidding does not mitigate the WC, the underlying reason for the pro-competitive effect. Thus, our last observation does not, by itself, invalidate the pro-competitive benefit from the inference/precision effect developed for a single unit auction when pooling information does lead to WC reduction.

### 3.2 $M_2$ Under $S$: 2
Consider the following bidding strategy for a pair who observes $P$: $\hat{x}, q_1$

\[ \mathcal{G}_2 \mathcal{Y}_x, q_1: \hat{x}, x \hat{a} \]

**Proposition** When supply, $S$: 2: The bidding strategy $\mathcal{G}_2 \mathcal{Y}_x, q_1: \hat{x}, x \hat{a}$ is the unique SNE to $M_2$.

**proof** The proof mimics the proof to Proposition 1.

Under $S$: 2, the price in $M_2$ is determined by the third highest bid, $X^2$, which is equal to $Y_2$ or $Y_3$ with probabilities of $\frac{n^2}{n^1}$ and $\frac{1}{n^1}$ respectively. Thus, conditional on $Y_3: y_3$, the expected price is: $E[R_3^2 | Y_3 : y_3 \hat{a}]: \frac{n^2}{n^1} E[R_2^2 | Y_3 : y_3 \hat{a}] + \frac{1}{n^1} E[R_3^2 | Y_3 : y_3 \hat{a}]$ and with some tedious simplification, we have:

\[ \mathcal{G}_2 \mathcal{Y}_x, q_1: \hat{x}, x \hat{a} \]

\[ - J 5 \mathcal{B}_2 \frac{n^2}{n^1}, 1 \hat{a} E[R_2^2 | Y_3 : y_3 \hat{a}] 9 0 \]

We summarize these observations by:

**Proposition** There exist 0 9 $J \frac{n^2}{n^1} 9 1$, such that: 2. $- J 5 \mathcal{B}_2 \frac{n^2}{n^1}, 1 \hat{a} E[R_3^2]; E[R_3^2]; E[R_1^2]$

**proof** 1. For any given realization of $Y_3 : y_3 9 1$, and $J : 0$, $\mathcal{G}_2 \mathcal{Y}_x$ implies that $E[R_2^2 | Y_3 : y_3 \hat{a}] 9 0$, which together with its continuity in $J$, establishes the proof. 2.
When $J : 0$, $F_2 \hat{Y}_k \hat{P} : x$ bidding in $M2$ is more aggressive than in $M1$: Any two signals $x ; q$, held by a pair in $M2$ induce the pair to bid $x$ for each unit. The same two signals held by two different bidders in $M1$ induce the bids $x$ and $q$, $Y_i = 9 \ x \ P$. It is surprising then that in such a case (or more generally for $J : 5 \ \hat{B}, J \ \bar{D} \ \hat{a}$, the price in $M2$ is higher, despite the fact that the information pooling in $M2$ does not mitigate the WC. In fact, pooling information worsens the WC in the following sense: In the symmetric equilibrium of $M1$ with $S : 2$, the willingness to pay of a bidder holding $X_i$ is guided by the condition that $Y_i ? 2 \ P$ signals are not higher than that. Yet, in the symmetric equilibrium of $M2$, a winner can infer that $Y_i ? 1 \ P$ $Y_i = 9 \ P$ signals are not higher than $X_i$. Thus, the last proposition suggests that the inference effect intuition neglects an important difference between the response to an increase in supply in a single unit auction without joint bidding and the response to an increase in supply in an auction with (constrained) joint bidding.

The following example sharply illustrates this point and uses a standard CV case. Let there be four bidders, $n : 4$, and let $V : > \ dfrac{n}{2} Y_i/4$, i.e., $V$ is the average of the four signals. Denote the number of units supplied by $S : \hat{a}1, 2, \hat{a}$. Let each $X_i$ be i.i.d. from the generalized uniform distribution function $f_{Y_i} : \hat{X}^1, \hat{X}^2, \hat{X}^3, \hat{X}^4$. Let $F_1 \hat{Y}_i, R \hat{P}$ represent the symmetric equilibrium of $M1$ given $S$ and $R$. It is well known that these $F_1 \hat{Y}_i, R \hat{P}$ are:

$F_1 \hat{Y}_i, R \hat{P} : E_{R \hat{P}} Y_i \ | Y_i \ : \ Y_i \ \hat{a}1 + \hat{a}2 \ \hat{a}2 ; \ Y_i \ \hat{a}3 \ \hat{a}4 \ \hat{a}4 = \ Y_i \ \hat{a}1 \ \hat{a}2 \ \hat{a}4 \ \hat{a}4$, and $F_2 \hat{Y}_i, R \hat{P} : E_{R \hat{P}} Y_i \ | Y_i \ : \ Y_i \ \hat{a}1 \ \hat{a}2 \ \hat{a}4 \ \hat{a}4$. We form two pairs by randomly matching two bidders and denote by $P_k = \hat{a} \hat{X}^1, \hat{X}^k \ \hat{a}, k : 1, 2$.

Each pair’s information set, by $\hat{X}^k : \dfrac{\hat{X}^1 + \hat{X}^2 + \hat{X}^3 + \hat{X}^4}{2}$, the average of the two signals in $P_k$ and by $\hat{x}^k$ its realization.

**Proposition** $G \hat{Y}_k \hat{P} : \hat{x}^k$ and $G \hat{Y}_k \hat{P} : \hat{a} \hat{x}^k, \hat{x}^k \ \hat{a}$ are the unique symmetric equilibrium of $M2$ with $S : 1$ and $S : 2$, respectively. This is, each pair submits one, or two bids, equal to their average signal under $S : 1$ and $S : 2$, respectively.

**proof** Start with $S : 2$. Consider pair 1 and assume their rivals, pair 2, follow the prescribed strategy. In event $A$, $\hat{x}^1 ; \hat{x}^2$, pair 1 wins 2 units at a price determined by the third highest bid, $\hat{x}^2$, set by pair 2. Pair 1’s profit per unit is $\dfrac{\hat{x}^1 + \hat{x}^2 + \hat{x}^3 + \hat{x}^4}{4}$ $\hat{x}^2 : \dfrac{\hat{x}^1 \ \hat{x}^2 \ \hat{x}^4}{2} ; 0$. Bidding higher does not affect the result. By lowering the bids, even on one unit, there is a strictly positive probability of losing at least one unit to pair 2, resulting in zero profits on the unit(s) lost. We conclude that in this event pair 1 does not want to deviate from the proposed strategy. In event $B$, $\hat{x}^2 ; \hat{x}^1$ (we ignore ties), pair 1 wins 0 units. Deviating and raising their bid in order to win one unit or two will cause pair 2 to set the price per unit at $\hat{x}^2$ with profit for pair 1 for each unit won as a result of the deviation equal to $\dfrac{\hat{x}^1 + \hat{x}^2 + \hat{x}^3 + \hat{x}^4}{4}$ $\hat{x}^2 : \dfrac{\hat{x}^1 \ \hat{x}^2 \ \hat{x}^4}{2} ; 0$. That completes the proof that the proposed strategy is a symmetric equilibrium. Uniqueness in the class of continuous and
monotonic symmetric equilibrium is proven with the same arguments as in Levin and Harstad (1986). The proof is similar for $S = 1$.

Note that $G_{1} \hat{Y}_{k}^{P_{1}}: \hat{x}^{k}$ and $G_{2} \hat{Y}_{k}^{P_{2}}: \hat{a}^{k}, \hat{x}^{k}, \hat{a}$, are independent of $F(\theta)$ and bid per unit does not increase with an increase in supply as in $M1$. Also note that pooling information by $\bar{x}^{k}$ satisfies the two intuitive properties reported. Namely, with $\bar{x}^{k}$, estimation of $V$ is more precise than with a single $x_{i}$ and there is less WC in winning with the highest $x^{k}$ in M2 than with the highest $x_{i}$ in M1. Let $\bar{x}: x$ and consider $\mathcal{F}_{S} \hat{Y}_{k}; \mathcal{R} \hat{P}_{1}^{R_{k}}: x \hat{P}_{k}$ for $S = 1$, this expression is strictly negative for all $x$ and $R$, providing support for the inference effect intuition. With $S = 2$, $R = 1$ and $R = 2$, $F_{2} \hat{Y}_{k}; 1 \hat{P}_{k}: \frac{3}{4} x + \frac{1}{8}$ and $F_{2} \hat{Y}_{k}; 2 \hat{P}_{k}: \frac{1}{6} \frac{5 x^{k} + 4 x^{k} + 1}{4 x^{k}}$ so that $\mathcal{F}_{2} \hat{Y}_{k}; 1 \hat{P}^{R_{k}}: G_{2} \hat{Y}_{k}^{P_{2}}: x \hat{P}_{k}: \frac{8 \overline{7}^{2} x^{k}}{8}$; 0 for $x 9 1 / 2$ and $\mathcal{F}_{2} \hat{Y}_{k}; 2 \hat{P}^{R_{k}}: G_{2} \hat{Y}_{k}^{P_{2}}: x \hat{P}_{k}: \frac{1}{6} \overline{7}^{2} \frac{1}{4} \hat{a}$; 0 for $x 9 \frac{17}{12} : 0.618$. As $R$ increases without a bound $F_{2} \hat{Y}_{k}; \mathcal{R} \hat{P}_{k}: F_{2} \hat{Y}_{k}; \mathcal{K} \hat{P}_{k}: \frac{3 x^{k} + 4}{4}, \hat{a}$, and $\mathcal{F}_{2} \hat{Y}_{k}; \mathcal{K} \hat{P}_{k}: G_{2} \hat{Y}_{k}^{P_{2}}: x \hat{P}_{k}: \frac{17 x^{k}}{4}$; 0, $- x 5 \overline{4}, 1 \hat{P}_{k}$.

Thus, our example and the analysis that followed it establishes:

**Proposition** The inference/precision effect intuition does not generalize to joint bidding auctions with multi-unit demand even in the absence of DR: they fail to raise bidding and expected price. footnote

The inference/precision argument fails to explain less aggressive bidding even without DR in a multi-unit auction. An increase in the number of units supplied in a single unit demand without joint bidding mitigates the WC and results in more aggressive bidding with the same private signal. In contrast, with joint bidding and information pooling, pairs do not increase their bid per unit as supply increases. Had we not assumed that as supply increases in M2, pairs’ demand also increases to two units (which is reasonable in our context), the above result would have been even more dramatic as the unique symmetric equilibrium in M2 with one unit demand has each pair bidding zero! Thus, our last example demonstrates that although bidding is necessarily higher with information pooling and joint bidding, in the single unit auction, the ranking is reversed in a multi-unit environment. We show below that allowing DR may reverse revenue ranking even the inference/precision intuition does hold.

Finally, we ask whether the introduction of a (an optimal) minimum bid would restore the inference effect intuition of single unit auctions. When $J = 0$, the negative answer follows immediately. Under $S = 1$, the seller extracts full surplus in M1, but not in M2. In this case, the seller cannot do any better and will not use any reserve price in M1. Optimal reserve price may help in this case improve revenue in M2. However, if effective it must be binding and thus cannot restore full surplus extraction. By continuity in $J$ it is safe to argue that at least for small enough $J^{\prime} s$, (optimal) reserve price will not restore the ranking $E \mathcal{R}_{1}^{J} \hat{a}$; $E \mathcal{R}_{1}^{J} \hat{a}$

**Allowing Demand Reduction**

In our model that allows demand reduction, $[M3]$, we assume that $S = 2$, as DR is meaningless with $S = 1$, otherwise, everything is the same as in M2. Consider a model with
general, even number, \( n : 2m \) bidders, general signal distribution, \( F \) continuous on \( \mathbb{R} \), and as before, \( V : Y_1 + \gamma \ ? P^2 \). Joint bidders can submit two bids in an unconstrained fashion. As before, let \( P_i \sim \gamma \), \( q \) denote the greater and lesser signals, respectively, of generic pair \( i \) \( 1, 2, \ldots, m \), and let \( X_i^1, X_i^2, \ldots, X_i^m \) denote the ordered values of \( x \) for all pairs \( j \) other than \( i \). Even in such a simplified environment, where each merged entity demands just two units, it is very difficult to characterize and solve for the whole set of equilibria with DR. The reason is that in general each bid may depend on both signals and would therefore seem to be the solution to a thorny multi-dimensional signal problem. Richard Engelbrecht-Wiggans and Charles Kahn, (1988, EW&K), faced such a problem in a uniform-price, private-value, environment and resort to “brute-force” to derive equilibria. footnote The fact that ours is a common-value auction model changes the derivations but does not make the general solution any simpler. Thus, and as in EW&K, we focus on characterizing an equilibrium where each pair bid their relevant willingness to pay on the first unit and bid low enough on the second unit, effectively avoiding winning it. Such equilibria have lower revenues than equilibria where DR is less drastic and may serve as a useful boundary to assess the competitiveness of such joint bidding. Thus, we study below under what conditions on \( F \) the following symmetric strategy profile \( \Delta Y, q \mid \Delta Y, q \) where,

\[
\hat{Y}^1 \triangleq \text{H} Y, q \triangleq \text{H} Y, q \triangleq \text{H} Y, q \triangleq 0,
\]

is an equilibrium. First, note that given that other pairs are using this strategy, the proposed \( H Y, q \) is a best response, via the usual arguments. (This is, deviating upwards or downwards if matters is necessarily regrettable.) The only question therefore is whether the proposed \( H Y, q \) is 0 is a best response as well. Next, note that for two types \( Y, q \) and \( \hat{Y}, q \) with \( q \) \( q \) type \( \hat{Y}, q \) gains strictly more by deviating and submitting a second bid (easy to show formally). Thus, for a given x, it is necessary and sufficient that type \( \hat{Y}, q \) does not want to deviate. Let \( G \) be the distribution of a given pair’s high bid induced by the proposed \( H Y, q \) in \( \hat{Y} \) and let \( \Delta Y, b \) denote the difference in payoffs of bidder \( i \) who observes \( Y, x \) and “defects” and uses \( \Delta Y, b \) as proposed in \( \hat{Y} \) while all other \( \hat{Y} \) \( 1 \) pairs use the proposed strategy. We first derive the function \( \Delta Y, b \) and then \( \Delta Y, b \) of \( b \). Consider a bidder \( i \) who observes a pair of signals \( Y, x \). Since the lowest possible higher bid by each of the \( \hat{Y} \) ? 1 rivals’ pairs is \( \Delta Y, b \) \( 0 \), any \( b \) \( 9 \) \( b \) never and \( b \) \( 5 \) \( b \), \( H \) : footnote Using a lower bid of \( b \) \( 5 \) \( b \), \( H \) \( 0 \) rather than \( b \) \( b \), matters in two events: When \( \Delta Y, b \) \( 9 \) \( b \), (WLOG ties are ignored) using such \( b \) \( b \) earns bidder \( i \) two units rather than one at a price of \( E \Delta Y, b \) \( 9 \) \( b \) \( 1 \) per unit rather than one at a price of \( E \Delta Y, b \) \( 9 \) \( b \) \( 1 \) \( b \). In this event bidder \( i \) may benefit from such deviating. Since although she pays a higher price, she earns twice the surplus per unit. Also note, that in this event implies that bidder \( i \)’s pair of signals \( Y, x \) must be the two highest signals and thus the common-value

\[
V : Y + \gamma \ ? P^2 : x.
\]

The other event that matters is when \( \Delta Y, b \) \( 9 \) \( b \), \( H \) \( Y \) \( 1 \) \( b \) \( 2 \), \( H \) \( Y \) \( 1 \) \( b \). In this
events using $b$; $b$ is always regrettable as it just raise the price from $E^{h_{X_{1}}, h_{X_{2}}}$ to $b$. Thus, $=\dot{Y}, b\P$ can be written as:

$$=\dot{Y}, b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$$

Thus, we can write:

$$\dot{Y}1 \mathbin{\leftarrow} =\dot{Y}, b\P: x \mathbin{\leftarrow} \dot{Y}, b\P \mathbin{\leftarrow} 1 - G_{b}(x)$$

Differentiating $=\dot{Y}, b\P$ with respect to $b$ yields:

$$=\dot{Y}, b\P \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$$

Thus, we can finally write:

$$\dot{Y}2 \mathbin{\leftarrow} =\dot{Y}, b\P \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$$

A sufficient condition for the proposed profile of strategies to be an equilibrium is that this derivative be nonpositive for all $x$ and all $b^2$. Clearly $=\dot{Y}, b\P \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$. Thus, a sufficient condition is that $=\dot{Y}, b\P \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$. The event that $\dot{Y}2 b$ is equivalent to the event $\dot{Y}b\P$, where $h\P$ is the inverse function of $\dot{Y}b\P$. When a random variable, $X$, is the highest of two signals $i.i.d$ from a distribution function $F_{X}$, then its distribution function is given by $F_{X}$. Thus, in terms of $F_{X}$, the distribution function of the signals, we can write the sufficient condition as:

$$\dot{Y}3 \mathbin{\leftarrow} =\dot{Y}, b\P \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$$

Note that $D_{\dot{Y}b\P} \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$. Thus, a sufficient condition for $D_{\dot{Y}b\P} \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$. However, since $h\P = \frac{1}{H\P}$, we can also rewrite $D_{\dot{Y}b\P} \mathbin{\leftarrow} b\P$ as:

$$\dot{Y}4 \mathbin{\leftarrow} =\dot{Y}, b\P \mathbin{\leftarrow} b\P: h\P \mathbin{\leftarrow} H\P$$

Furthermore, since our definition of $H\P$ implies that $=\dot{Y}, b\P \mathbin{\leftarrow} b\P: \int_{b}^{1} \frac{1}{1 - G_{b}(x)} dx + \int_{0}^{1} \frac{1}{1 - G_{b}(x)} dx$. However, since
The Strategy profile \( Y_{10}P \) is an equilibrium if \( \frac{b}{h} \) is a convex function.

Note that \( \frac{b}{h} \) can be convex even when \( \beta \) itself is a concave function. E.g., \( F \cdot \beta : x^5, \) is strictly concave - \( R \) yet, \( F \cdot \beta' : x^2R, \) is convex as long as \( R = \frac{1}{2}. \)

Also note that the convexity condition is stronger than needed. (Possible fn. It is clear that the construction of the profile in \( Y_{10}P \) and the proof could be use for joint bidding or DR analysis in the IPV case as well with \( J \cdot \beta \). Finally, note that because \( H_{10}P \); \( 0 \) it is not strictly necessary that \( H_{10} \), \( q \cdot \) will do, as it will never win or set the price. The proposed strategy profile is the only candidate (up to the trivial indeterminacy of \( H_{10} \), \( q \cdot \) for a symmetric equilibrium in which \( H_{10} \), \( q \cdot \) wins with probability \( 0 \) for all \( Y_{10} \), \( q \cdot \). If this is not an equilibrium, then in general \( H_{10} \), \( q \cdot \) will depend on both \( x \) and \( q \), and would therefore seem to be the solution to a thorny multi-dimensional signal problem.

Proposition 7 informs us that when \( J \) is small enough, joint bidding necessarily raises bidding and expected price per unit, i.e., \( E \beta_1 \); \( E \beta_1 \); \( 0 \). From equation \( \beta \) we learn that the last difference is maximized with \( J = 0 \). But, is it possible that allowing DR can reverse this ranking to so that \( E \beta_1 \); \( E \beta_1 \) in the most favorable case for joint bidding? The trade-off is simple; in M1 and with \( J = 0 \), \( F \cdot \beta : x \) (see equation 4), so that the resulting price is \( E \beta_2 \). In contrast, in M3 and with \( J = 0 \), \( H \); \( x \), but with probabilities \( \frac{n_4}{n_1} \), \( \frac{3n_4}{n_1} \) and \( \frac{3n_3}{n_1} \), \( \frac{3n_3}{n_1} \), \( \frac{3n_3}{n_1} \), \( \frac{3n_3}{n_1} \) the price is set by the pair holding \( Y_3 \), \( Y_4 \) or \( Y_5 \), respectively. When \( F \cdot \beta \) is a uniform, distribution, \( H \); \( Y_5 \cdot x + 1 \); \( x \), but as \( H \cdot \beta \) is convex in this case, \( E \beta_3 \); \( E \beta_3 \); \( 0 \). \( R \); \( H \cdot \beta \); \( 0 \) ; \( \beta_2 \); \( 2 \cdot \beta_3 \); \( 0 \). \( \beta_2 \); \( 2 \cdot \beta_3 \); \( 0 \). \( \beta_2 \); \( 2 \cdot \beta_3 \); \( 0 \). \( \beta_2 \); \( 2 \cdot \beta_3 \); \( 0 \).

This demonstrates that DR cannot be ignored as even in the most favorable, “pro-revenue,” case for joint bidding in our model, with \( J = 0 \), there are plausible symmetric equilibria in M3 that places \( E \beta_3 \) below \( E \beta_1 \).

**Summary and Conclusion**

The intuition of the inference/precision effect is based on simple statistical properties of pooling information that ignore strategic considerations affecting equilibrium in joint bidding with multiple-units. Consequently, joint bidding with its improved inference and precision does not imply more aggressive bidding even when DR is not allowed. In contrast to Krishna and Morgan conclusion, less aggressive bidding resulting from a relaxation of restrictions on joint bidding is consistent with non-cooperative bidding behavior, at least in a multi-unit auctions. The other conclusion is that increased monopsony power could strengthen anti-competitive behavior. Even when the inference/precision effect has a pro-competitive impact on auctions’ revenue in the absence of DR, including it lead to lower revenues.

Analyzing the impact of joint bidding on bidders’ behavior and revenue in discriminatory-price auctions is interesting but harder. In such auctions, the inference effect, increased monopsony power, and reduction in number of bidders interact in a complicated
way. However, such analysis is beyond the scope of this work and will be attempted in the future.


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