Lecture notes for Math 3 (Calculus)
Dartmouth College, Fall term 2023
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Disclaimer: these lecture notes were made on-the-fly and are unedited. As such, they may contain minor errors.

Math 3

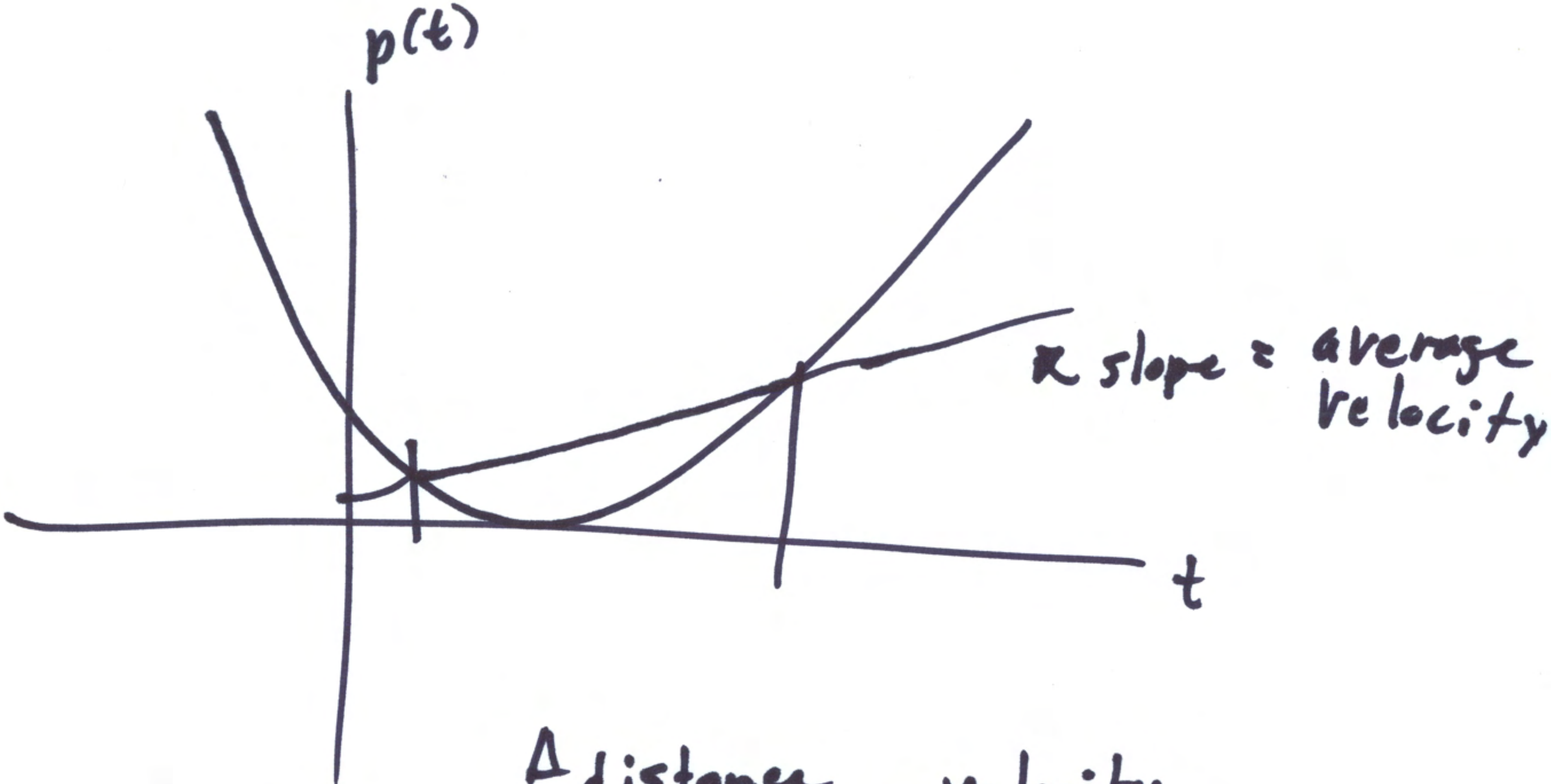
Lecture #2

9/13/23

Jonathan Lindbloom

Note on Velocity

$p(t)$



Why?. slope $= \dfrac{\Delta \text{distance}}{\Delta \text{time}} =$ velocity

This is why slope of tangent line = instantaneous velocity

Two problems:

① Tangent line: find equation of tangent line to curve $y=f(x)$ at point x.

② Velocity problem: given $p(t)$, find the function $v(t)$ that gives the instantaneous velocity. (slope of tangent to $p(t)$ at time t)

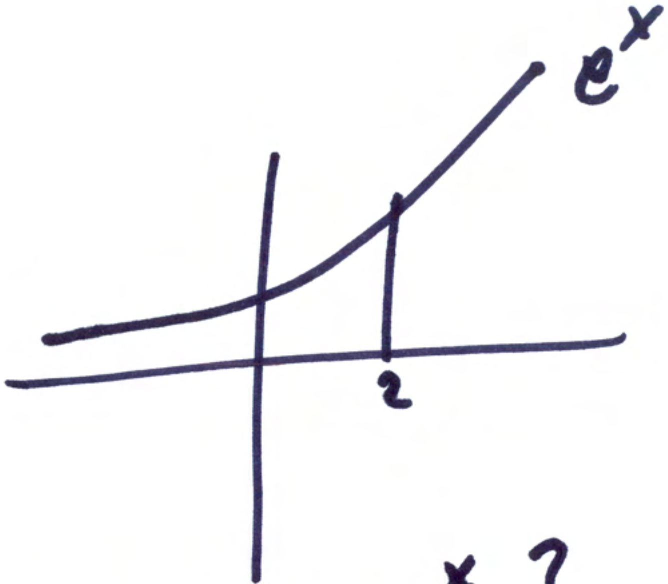
Limits

Def (Limit): Let $f(x)$ be defined on an open interval containing a, except maybe at a itself. Then we write

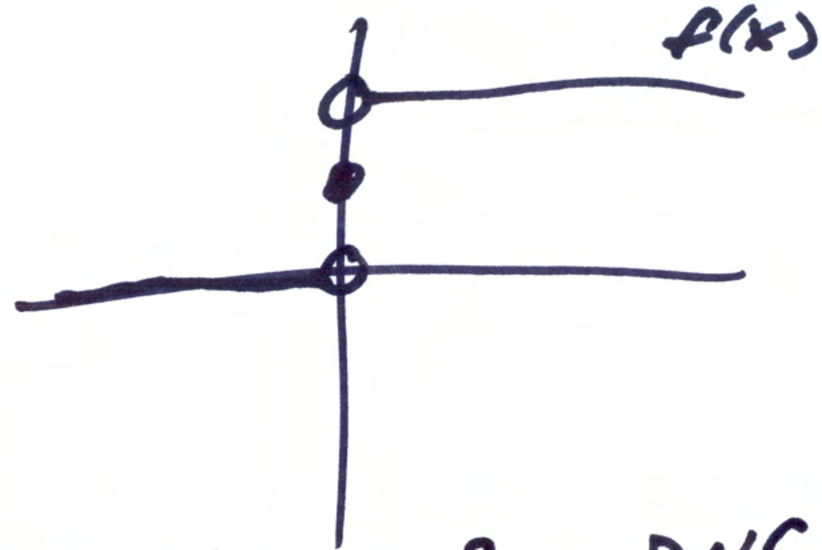
$$\lim_{x \to a} f(x) = L$$

if we can make $f(x)$ as close as we like to L by making x sufficiently close to a.

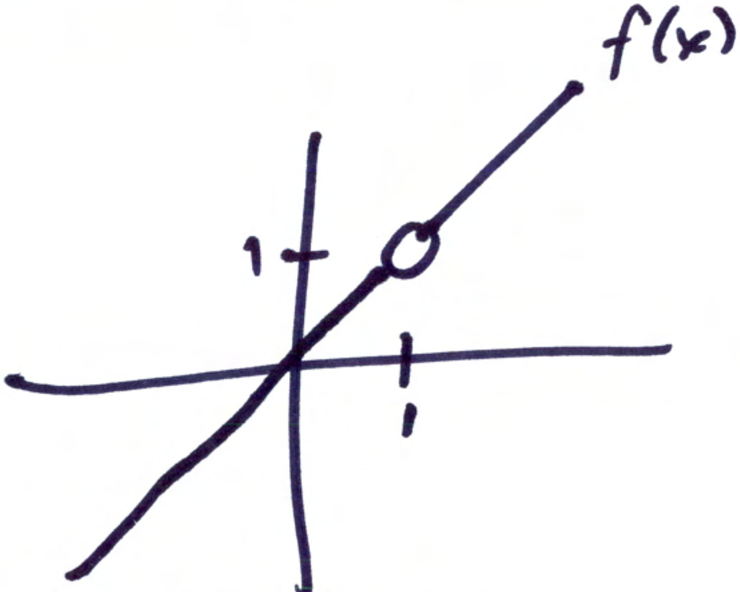
We also denote this by $f(x) \to L$ as $x \to a$.



What is $\lim_{x\to 2} e^x$?



$\lim_{x\to 0} f(x) = ?$ DNE



$\lim_{x\to 1} f(x) = 1.$

$\lim_{x\to 3} f(x) = ?$ $= +\infty$

(one-sided)

Left-right
sided limits

Def: (one-sided limits)

same as limit definition, but:

(left) We can get $f(x)$ as close to L by making x suffic. close to a but with $x < a$, and write

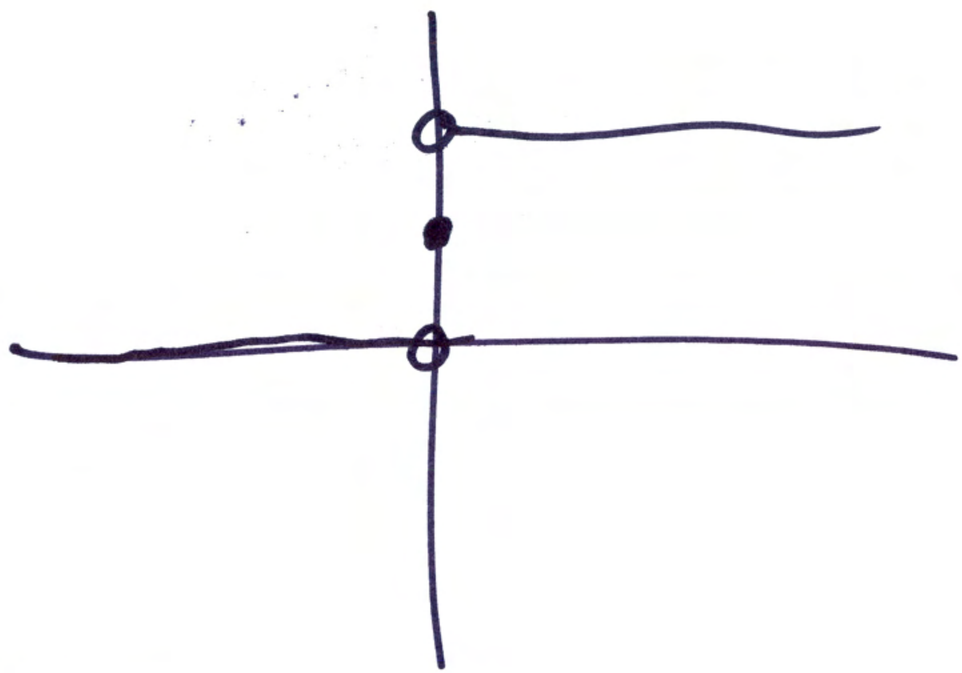
$$\lim_{x \to a^-} f(x) = L$$

(right) can get $f(x)$ as close to L by making x sufficiently close to a but with $x > a$, and write $\lim_{x \to a^+} f(x) = L$

$$\lim_{x \to a} f(x) = L$$

iff

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$



$$\lim_{x \to 0^-} f(x) = 0$$

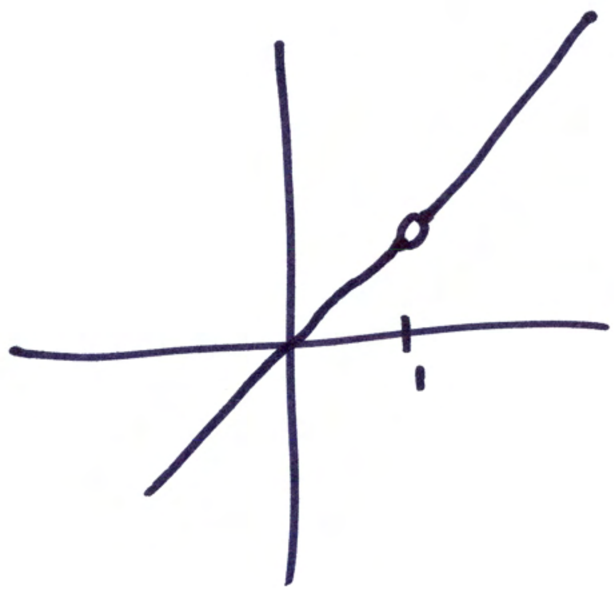
$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0} f(x) = DNE$$

$$f(x) = \frac{x(x-1)}{(x-1)}$$

$$\quad = x$$

$$f(x) = \begin{cases} x & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



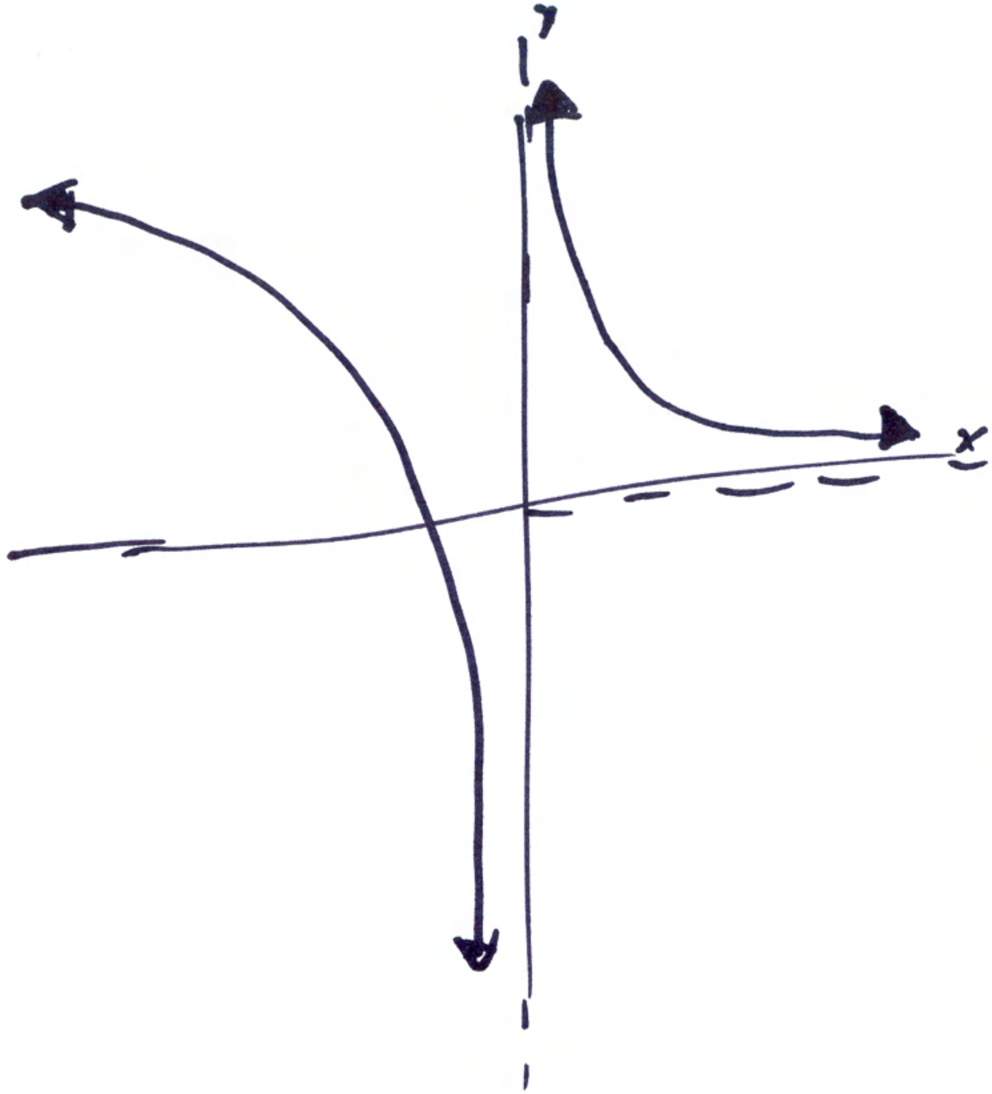
$$\lim_{x \to 1^-} f(x) = 1$$

$$\lim_{x \to 1^+} f(x) = 1$$

$$\lim_{x \to 1} f(x) = 1$$

but $f(1)$ undefined.

ok.



$$\lim_{x \to -\infty} f(x) = +\infty$$

$$\lim_{x \to 0^-} f(x) = -\infty$$

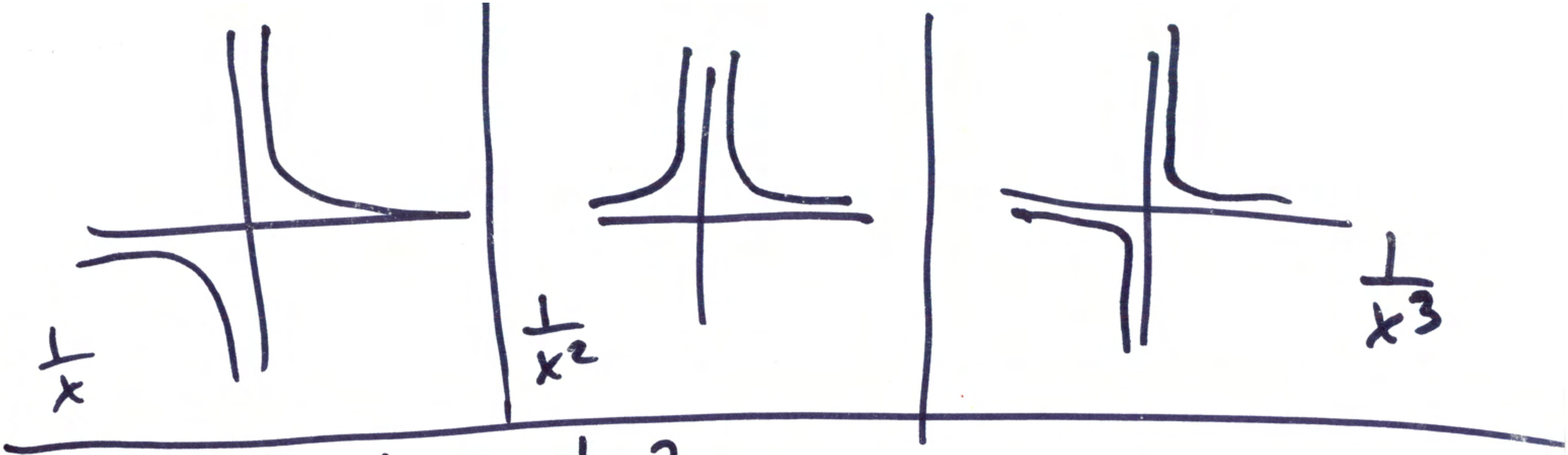
$$\lim_{x \to 0^+} f(x) = +\infty$$

$$\lim_{x \to \infty} f(x) = 0$$

$$\lim_{x \to 0} f(x) = DNE$$

Vertical Asymptotes

Def: (Vertical asymptotes): If the limit or any one-sided limit of $f(x)$ at $x=a$ is $\pm\infty$, then the lim $x=a$ is a vertical asymptote of $y=f(x)$.



- What is $\lim_{x\to 0} \frac{1}{x}$? $(= \text{DNE})$
- What is $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$?
  $(" +\infty")$ $(" -\infty")$
- What is $\lim_{x\to 0} \frac{1}{x^2}$? $= +\infty$
- What is $\lim_{x\to 0} \frac{1}{x^3} =$? DNE
- What is $\lim_{x\to 0} \frac{1}{x^n}$? $\lim_{x\to 0^-} \frac{1}{x^n}$? $\lim_{x\to 0^+} \frac{1}{x^n}$?

It is:

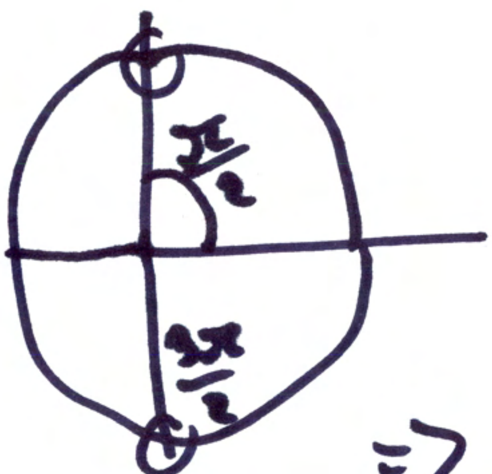
$$\lim_{x \to 0} \frac{1}{x^n} = \begin{cases} \text{DNE} & n \text{ odd} \\ +\infty & n \text{ even} \end{cases}$$

$$\lim_{x \to 0^-} \frac{1}{x^n} = \begin{cases} -\infty & n \text{ odd} \\ +\infty & n \text{ even} \end{cases}$$

$$\lim_{x \to 0^+} \frac{1}{x^n} = \begin{cases} +\infty & n \text{ odd} \\ +\infty & n \text{ even} \end{cases}$$

Ex. 8: Find vertical asymptotes of $f(x) = \tan(x)$.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$ When does $\cos(x) = 0$?



When $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

OR just $x = \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$.

$\Rightarrow$ vertical asymptotes at

$$x = \frac{(2k+1)\pi}{2}, \; k \in \mathbb{Z}. \text{ (tan)}$$

Q: What is $\lim_{x \to \frac{(2k+1)\pi}{2}} \tan(x)$? (DNE). $\lim_{x \to \frac{(2k+1)\pi}{2}^-}$? $+$? $(-\infty)$

Limit Laws

Limit Laws:

Suppose $c \in \mathbb{R}$ and $\lim\limits_{x \to a} f(x)$ & $\lim\limits_{x \to a} g(x)$ exist.

Then

1. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$.

2. $\lim\limits_{x \to a} [c f(x)] = c \lim\limits_{x \to a} f(x)$.

3. $\lim\limits_{x \to a} [f(x) g(x)] = \left(\lim\limits_{x \to a} f(x)\right) \cdot \left(\lim\limits_{x \to a} g(x)\right)$

4. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ if $\lim\limits_{x \to a} g(x) \neq 0$.

Given only $\lim_{x \to a} x = a$, + limit laws,

What is $\lim_{x \to a} p_n(x) = \lim_{x \to a} \left[c_0 + c_1 x^1 + \cdots + c_n x^n \right]$?

Using sum, constant, + product properties we get

$$\lim_{x \to a} p_n(x) = c_0 \left[\lim_{x \to a} 1 \right] + c_1 \left[\lim_{x \to a} x \right] + \cdots + c_n \left[\lim_{x \to a} x^n \right]$$

$$= c_0 + c_1 a + \cdots + c_n a^n$$

$$= p_n(a).$$

polynomials you can just plug in!

Using just $\lim_{x \to a} x = a$, & limit laws, $(a > 0)$

what is $\lim_{x \to a} \sqrt{x}$?

Whatever it is, let $L = \lim_{x \to a} \sqrt{x}$.

Then
$$L^2 = \left(\lim_{x \to a} \sqrt{x}\right)\left(\lim_{x \to a} \sqrt{x}\right)$$
$$= \lim_{x \to a} \sqrt{x}\sqrt{x}$$
$$= \lim_{x \to a} x$$
$$= a$$

$\Rightarrow L^2 = a \Rightarrow L = \pm\sqrt{a}$?

No, $L = \sqrt{a}$. $(= f(a))$.

So for roots, just plug in

If $\forall a \in D$ $f(a) = \lim_{x \to a} f(x)$, then

$f(x)$ is said to be continuous on D.

Most "simple" functions you know are continuous.

swap

If $f(a) = \lim_{x \to a} f(x)$, then

$f(x)$ is said to be continuous at $x = a$.

"Continuous functions are ones you can draw without picking up pencil" valuate $\lim_{x \to 1} \frac{x(x+1)}{x^2-1}$ } $\frac{x(x-1)}{x^2-1}$

$$\frac{(9-x)(3+\sqrt{x})}{9 - x + 3\sqrt{x} - 3\sqrt{x}}$$

$$\frac{x+1)}{-1} = \frac{x\cancel{(x+1)}}{\cancel{(x+1)}(x-1)} = \frac{x}{x+1} \quad \text{if } x \neq 1$$

$3 + \sqrt{x}$ if $x \neq 9$

$$\lim_{x \to 1} \frac{x(x+1)}{x^2-1} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}.$$

$3 + 3 = 6.$

Math 3

Lecture #3

9/15/23

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Warmup

Limit

Example

Evaluate $\displaystyle\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$.

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \frac{2hx + h^2}{h} = 2x + h \quad \text{when } h \neq 0$$

$$\Rightarrow \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} 2x + h = 2x.$$

Some limits to know:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x \to 0} \frac{\cos(x)}{x} = DNE$$

$$\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0$$

What is $\lim_{x \to 0} \frac{\sin(7x)}{x}$?

$$= \frac{7}{7} \lim_{x \to 0} \frac{\sin(7x)}{x}$$

$$= 7 \lim_{x \to 0} \frac{\sin(7x)}{7x}$$

let $u = 7x$ $\downarrow$

$$= 7 \lim_{u \to 0} \frac{\sin(u)}{u}$$

$$= 7 \cdot 1 = 7$$

Squeeze

Theorem

Squeeze Thm:

If $f(x) \le g(x) \le h(x)$ ~~when~~ on an open interval containing a (except maybe at a),

~~then~~ and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

t is

$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$?. $(=0)$.

ph using desmos!

ote: $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad \forall x$.

t both sides by x^2:

$$-x^2 \leq x^2\cos\left(\frac{1}{x}\right) \leq x^2$$

$$\underset{f(x)}{\overset{\shortparallel}{}} \leq \underset{g(x)}{\overset{\shortparallel}{}} \leq \underset{h(x)}{\overset{\shortparallel}{}}$$

Formal Definition of Limit

Def: (Limits, formal) We say that

$$\lim_{x \to a} f(x) = L \quad \text{if} \quad \forall \varepsilon > 0, \ \exists \delta > 0$$

s.t.

$$|x-a| < \delta \implies |f(x) - L| < \varepsilon.$$

Ex: $a = -\frac{1}{2}$ (on board)

$f(x) = \cos(x)$

Ex:

Consider $\lim_{x \to a} x = a$ (since I told you).

Let $\varepsilon = 1$. Can you find δ s.t.

$|x-a| < \delta \Rightarrow |x-a| < 1$?

Yes, just $\delta = 1$.

What about $\varepsilon = \frac{1}{2}$? Yes, just $\delta = \frac{1}{2}$.

In general, pick $\delta = \varepsilon$.

Since this works for any $\varepsilon > 0$, we know that $\lim_{x \to a} x = a$ by formal definition.

Ex: Prove $\lim_{x \to a} 7x = 7a$.

~~We want to~~ For any $\varepsilon > 0$, we want to find $\delta > 0$ s.t. $|x-a| < \delta \Rightarrow |7x - 7a| < \varepsilon$.

Idea: make the ε inequality "look like" the δ inequality. Do:

$$|7x - 7a| < \varepsilon \Rightarrow 7|x-a| < \varepsilon$$
$$\Rightarrow |x-a| < \frac{\varepsilon}{7},$$

Idea: Pick $\delta = \frac{\varepsilon}{7}$.

F Proof?

$\varepsilon > 0$, and let $\delta = \frac{\varepsilon}{7}$. Then

$$|x-a| < \delta \Rightarrow |x-a| < \frac{\varepsilon}{7}$$

$$\Rightarrow 7|x-a| < \varepsilon$$

$$\Rightarrow |7x-7a| < \varepsilon.$$

we have shown by def. that

$7x = 7a$.

Ex: Prove $\lim_{x \to 2} 4x - 3 = 5$.

For any $\varepsilon > 0$,
We need to find $\delta > 0$ s.t.

$$|x-2| < \delta \Rightarrow |(4x-3)-5| < \varepsilon.$$

$$|(4x-3)-5| < \varepsilon$$

$$\Rightarrow |4x-8| < \varepsilon$$

$$\Rightarrow 4|x-2| < \varepsilon$$

$$\Rightarrow |x-2| < \frac{\varepsilon}{4},$$

so pick $\delta = \frac{\varepsilon}{4}$.

How to write proof?

Let $\varepsilon > 0$. Pick $\delta = \frac{\varepsilon}{4}$.

Then

$$|x-2| < \delta \Rightarrow |x-2| < \frac{\varepsilon}{4}$$

$$\Rightarrow 4|x-2| < \varepsilon$$

$$\Rightarrow |4x-8| < \varepsilon,$$

so by definition $\lim_{x \to 2} 4x-3 = 5$.

Ex: Prove that $\lim_{x \to 4} x^2 = 16$.

(scratch)

Start with $|x^2 - 16| < \varepsilon$.

$\Rightarrow |(x-4)(x+4)| < \varepsilon$

$\Rightarrow |x-4||x+4| < \varepsilon$

$\Rightarrow$? can't get $|x-4| <$ "some constant".

Idea: Make extra assumptions about x. Since limit occurs when x near 4, just assume that $|x-4| < 1$. $\Rightarrow -1 < x-4 < 1$

$\Rightarrow 7 < x+4 < 9 \Rightarrow |x+4| < 9$. So we get $|x-4||x+4| < \varepsilon$

$\Rightarrow 9|x-4| < \varepsilon \Rightarrow \boxed{|x-4| < \frac{\varepsilon}{9}}$.

So do we just pick $\delta = \frac{\varepsilon}{9}$?

No! We also made assumption that $|x-4|<1$, so we also require that $\delta < 1$.

How do we pick δ s.t. both

$\delta < 1$ and $\delta < \frac{\varepsilon}{9}$?

Pick $\delta = \min\{1, \frac{\varepsilon}{9}\}$.

Ex: Prove $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

Let $L_1 = \lim_{x \to a} f(x)$, $L_2 = \lim_{x \to a} g(x)$,

For any $\varepsilon > 0$, we need to find $\delta > 0$
s.t.
$$|x - a| < \delta \implies |f(x) + g(x) - (L_1 + L_2)| < \varepsilon.$$

Using triangle inequality $|a + b| \le |a| + |b|$,
we see
$$|f(x) + g(x) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)|$$
$$\le |f(x) - L_1| + |g(x) - L_2|.$$

Idea: Pick δ_1 + δ_2 s.t.

$$|x-a| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$$

+ $$|x-a| < \delta_2 \Rightarrow |g(x) - L_2| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ to combine these, which gives

$$|x-a| < \delta \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$$
$$\text{and } |g(x) - L_2| < \frac{\varepsilon}{2}.$$

So, given any $\varepsilon > 0$, we can pick these δ_1 δ_2 + δ which gives us

$$|x-a| < \delta \Rightarrow |f(x) + g(x) - (L_1 + L_2)| \quad \checkmark.$$
$$\leq |f(x) - L_1| + |g(x) - L_2| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

<u>Ex:</u> prove $\lim_{x \to 9} \sqrt{x} = 3$.

(scratch) start with $|\sqrt{x} - 3| < \varepsilon$. Can we make it "look like" δ inequality $|x - 9| < \_\_\_\_$?.

$$|\sqrt{x} - 3| < \varepsilon$$

$$\Downarrow$$

$$-\varepsilon < \sqrt{x} - 3 < \varepsilon \implies 3 - \varepsilon < \sqrt{x} < 3 + \varepsilon$$

$$\implies (3-\varepsilon)^2 < x < (3+\varepsilon)^2$$

$$\implies 9 - 6\varepsilon + \varepsilon^2 < x < 9 + 6\varepsilon + \varepsilon^2$$

$$\implies -6\varepsilon + \varepsilon^2 < x - 9 < 6\varepsilon + \varepsilon^2$$

$$-6\varepsilon - \varepsilon^2 < -6\varepsilon + \varepsilon^2$$

$$\implies |x - 9| < 6\varepsilon + \varepsilon^2$$

Pick $\delta = 6\varepsilon + \varepsilon^2$.

Def: (infinite limits)

We say that $\lim_{x \to a} f(x) = \infty$

if $\forall M > 0$, $\exists \delta > 0$ s.t.

$$|x-a| < \delta \Rightarrow f(x) > M.$$

Ex: Prove $\lim_{x \to 0} \frac{1}{x^4} = +\infty$.

For any $M > 0$, we need to find $\delta > 0$ s.t. $\underbrace{|x - 0|}_{|x|} < \delta \Rightarrow \frac{1}{x^4} > M$.

Can we take $\frac{1}{x^4} > M$ and set an inequality $|x| < \underset{\text{delta}}{\underline{\qquad}}$?

$\frac{1}{x^4} > M \Rightarrow x^4 < \frac{1}{M} \Rightarrow |x| < \sqrt[4]{\frac{1}{M}}$

So pick $\delta = \sqrt[4]{\frac{1}{M}}$.

Can we make a similar argument

how $\lim_{x \to 0} \frac{1}{x^3} = +\infty$? No!

isn't true. But it is true

$\lim_{x \to 0^+} \frac{1}{x^3} = +\infty$. What is

formal definition of one-sided

ts?

D We say that $\lim_{x \to a^+} f(x) = L$ if

$\forall$, $\exists \delta > 0$ s.t.

$0 < x - a < \delta \Rightarrow |f(x) - L| < \varepsilon$.

For $_{a^-} f(x) = L$, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$-\delta < x - a < 0 \Rightarrow |f(x) - L| < \varepsilon$.

<u>Ex</u>: Prove $\displaystyle\lim_{x \to 0^+} \frac{1}{x^3} = +\infty$.

For any $M > 0$, we need to find $\delta > 0$

s.t. $|x| < \delta \Rightarrow \dfrac{1}{x^3} > M$.

$$\frac{1}{x^3} > M \Rightarrow x^3 < \frac{1}{M} \quad \cancel{x < \frac{1}{\sqrt[3]{M}}}$$

$\Big\downarrow$ since $x > 0$!

$$0 < x < \frac{1}{\sqrt[3]{M}}.$$

So pick $\delta = \dfrac{1}{\sqrt[3]{M}}$.

Negative infinite limits?

$$\lim_{x \to a} f(x) = \cancel{L} -\infty$$

if $\forall N < 0$, $\exists \delta > 0$ s.t.

$$|x-a| < \delta \Rightarrow f(x) < N.$$

<u>Ex:</u> Prove $\displaystyle\lim_{x \to 0^-} \frac{1}{x^3} = -\infty$.

$$\frac{1}{x^3} < N \Rightarrow x^3 > \frac{1}{N} \Rightarrow x > \sqrt[3]{\frac{1}{N}}.$$

But we want $|x| < \underline{\quad\quad}$? What do we do? δ needs to be positive, but $\sqrt[3]{\frac{1}{N}}$ is a negative number. So pick

$$x > \sqrt[3]{\frac{1}{N}} \Rightarrow -x < -\sqrt[3]{\frac{1}{N}}$$

$$\Rightarrow |x| < -\sqrt[3]{\frac{1}{N}} = \delta.$$

MATH 3

Lecture #4

9/18/23

Q: Prove that $\lim_{x \to 0} \frac{1}{x}$ is not finite.

Suppose that $\lim_{x \to 0} \frac{1}{x} = L$ for some $L \in \mathbb{R}$.

Consider $\varepsilon = 1$. Is there some $\delta > 0$ s.t. $|x| < \delta \Rightarrow \left|\frac{1}{x} - L\right| < \varepsilon$? No. For any $\delta > 0$, consider $x_0 = \min\{\delta, \frac{1}{L+2}\}$. Then we have both $|x_0| < \delta$ and $x_0 < \frac{1}{L+2} \Rightarrow \frac{1}{x_0} > L+2$.

So for $\varepsilon = 1$, there is no $\delta > 0$ s.t. $|x| < \delta \Rightarrow \left|\frac{1}{x} - L\right| < \varepsilon$.

$\Rightarrow \lim_{x \to 0} \frac{1}{x}$ cannot be finite.

Def: We say that $\lim_{x \to a^+} f(x) = +\infty$ if $\forall N > 0,\ \exists \delta > 0$ such that $a < x < a + \delta \Rightarrow f(x) > N$.

Ex: Prove that $\lim_{x \to 0^+} \frac{1}{x} = +\infty$.

Proof: Let $N > 0$. Pick $\delta = \frac{1}{N+1}$.

Then $0 < x < \delta \Rightarrow \frac{1}{x} > N + 1 > N$. $\blacksquare$

Ex: Prove that $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

Proof: Let $N < 0$. Pick $\delta = -\frac{1}{N-1}$.

Then $-\delta < x < 0 \Rightarrow \frac{1}{x} < N - 1 < N$. $\blacksquare$

Continuity

<u>Def</u> (continuity):

A function $f(x)$ is continuous at $x=a$ if $\lim_{x \to a} f(x) = f(a)$.

If $f(x)$ is continuous $\forall x \in D$, then $f(x)$ is continuous on D.

<u>Q</u>: Is $3x^2 + 2x + 1$ continuous on $\mathbb{R}$? Yes!

<u>Q</u>: Is $\ln(x)$ continuous on $\mathbb{R}$?

No!

(ε-δ def of continuity):

$f(x)$ is continuous at $x=a$ if

$\forall \varepsilon > 0, \quad \exists \delta > 0$ s.t.

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

$f(x)$ is continuous on D if

$\forall x \in D, \ \forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

<u>Q:</u> Prove that if $f(x)$ is continuous on D, then $g(x) := cf(x)$ is continuous on D as well.

<u>Proof:</u> We need to show that $\forall a \in D \ \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |cf(x) - cf(a)| < \varepsilon$.

Note that $|cf(x) - cf(a)| < \varepsilon \Rightarrow |f(x) - f(a)| < \underbrace{\frac{\varepsilon}{|c|}}_{:= \tilde{\varepsilon}}$.

Since $f(x)$ is continuous on D, we know that $\forall a \in D \ \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

So we can just pick $\varepsilon = \hat{\varepsilon}$ into this definition, which tells us $\exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \tilde{\varepsilon}$

$$\Rightarrow |cf(x) - cf(a)| < \varepsilon.$$

This proves that $g(x) := cf(x)$ is also continuous on D.

∎

Def: $f(x)$ is continuous from the **right** at a if $\lim_{x \to a^+} f(x) = f(a)$.

continuous from the **left** if $\lim_{x \to a^-} f(x) = f(a)$.

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$$

Is this continuous from left or right?

Continuity laws: Let $c \in \mathbb{R}$, and

Let $f(x)$ & $g(x)$ be continuous at a.

Then the following functions are continuous at a:

① $f(x) + g(x)$

② $f(x) - g(x)$

③ $c f(x)$

④ $f(x) g(x)$

⑤ $\dfrac{f(x)}{g(x)}$ if $g(a) \neq 0$.

<u>Thm:</u> ① If $f(x)$ is a polynomial, then $f(x)$ is continuous on $\mathbb{R}$.

② If $f(x)$ is a rational function, then $f(x)$ is continuous wherever it is defined.

Where is $f(x) = \dfrac{x^2-9}{x+3}$

fined? Everywhere except $x=-3$,
domain is

$$D = (-\infty, -3) \cup (3, +\infty).$$

re is $f(x)$ continuous? Same
D, by the theorem.

<u>Q:</u> Where is $f(x) = \dfrac{x+3}{x^2+x-2}$ defined? Everywhere except $x=-2$ & $x=1$.

$\Rightarrow D = (-\infty,-2) \cup (-2,1) \cup (1,\infty)$.

Where is $f(x)$ continuous?

Same, by the theorem.

as defined

n: These functions are continuous
r their domains:

Polynomials

trig functions

Exponential Functions

Rational Functions

Inverse trig functions
sin(x), arccos(x), arctan(x))

(6) Root functions

(7) Logarithmic functions

<u>Thm</u>: If $f(x)$ is continuous at $x = b$ and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f(b), \quad \text{i.e.,}$$

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

Thm: If $g(x)$ is continuous at $x=a$ and $f(x)$ is continuous at $g(a)$, then $(f \circ g)(x) = f(g(x))$ is continuous at $x=a$.

What is $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$?

$\lim_{x \to 0} \frac{1}{x}$ DNE, so we can't apply the theorem. In fact, $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ DNE as well.

(ε-δ) Proof that $\lim_{x \to 0} f(x) = f(0) = 0$

Let $\varepsilon > 0$. Pick $\delta = \sqrt[4]{\varepsilon}$. Then

$$|x| < \delta \Rightarrow |x| < \sqrt[4]{\varepsilon} \Rightarrow |x|^4 < \varepsilon$$

since $-x^4 \le x^4 \sin(\frac{1}{x}) \le x^4 \Rightarrow |x^4 \sin(\frac{1}{x})| \le x^4$, $= |x|^4$

$$\Rightarrow |x^4 \sin(\tfrac{1}{x})| \le |x|^4 < \varepsilon.$$

This proves that $\lim_{x \to 0} f(x) = f(0) = 0$. $\blacksquare$

Show that $f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

continuous on $\mathbb{R}$.

---

he theorem, we know that $f(x)$ is

uous on $(-\infty, 0) \cup (0, +\infty)$. So we

need to prove that $\lim_{x \to 0} f(x) = f(0) = 0$.

we can just invoke squeeze theorem,

lower bound $\ell(x) = -x^4$

upper bound $u(x) = x^4$.

can also formally prove this (see next page).

MATH 3

Lecture #5

9/20/23

Jonathan Lindblom

Intermediate Value Theorem

Intermediate Value Theorem:

Suppose that $f(x)$ is continuous on the closed interval $[a,b]$ and let N be any number $f(a) < N < f(b)$, with $f(a) \neq f(b)$.

Then $\exists\, c \in (a,b)$ such that

$$f(c) = N.$$

<u>Ex</u>: Show there is a solution $x^* \in [-1,1]$ to $x = 0$.

$f(x) = x$ is continuous on $[-1,1]$, and $f(-1) = -1$, $f(1) = 1$. Then, by IVT $\forall N \in (-1,1)$ $\exists c \in (-1,1)$ s.t. $f(c) = N$.

Setting $N = 0$, $\Rightarrow \exists x^* \in (-1,1)$ s.t. $f(x^*) = 0$. ✓

Ex: Show that $f(x) = \cos(e^x)$ has a root in $(0,1)$.

First, graph. Pick $a=0$, $b=1$.

By a Thm, we know that $f(x)$ is continuous on $[0,1]$. $f(0) = \underset{(\approx 0.54)}{\cos(1)} > 0$, and $f(1) = \underset{(\approx -0.9)}{\cos(e)} < 0$. By IVT, $\exists\, c \in (0,1)$ s.t. $f(c) = 0 \in (\cos(e), \cos(1))$.

<u>Ex</u>: Show that there is a solution to $\cos(x) = x$ on the interval $[0,1]$.

<u>A</u>: A solution to $\cos(x) = x$ is also a root of $f(x) = \cos(x) - x$, which is continuous on $[0,1]$. Note that $f(0) = 1$, $f(1) = \cos(1) - 1 \approx -0.46 < 0$. Thus by the IVT $\exists\, c \in (0,1)$ s.t. $f(c) = 0$.

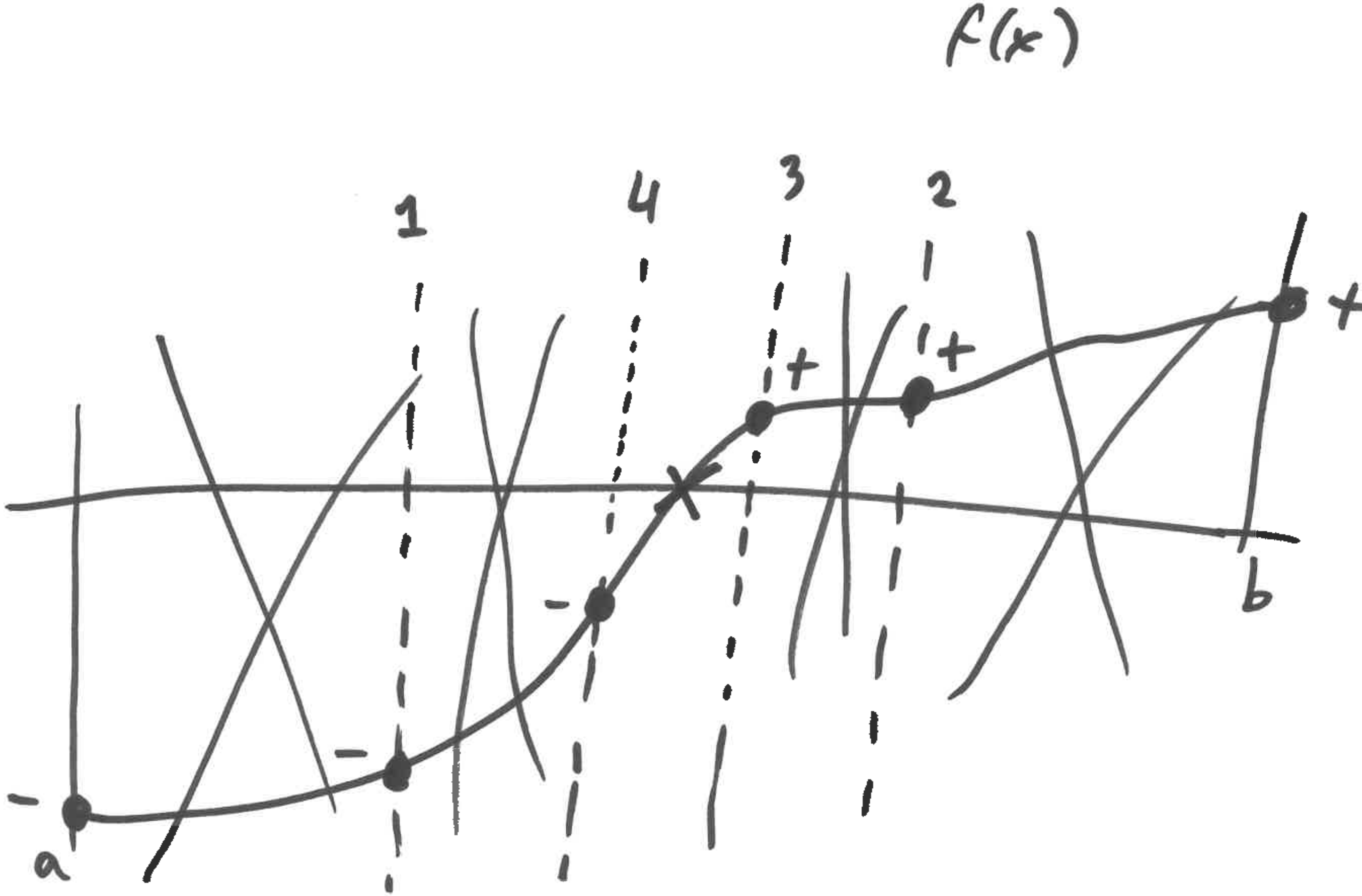
$\Rightarrow \cos(c) - c = 0 \Rightarrow \cos(c) = c$,

so there is a solution on the interval $[0,1]$.

Application: Bisection root-finding

Let $f(x)$ be continuous on $[a,b]$, and suppose $f(a) < 0$, $f(b) > 0$. By IVT, this means there is a root $c \in (a,b)$. How can we find it?

Idea: cut the interval (a,b) into two parts, then check endpoints of subintervals to check what subinterval the root is in. Rinse and repeat.



f(x)
1
4
3
2
a
b

Horizonal Asymptotes

Def: (limit at ∞)

Let $f(x)$ be defined on an interval (a, ∞). Then we say that $\lim_{x \to \infty} f(x) = L$ if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x > \delta \Rightarrow |f(x) - L| < \varepsilon$.

We say $\lim_{x \to -\infty} f(x) = L$ if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x < -\delta \Rightarrow |f(x) - L| < \varepsilon$.

Def: If $\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to +\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of $f(x)$.

Q: Does $f(x) = x^2+1$ have any horizontal asymptotes? **No.**

Q: Does $f(x) = \frac{1}{x}$ have any horizontal asymptotes? Yes, $y=0$.

Q: What are the horizontal asymptotes of $f(x) = \begin{cases} \frac{1}{x}+1 & x>0 \\ \frac{1}{x}-3 & x<0 \end{cases}$?

$y=1$ & $y=-3$.

Q: What are the horizontal asymptotes of $f(x) = \dfrac{3x^2+4}{4x^2-9}$?

$$\lim_{x\to\infty} \frac{3x^2+4}{4x^2-9} = \lim_{x\to\infty} \frac{3+\frac{4}{x^2}}{4-\frac{9}{x^2}}$$

$$= \frac{3+\lim_{x\to\infty}\frac{4}{x^2}}{4-\lim_{x\to\infty}\frac{9}{x^2}}$$

$$= \frac{3}{4}$$

Same for $\displaystyle\lim_{x\to-\infty} \frac{3x^2+4}{4x^2-9} = \frac{3}{4}$.

Q: Does $f(x) = \dfrac{x+1}{x^2-1}$ have any horizontal asymptotes?

$$\lim_{x\to\infty} \frac{x+1}{x^2-1} = \lim_{x\to\infty} \frac{(x+1)}{(x+1)(x-1)}$$

$$= \lim_{x\to\infty} \frac{1}{x-1}$$

$$= 0$$

Same for $\lim_{x\to-\infty} f(x) = 0$.

So yes, at $y = 0$.

Ex: Does $f(x) = 3^x$ have any horizontal asymptotes?

$$\lim_{x \to +\infty} f(x) = +\infty, \text{ but}$$

$$\lim_{x \to -\infty} f(x) = 0, \text{ so } y = 0 \text{ is}$$

an asymptote.

Common pattern for rational functions?

① If numerator has higher degree than denominator, then <u>no</u> horizontal asymptotes.

② If numerator has lower degree than denominator, then $y=0$ is a horizontal asymptote.

③ If numerator is same degree as denominator, then the ratio of leading coefficients.

Ex: Horizontal asymptotes of

$$f(x) = \frac{3x^3 - 2x + 8x + 7}{7 - x^2 - 8x^3 + x} ?$$

Just $y = -\frac{3}{8}$.

Ex: Does $f(x) = \dfrac{\sqrt[27]{7x^{27}+3x+1}}{5x+8}$

have any horizontal asymptotes?

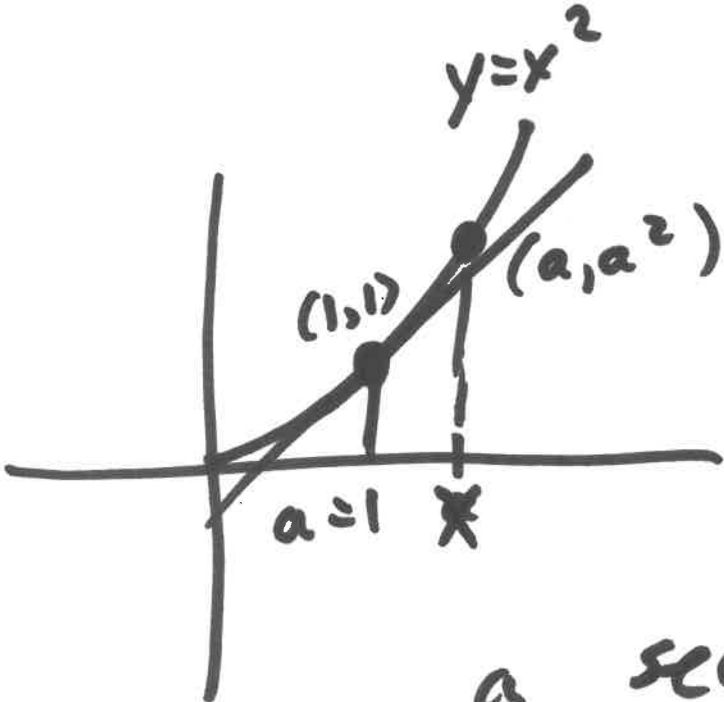
$$\lim_{x\to+\infty} \frac{\sqrt[27]{7x^{27}+3x+1}}{5x+8} = \lim_{x\to+\infty} \frac{\dfrac{\sqrt[27]{7x^{27}+3x+1}}{x}}{\dfrac{5x+8}{x}}$$

$$= \lim_{x\to+\infty} \frac{\dfrac{\sqrt[27]{7x^{27}+3x+1}}{\sqrt[27]{x^{27}}}}{\dfrac{5x+8}{x}} = \lim_{x\to+\infty} \frac{\sqrt[27]{\dfrac{7x^{27}+3x+1}{x^{27}}}}{\dfrac{5x+8}{x}}$$

$$= \frac{\sqrt[27]{7}}{5}.$$

So $y = \dfrac{\sqrt[27]{7}}{5}$.

The Derivative



How to find slope of tangent line?

Take limit as secant point defining a secant line approaches the point of interest, in this case, $a=1$. So, compute

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x+a)(x-a)}{(x-a)}$$

$$= \lim_{x\to a} x+a = 2a,$$ with $a=1 \Rightarrow m=2$.

__Def__: the tangent line to the curve $y = f(x)$ at point $(a, f(a))$ is the line through $(a, f(a))$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(if the limit exists)

<u>Def</u>: In general, we define the derivative of $f(x)$ at $x=a$ as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

We may also write this as $f'(a) = \left.\frac{df}{dx}\right|_{x=a}.$

<u>Def</u>: We can also view the derivative as a function of x. For $f(x)$, we define its derivative function $f'(x)$ as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x}.$$

We may also write this as

$$f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x).$$

Def: A function $f(x)$ is differentiable at $x=a$ if $f'(a)$ exists. $f(x)$ is differentiable on (a,b) if it is differentiable everywhere in (a,b).

Instantaneous Velocity: $(p(t): \mathbb{R} \to \mathbb{R})$

Let $p(t)$ be a function denoting position as a function of time. Then the instantaneous velocity $v(t)$ is given by

$$v(t) = p'(t) = \lim_{h \to 0} \frac{p(t+h) - p(t)}{h}.$$

<u>Claim</u>: Consider $f(x)$ and $g(x) := f(x) + C$ for some $C \in \mathbb{R}$.

Then $f'(x) = g'(x)$.

<u>Proof</u>:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) + C] - [f(x) + C]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'(x).$$

$\Rightarrow$ Adding constants to functions does not change their derivatives.

Q: What is the derivative $\frac{d}{dx} x^3$?

A: $\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{(x+h)(x^2+2hx+h^2) - x^3}{h}$

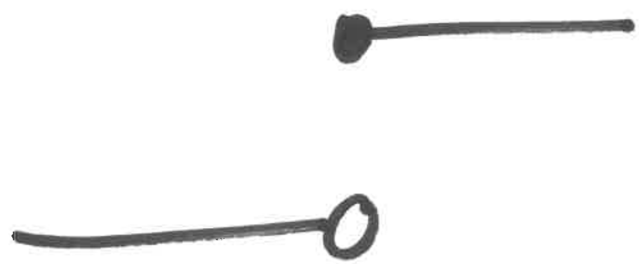
$= \lim_{h \to 0} \frac{x^3 + 2hx^2 + h^2x + hx^2 + 2h^2x + h^3 - x^3}{h}$

$= \lim_{h \to 0} 2x^2 + hx + x^2 + 2hx + h^2$

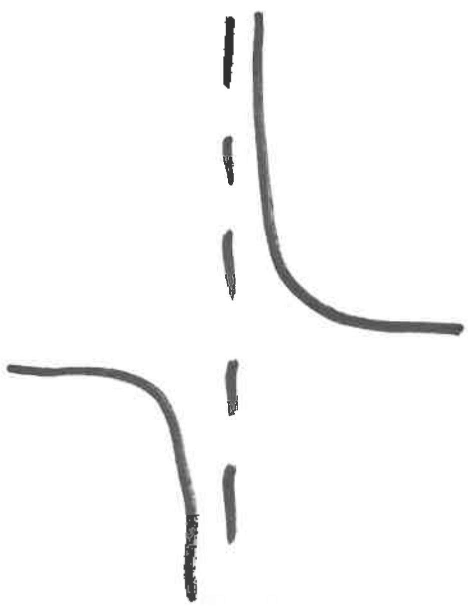
$= 2x^2 + x^2 = 3x^2.$

Examples of non-differentiability:

cusps/corners



discontinuities



vertical asymptotes

Q: What is the derivative of
$f(x) = x^2 + x + 1$?

A: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{[(x+h)^2 + (x+h) + 1] - [x^2 + x + 1]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 + x + h - 1 - x^2 - x - 1}{h}$$

$$= \lim_{h \to 0} \frac{2hx + h^2 + h}{h}$$

$$= \lim_{h \to 0} 2x + h + 1 = 2x + 1.$$

Q: What is the tangent line to $f(x) = x^2 + x + 1$ at $x = 3$?

Point: $(3, f(3))$, $f(3) = 13$

We just saw $f'(x) = 2x + 1$,

so slope is $m = f'(3) = 7$.

Using point-slope form,

$$y - 13 = 7(x - 3)$$

$$\Rightarrow y = 7x - 21 + 13$$

$$\Rightarrow y = 7x - 8$$

is the tangent line.

MATH 3

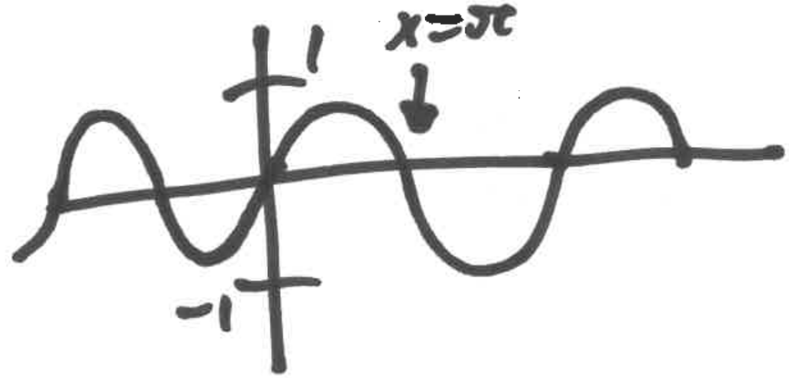
Trigonometry Review

9/21/23

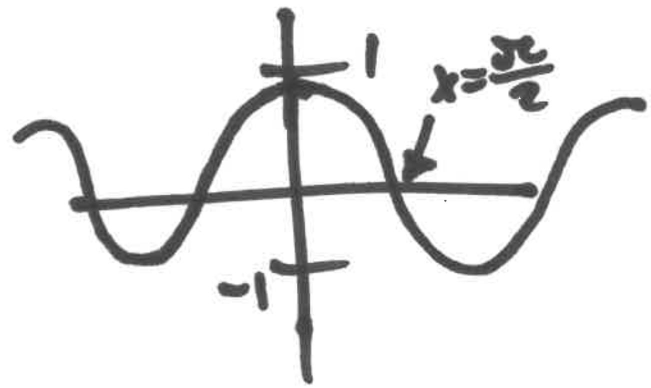
Jonathan Lindblom

3 basic functions

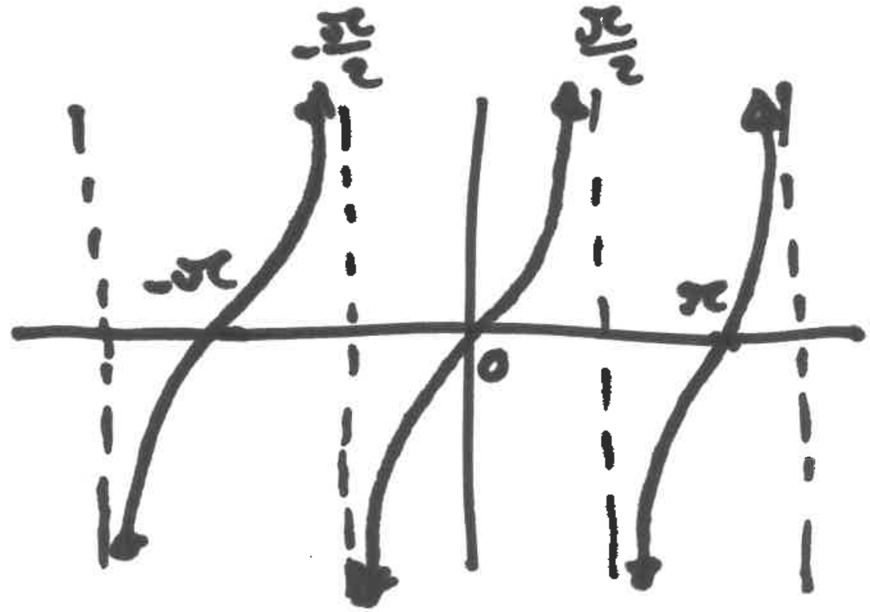
sin(x)



cos(x)



tan(x)



What is....?

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\tan\left(-\frac{3\pi}{4}\right) = 1$

$\sin\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

$= -\frac{\sqrt{3}}{3}$

$\sin\left(-\frac{5\pi}{2}\right) = -1$

Q: Solve $2\cos^2(x) + \cos(x) - 1 = 0$.

Let $y = \cos(x)$. Then solve

$$2y^2 + y - 1 = 0$$

$$\Rightarrow \quad y^2 + \frac{1}{2}y - \frac{1}{2} = 0$$

$$\Rightarrow \quad (y+1)(2y-1) = 0$$

$$\Rightarrow \quad y = -1, \quad y = \frac{1}{2}$$

But we want x. Note:

$y = -1$ + $y = \frac{1}{2}$

$\Rightarrow \cos(x) = -1$

$\Rightarrow x = k\pi$, k an odd integer

$\Rightarrow \cos(x) = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$

or $x = -\frac{\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$

So the solution set is:

$$x = \begin{cases} k\pi, & k \text{ an odd integer} \\ -\frac{\pi}{3} + 2\pi k, & k \in \mathbb{Z} \\ \frac{\pi}{3} + 2\pi k, & k \in \mathbb{Z} \end{cases}$$

Other trig functions:

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)} = \csc(x)$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Some basic identities:

$\forall x \in \mathbb{R}$, $\sin^2(x) + \cos^2(x) = 1$

$\forall x \in \mathbb{R}$, $1 + \tan^2(x) = \sec^2(x)$

$\forall x \in \mathbb{R}$, $1 + \cot^2(x) = \csc^2(x)$

$\forall x \in \mathbb{R}$, $\tan(x \pm \pi) = \tan(x)$

$\forall x \in \mathbb{R}$, $\sin(x + \frac{\pi}{2}) = \cos(x)$

Half / Double Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

Q: What is $\lim_{x \to 0} \dfrac{\sin(8x)}{\sin(4x)}$?

$$\frac{\sin(8x)}{\sin(4x)} = \frac{2\cos(4x)\cancel{\sin(4x)}}{\cancel{\sin(4x)}}$$

$$= 2\cos(4x)$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin(8x)}{\sin(4x)} = \lim_{x \to 0} 2\cos(4x) = 2.$$

Sum/Difference Formulas:

$$\sin(x) \pm \sin(y) = 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

Q: What is $\frac{d}{dx}\sin(x)$?

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}.$$

We know that $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$, so inserting we get

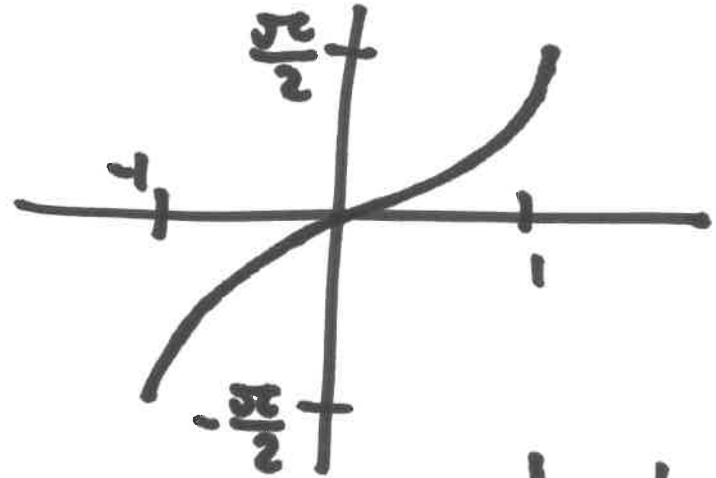
$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \cos(x)\left[\overbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}^{1}\right] + \sin(x)\left[\overbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}^{0}\right]$$

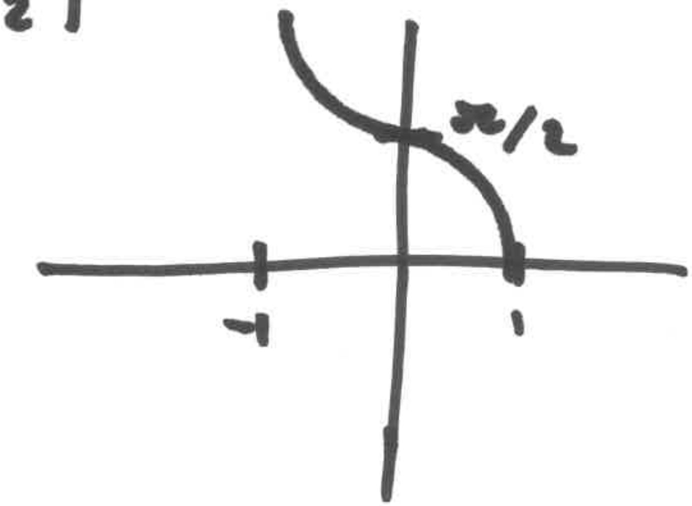
$= \cos(x)$, so $\boxed{\frac{d}{dx}\sin(x) = \cos(x)}$

Inverse trig functions:

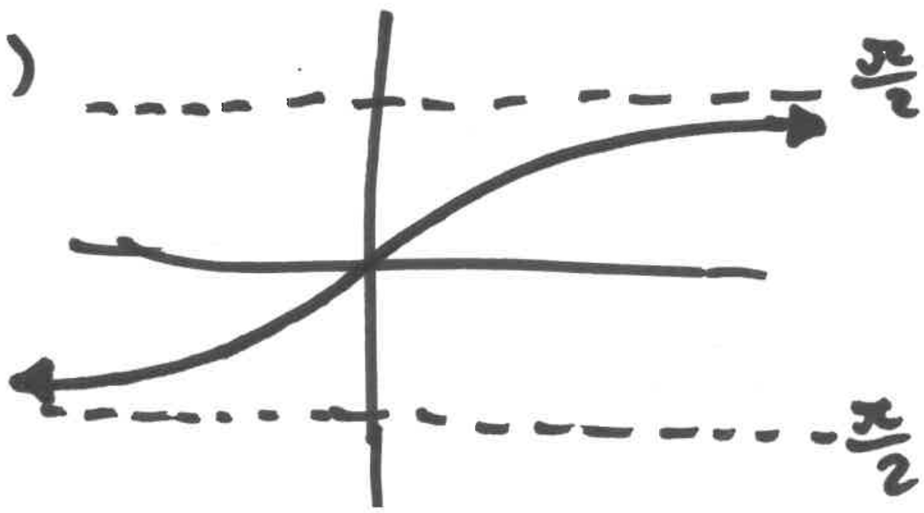
$\sin^{-1}(x) = \arcsin(x)$



$\cos^{-1}(x) = \arccos(x)$



$\tan^{-1}(x) = \arctan(x)$



What is....?

$$\tan^{-1}(1) = \frac{\pi}{4}$$

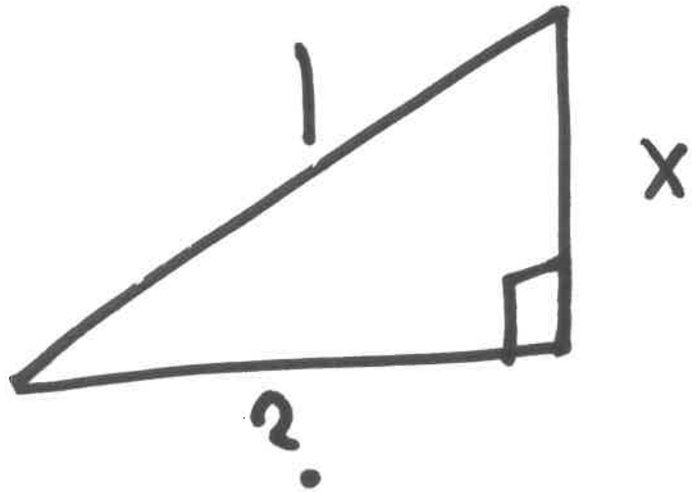
$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

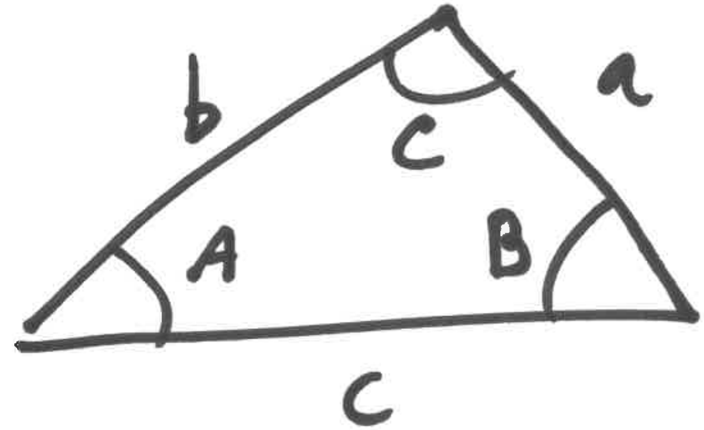
$$\cos^{-1}(0) = \frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\sin^{-1}(0) = 0$$

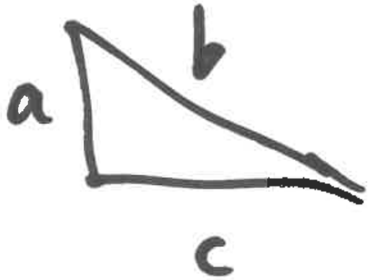




Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos(C)$

Law of sines: $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$

Heron's formula:



Let s be half the perimeter, $s = \frac{a+b+c}{2}$.

Then area $= \sqrt{s(s-a)(s-b)(s-c)}$.

Bonus Material

Complex numbers:

We say that $z \in \mathbb{C}$ is a complex number if it can be written as $z = a + bi$ for some $a, b \in \mathbb{R}$. Here $i = \sqrt{-1}$.

a is the "real part", $\mathrm{Re}(z) = a$.

b is the "imaginary part", $\mathrm{Im}(z) = b$.

Equality on complex numbers:

If $z_1 = a_1 + b_1 i$ & $z_2 = a_2 + b_2 i$,

then

$$z_1 = z_2$$

iff

$$\begin{cases} a_1 = a_2 \\ b_1 = b_2 \end{cases}$$

Any complex number can alternatively be written in polar form

$$z = re^{i\theta}, \quad r \geq 0 \text{ \& } \theta \in \mathbb{R}$$

If $z = a + bi$ also, then

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Note: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$= \text{cis}(\theta)$

__De Moivre's Thm:__

$$\left[e^{i\theta}\right]^n = e^{in\theta}$$

__OR__

$$\left[\cos(\theta) + i\sin(\theta)\right]^n = \cos(n\theta) + i\sin(n\theta)$$

Special case:

$$e^{i\pi} = -1. \quad \text{Why?}$$

Application: Prove the double-angle trig identities.

Consider $e^{i2\theta}$. By De Moivre's, we have

$$(e^{i\theta})^2 = e^{i2\theta}$$

$$= (\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta + i(2\cos(\theta)\sin\theta) - \sin^2(\theta)$$

$$= \cos(2\theta) + i\sin(2\theta).$$

Equating real + imaginary parts of both sides gives

$$\begin{cases} \sin(2\theta) = 2\cos(\theta)\sin(\theta) \\ \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta). \end{cases}$$

MATH 3

Lecture #6

9/22/23

Jonathan Lindblom

Q: Where is the Heaviside function

$$f(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$$

differentiable?

A: For any $x < 0$, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{0 - 0}{h} = \lim_{h \to 0} \frac{0}{h} = 0$,

and similarly for any $x > 0$, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$.

What about for $x = 0$? The limit

$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ DNE, so $f(x)$ is

not differentiable at $x = 0$.

So $f(x)$ __is__ differentiable on $(-\infty, 0) \cup (0, +\infty)$.

Q: What is the derivative of $f(x) = \sqrt{x}$?

A: $$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}}.$$

Q: What is $\frac{d}{dx}\frac{1}{x}$?

$$\frac{d}{dx}\frac{1}{x} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} - \frac{\frac{h}{x(x+h)}}{h} = \lim_{h\to 0} - \frac{1}{x(x+h)}$$

$$= -\frac{1}{x^2},$$

So $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.

Q: What is $\frac{d}{dx}\frac{1}{\sqrt{x}}$?

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \times \frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{hx\sqrt{x+h} + h\sqrt{x}(x+h)} = \lim_{h\to 0} - \frac{h}{h(x\sqrt{x+h} + \sqrt{x}(x+h))}$$

$$= \lim_{h\to 0} - \frac{1}{x\sqrt{x+h} + \sqrt{x}(x+h)} = -\frac{1}{x\sqrt{x} + x\sqrt{x}}$$

$$= -\frac{1}{2x\sqrt{x}}, \text{ so } \frac{d}{dx}\frac{1}{\sqrt{x}} = -\frac{1}{2x\sqrt{x}}$$

Thm: If $f(x)$ is differentiable at $x=a$, then $f(x)$ is continuous at $x=a$.

Proof: Since $f(x)$ is diff. at $x=a$, we know $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists. We want to show that $\lim_{h\to 0} f(a+h) = f(a)$.

Note that $\lim_{h\to 0} [f(a+h)-f(a)]$

$$= \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} \cdot \frac{h}{1} \right] = \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h}\right] \cdot 0$$

$$= f'(a) \cdot 0 = 0.$$

$$\Rightarrow \lim_{h\to 0} f(a+h) = f(a).$$

Higher-order Derivatives:

Given $f(x)$, $f'(x)$ denotes its 1st derivative.

$f''(x) = \frac{d}{dx} f'(x)$ denotes its 2nd derivative,

and $f^{\underbrace{'\cdots'}_{k}}(x) = \frac{d}{dx} f^{\overbrace{'\cdots'}^{k-1}}(x)$ denotes

its kth derivative. We also write it

as $f^{(k)}(x)$, or $\frac{d^k f}{dx^k}$.

Ex: $f(x) = x^2 \rightarrow f'(x) = 2x \rightarrow f''(x) = 2$

$\rightarrow f'''(x) = 0$. $f^{(3)}(x) = 0$, $\frac{d^3 f}{dx^3} = 0$.

Some basic differentiation rules:

① $$\frac{d}{dx}\left[c f(x)\right] = c \frac{d}{dx} f(x)$$

② $$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \left[\frac{d}{dx} f(x)\right] + \left[\frac{d}{dx} g(x)\right]$$

Does $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx} f(x)}{\frac{d}{dx} g(x)}$$

& $$\frac{d}{dx}\left[f(x) g(x)\right] = \left[\frac{d}{dx} f(x)\right]\left[\frac{d}{dx} g(x)\right] ?$$

NO!!!

Q: Find eqn of tangent line to $f(x) = \sin(x)$ at $x=0$.

A: We know that $f'(x) = \cos(x)$, so the slope is
$$
\begin{aligned}
m &= f'(0) \\
&= \cos(0) \\
&= 1.
\end{aligned}
$$

The point of interest is $(0,0)$, so inserting into point-slope form we obtain
$$y - 0 = 1(x-0) \Rightarrow y = x$$
is the tangent line to $f(x) = \sin(x)$ at $x=0$.

Q: What is the derivative of $f(x) = \sin(x)$?

From the definition, we have

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

⚠ Recall the angle sum identity which tells us that $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$.

Inserting, we get

$$\lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos(x)\sin(h)}{h} + \frac{\sin(x)[\cos(h) - 1]}{h} \right]$$

$$= \cos(x) + 0 = \cos(x).$$

Summation Notation:

Recall that

$$a_0 + a_1 + \cdots + a_{n-1} + a_n = \sum_{k=0}^{n} a_k$$

Generally,

$$f(0) + f(1) + \cdots + f(n-1) + f(n) = \sum_{k=0}^{n} f(k)$$

Q: What is the derivative of $f(x) = x^n$?

(when n is integer ≥ 1)

A: Binomial theorem, which states that

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad , \quad \text{where } \binom{n}{k} := \frac{n!}{k!\,(n-k)!}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h}$$

$$= \lim_{h \to 0} \frac{x^n + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h}$$

$$= \lim_{h \to 0} \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^{k-1}$$

In the sum $\sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^{k-1}$,

all terms vanish as $h \to 0$ except the only term that has "no" h in it, namely the term for $k=1 \to \binom{n}{1} x^{n-1} h^0$

$$= \binom{n}{1} x^{n-1}.$$

Since $\binom{n}{1} = n$,

$$\Rightarrow f'(x) = n x^{n-1}.$$

☆ We have just proven the power law $\frac{d}{dx} x^n = n x^{n-1}$ for n an integer ≥ 1.

Derivatives of polynomials:

This gives us everything we need to find the derivative of any polynomial.

$$\frac{d}{dx}\left[c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n\right]$$

$$= \left[\frac{d}{dx} c_0\right] + \left[\frac{d}{dx} c_1 x\right] + \cdots + \left[\frac{d}{dx} c_{n-1} x^{n-1}\right] + \left[\frac{d}{dx} c_n x^n\right]$$

$$= c_1 + 2c_2 x + \cdots + (n-1)c_{n-1}x^{n-2} + nc_n x^{n-1}$$

More succintly,

$$\frac{d}{dx}\sum_{k=0}^{n} c_k x^k = \sum_{k=1}^{n} k c_k x^{k-1}$$

Ex: What is $\frac{d}{dx}\, 3x^2 - 4x + 9$?

A: $\frac{d}{dx}\left[3x^2 - 4x + 9\right]$

$= 6x - 4$.

MATH 3

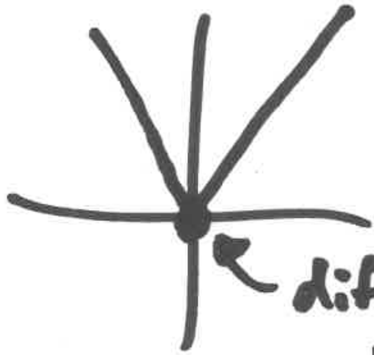
Lecture #7

9/25/23

Jonathan Lindblom

Some Examples

Ex: Where is $f(x) = |x|$ differentiable? Give a formula for $f'(x)$.

A: 

diff. everywhere except at $x=0$

How to get a formula for $f'(x)$?

Note

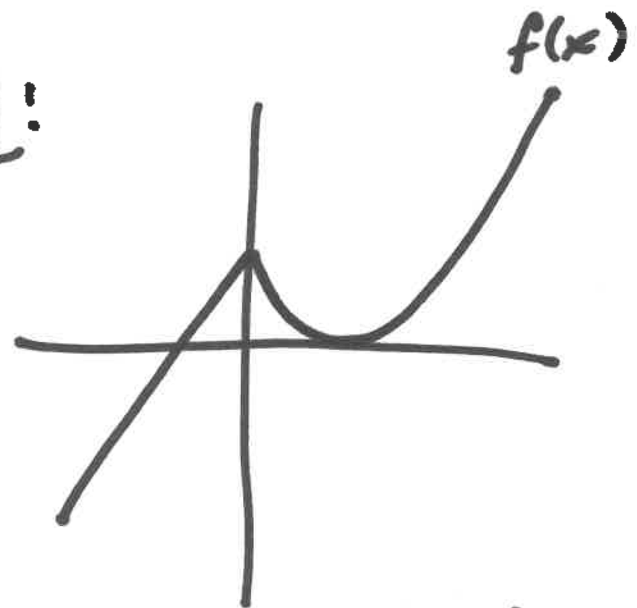
$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -1, & x < 0 \\ \text{DNE}, & x = 0 \\ 1, & x > 0 \end{cases}$$

Q: Let $f(x) = \begin{cases} 2x+1, & x < 0, \\ (x-1)^2, & x \geq 0. \end{cases}$

Where is $f(x)$ differentiable? Find a formula for $f'(x)$.

A:



Differentiable everywhere except $x = 0$,

so differentiable on $(-\infty, 0) \cup (0, \infty)$.

$$f'(x) = \begin{cases} 2, & x < 0, \\ \text{does not exist}, & x = 0, \\ 2x-2, & x > 0. \end{cases}$$

Q: At what points is the tangent line to $f(x) = 2x^2 + x - 3$ horizontal?

A: $f'(x) = 4x + 1$

set $= 0$: $4x + 1 = 0 \Rightarrow 4x = -1$

$\Rightarrow x = -\frac{1}{4}$

Derivatives of Polynomials

Last lecture, we proved the power rule

$$\frac{d}{dx} x^n = n x^{n-1}, \quad n \text{ a positive integer.}$$

Why does this immediatly give us a rule for taking derivatives of polynomials? This is beause of the sum/constant derivative properties.

$$\Rightarrow \frac{d}{dx} \sum_{k=0}^{n} c_k x^k = \sum_{k=1}^{n} k c_k x^{k-1}$$

__Note:__ We have not proven this yet, but actually n in the power rule can be any $n \in \mathbb{R}$,

$$\frac{d}{dx} x^n = n x^{n-1}, \quad n \in \mathbb{R}.$$

Q: What is $\frac{d}{dx}\left[x^{27} - 3x^4 + 2x - 100\right]$?

$$= \left[\frac{d}{dx} x^{27}\right] - 3\left[\frac{d}{dx} x^4\right] + 2\left[\frac{d}{dx} x\right] + \frac{d}{dx}\left[-100\right]$$

$$= 27x^{26} - 3 \times 4 x^3 + 2 + 0$$

$$= 27x^{26} - 12x^3 + 2.$$

Q: What is $\frac{d}{dx}\left[\frac{1}{x^3} + \frac{4}{x^{100}}\right]$?

$$= \frac{d}{dx}\left[x^{-3} + 4x^{-100}\right]$$

$$= \left[\frac{d}{dx}x^{-3}\right] + 4\left[\frac{d}{dx}x^{-100}\right]$$

$$= -3x^{-4} + 4\cdot -100\, x^{-101}$$

$$= -3x^{-4} - 400x^{-101}.$$

Q: Suppose $f(x)$ is quadratic with $f(0)=0$, $f'(1)=1$, $f''(2)=2$. What is $f(x)$?

A: f quadratic $\Rightarrow f(x) = ax^2+bx+c$ for some $a,b,c \in \mathbb{R}$.

$$0 = f(0) = a(0)^2 + b(0) + c = c \Rightarrow c = 0$$

So $f(x) = ax^2+bx$.

$$f'(x) = 2ax+b, \quad 1 = f'(1) = 2a(1)+b = 2a+b$$
$$\Rightarrow 2a+b = 1.$$

Also, $f''(x) = 2a$, $\quad 2 = f''(2) = 2a \Rightarrow a = 1$.

So $b = 1-2a = 1-2 = -1$.

So $f(x) = x^2 - x$.

Derivatives of Exponentials

What is $\frac{d}{dx} a^x$?

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \lim_{h \to 0} \frac{a^{(0+h)} - a^0}{h}$$

$$= a^x \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= a^x f'(0), \quad \text{where } f(x) = a^x.$$

$$\Rightarrow \frac{d}{dx} f(x) = a^x f'(0).$$

$$\begin{array}{c} \shortparallel \\ \frac{d}{dx} a^x \end{array}$$

$\Rightarrow$ "the derivative of an exponential function is proportional to the same exponential function."

Q: For exponential functions, we know that

$$\frac{d}{dx} a^x = a^x \cdot \left(\frac{d}{dx} a^x\right)\Bigg|_{x=0} .$$

Is there a base a such that

$$\left(\frac{d}{dx} a^x\right)\Bigg|_{x=0} = 1 \quad \Rightarrow \quad \frac{d}{dx} a^x = a^x \ ?$$

Yes, this number is $e \approx 2.718\ldots\ldots$

and is actually one possible way to define e. So, $\frac{d}{dx} e^x = e^x$.

But what about $\frac{d}{dx} a^x$?

In a few lectures from now, we will show that this is

$$\frac{d}{dx} a^x = a^x \cdot \ln(a) .$$

With $a = e$, this formula gives

$$\frac{d}{dx} e^x = e^x \cdot 1 = e^x$$

as expected.

Q: What functions are equal to their derivatives?

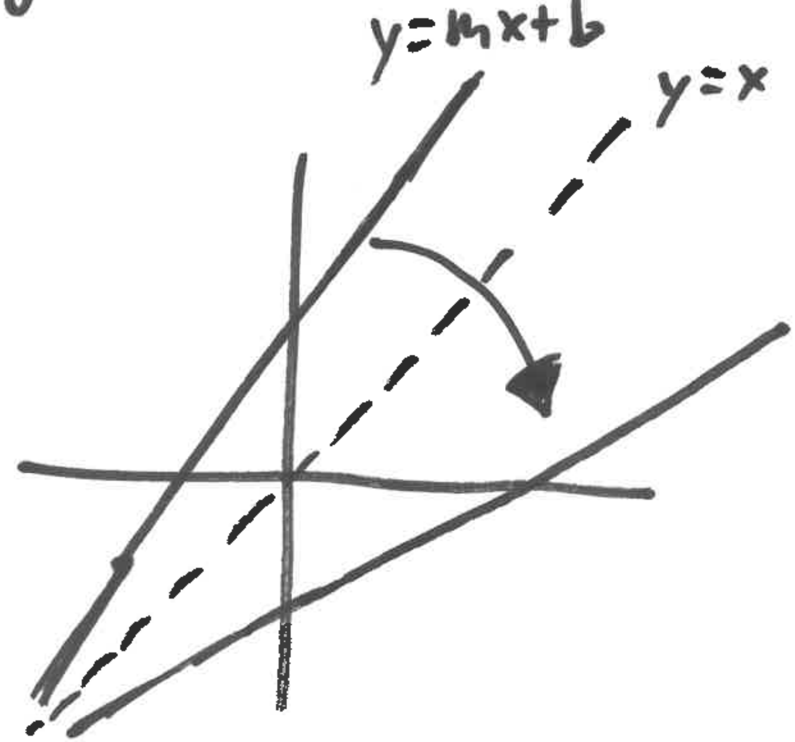
Using inverse functions to find derivatives

If we have trouble figuring out the derivative of $f(x)$, sometimes considering the inverse function $f^{-1}(x)$ can help. Why?

Examples: $\sqrt[3]{x}$, $\sqrt[4]{x}$, $\sqrt[n]{x}$, $\ln(x)$, $\log_a(x)$

Because there is a relationship between the tangent lines to $f(x)$ and the tangent lines to $f^{-1}(x)$.

Q: If $y = mx + b$, what is the equation of this line reflected across the line $y = x$?



A: The equation of the reflected line is $y = \frac{1}{m}x - \frac{b}{m}$.

☆ Note the slope of the reflected line is the reciprocal of the original.

For $y = x^{\frac{1}{3}}$:

$$(x, x^{\frac{1}{3}}) \xrightarrow[\text{about } y=x]{\text{reflect}} (x^{\frac{1}{3}}, x).$$

The derivative of $f^{-1}(x) = x^3$ is $g(x) = 3x^2$,

and evaluated at $x^{\frac{1}{3}}$ is $= 3x^{\frac{2}{3}}$.

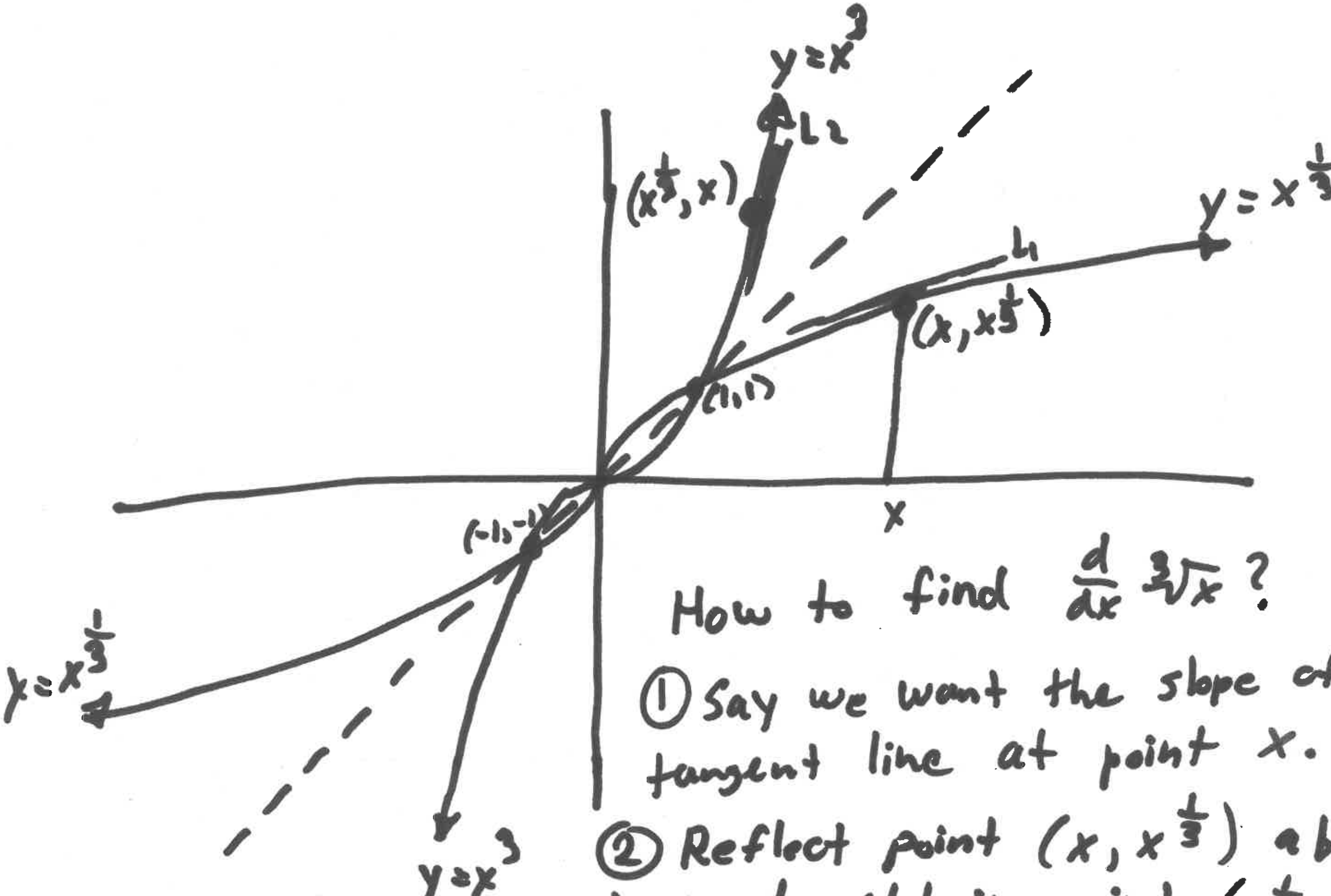
The reciprocal of this is $\frac{1}{3}x^{-\frac{2}{3}}$,

so

$$\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}}.$$

Note: power rule gives us $\frac{d}{dx}x^{\frac{1}{3}} = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}}$.

Consider $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, whose inverse is $f^{-1}(x) = x^3$.



How to find $\frac{d}{dx}\sqrt[3]{x}$?

① Say we want the slope of tangent line at point x.

② Reflect point $(x, x^{\frac{1}{3}})$ about $y=x$ to obtain point $(x^{\frac{1}{3}}, x)$.

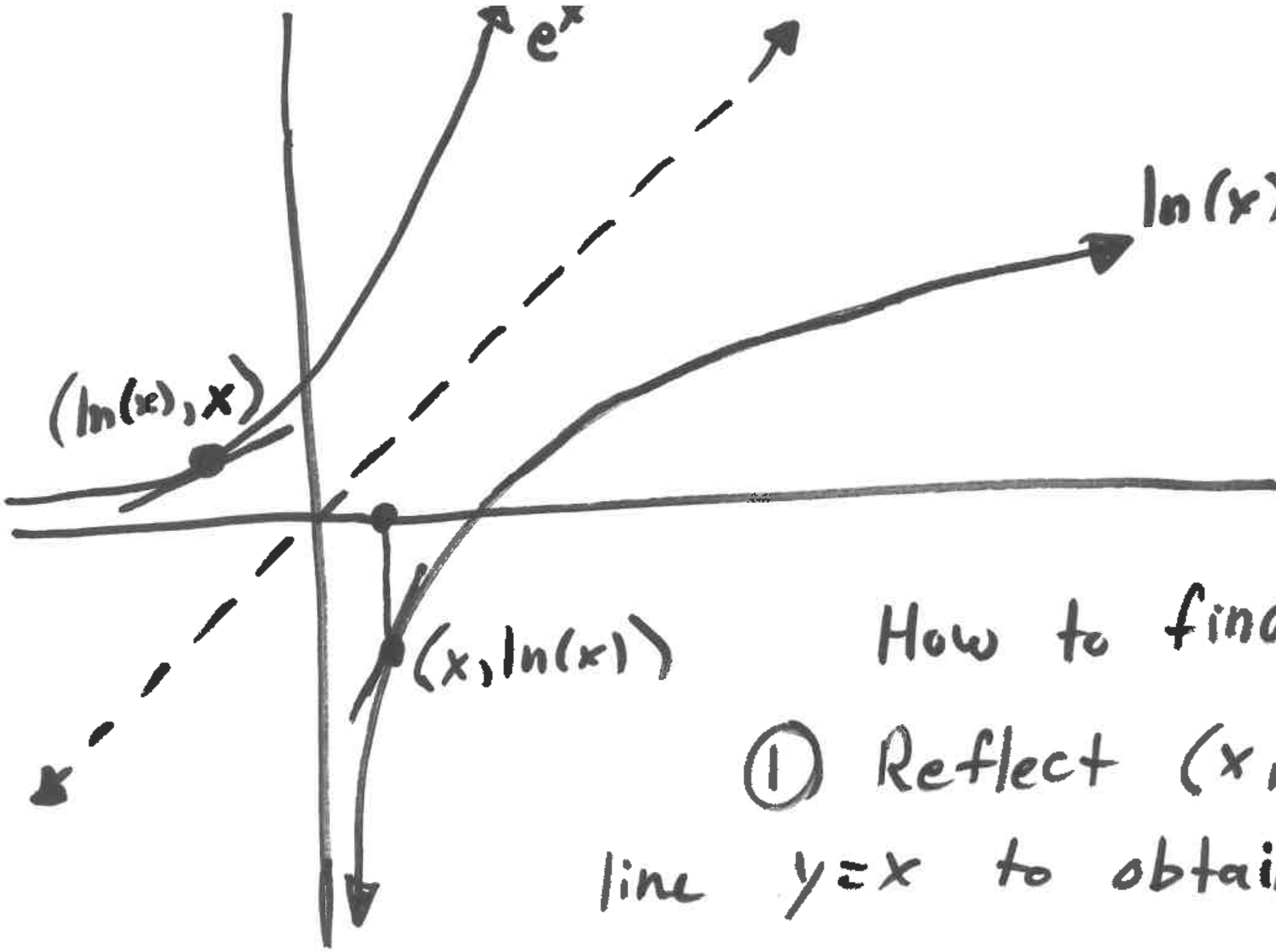
③ Compute the slope of the tangent line of $y = x^3$ at point $x^{\frac{1}{3}}$. The reciprocal is the slope of tangent to $y = x^{\frac{1}{3}}$.

Q: What about $f(x) = \ln(x)$?

Start with the point $(x, \ln(x))$.
Reflecting about $y = x$ gives $(\ln(x), x)$.
The inverse function is $f^{-1}(x) = e^x$, whose derivative is $\frac{d}{dx} f^{-1}(x) = e^x$. Evaluating at $\ln(x)$ we obtain $\left[\frac{d}{dx} f^{-1}(x)\right]\Big|_{\ln(x)}$

$= e^{\ln(x)} = x$, and the reciprocal is $\frac{1}{x}$.

$$\Rightarrow \boxed{\frac{d}{dx} \ln(x) = \frac{1}{x}}$$



How to find $\left(\frac{d}{dx}\ln(x)\right)\Big|_x$?

① Reflect $(x, \ln(x))$ across line $y=x$ to obtain $(\ln(x), x)$.

② Since $\ln(x)$ reflected about $y=x$ gives e^x and $\frac{d}{dx}e^x = e^x$, we know the slope of $y=e^x$ at point $\ln(x)$ is $m = e^{\ln(x)} = x$.

③ This slope is the reciprocal of the slope of this line reflected about $y=x$, which is $\tilde{m} = \frac{1}{x}$.

$\Rightarrow \frac{d}{dx}\ln(x) = \frac{1}{x}$.

General Rule:

$$\frac{d}{dx} f(x) = \frac{1}{\left[\frac{d}{dx} f^{-1}(x)\right]\Big|_{f(x)}}$$

Also goes in other direction:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

MATH 3

Lecture #8

9/27/23

Jonathan Lindbloom

Product Rule

What is $\frac{d}{dx}\left[f(x)g(x)\right]$? (Product Rule)

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right]}{h} + \lim_{h\to 0} \frac{g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$= \left(\lim_{h\to 0} f(x+h)\right)\left(\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right) + \left(\lim_{h\to 0} g(x)\right)\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)$$

$$= f(x)g'(x) + g(x)f'(x).$$

Q: What is $\frac{d}{dx}\left(e^x \sin(x)\right)$?

Let $f(x) = e^x$ and $g(x) = \sin(x)$. By product rule,

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$f'(x) = e^x$ and $g'(x) = \cos(x)$

$$\Rightarrow \frac{d}{dx}\left(e^x \sin(x)\right) = e^x \sin(x) + e^x \cos(x)$$

$$= e^x\left(\sin(x) + \cos(x)\right).$$

Q: What is $\frac{d}{d\theta}[\sin(2\theta)]$?

$$= \frac{d}{d\theta}\left[2\sin(\theta)\cos(\theta)\right]$$

$\Delta\ \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$$= 2\frac{d}{d\theta}\left[\sin(\theta)\cos(\theta)\right]$$

$$= 2\left(\left[\frac{d}{d\theta}\sin(\theta)\right]\cos(\theta) + \sin(\theta)\left[\frac{d}{d\theta}\cos(\theta)\right]\right)$$

$$= 2\left(\cos^2(\theta) - \sin^2(\theta)\right)$$

$$= 2\cos(2\theta)$$

$\Delta\ \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$

Q: Suppose $f(7)=8$ and $f'(7)=2$.

What is $\left[\frac{d}{dx}[f(x)]^2\right]\Big|_{x=7}$?

A:

$$\frac{d}{dx}\left[f(x)\right]^2 = \frac{d}{dx}\left[f(x)f(x)\right]$$

$$= f'f + ff' = 2f'f$$

at $x=7$, $\Rightarrow \left[\frac{d}{dx}[f(x)]^2\right]\Big|_{x=7} = 2f'(7)f(7)$

$$= 2\cdot 2\cdot 8$$

$$= \underline{32}.$$

Q:

What is the eqn of the tangent line to the curve $f(x) = \sqrt{x}\,\sin(x)$ at $x = \frac{\pi}{4}$?

A: $f'(x) = \left(\frac{d}{dx}\sqrt{x}\right)\sin(x) + \sqrt{x}\left(\frac{d}{dx}\sin(x)\right)$

$$= \frac{1}{2}x^{-\frac{1}{2}}\sin(x) + \sqrt{x}\cos(x)$$

Slope of tangent is $m = f'\left(\frac{\pi}{4}\right)$

$$= \frac{1}{2}\frac{1}{\sqrt{\frac{\pi}{4}}}\cdot\frac{\sqrt{2}}{2} + \sqrt{\frac{\pi}{4}}\,\frac{\sqrt{2}}{2}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2\pi}}{4}.$$

Plugging into point-slope form,

$$y - \frac{\sqrt{2\pi}}{4} = m\left(x - \frac{\pi}{4}\right)$$

$$\Rightarrow y = mx + \frac{\sqrt{2\pi}}{4} - \frac{\pi}{4}m$$

In general, tangent line is given by

$$y = \underbrace{f'(a)}_{\text{slope}}x + \underbrace{f(a) - \overset{f'(a)}{m}a}_{\text{intercept}}$$

<u>Q</u>: Evaluate the limit

$$\lim_{h \to 0} \frac{(x+h)^2 \cos(x+h) - x^2 \cos(x)}{h}.$$

<u>A</u>: Sometimes we can match a limit as the limit definition of some derivative, and this makes the problem easier. In this case, the limit is just $\frac{d}{dx}\left[x^2 \cos(x)\right]$

$$= \left[\frac{d}{dx} x^2\right] \cos(x) + x^2 \left[\frac{d}{dx} \cos(x)\right]$$

$$= 2x\cos(x) - x^2 \sin(x).$$

Q: Let $f(x)$ be some function, and suppose $f'(x) = e^x + xe^x$. Using the product rule, can you guess what $f(x)$ is?

A: $f'(x) = \underset{g'}{1} \cdot \underset{h}{e^x} + \underset{g}{x}\,\underset{h'}{e^x}$

$\Rightarrow f(x) = xe^x + C, \ C \in \mathbb{R}.$

Why the C?

Q: What is $\frac{d}{dx}\left[f(x)g(x)h(x)\right]$?

A: Using the product rule, we see

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = \frac{d}{dx}\left[\left(f(x)g(x)\right)h(x)\right]$$

$$= \left[\frac{d}{dx}\left[f(x)g(x)\right]\right]h(x) + f(x)g(x)\left[\frac{d}{dx}h(x)\right]$$

$$= g(x)h(x)\left[\frac{d}{dx}f(x)\right] + f(x)h(x)\left[\frac{d}{dx}g(x)\right] + f(x)g(x)\left[\frac{d}{dx}h(x)\right]$$

$$= f'gh + fg'h + fgh'.$$

Q: Let $F(x) = f(x)g(x)$. Assuming the derivatives needed exist, what is $F^{(n)}(x)$?

A: For $F^{(1)}(x) = F'(x)$ we see

$$F^{(1)}(x) = f'g + fg'.$$

For $F^{(2)}(x) = F''(x)$ we see

$$F^{(2)}(x) = f''g + 2f'g' + fg''.$$

For $F^{(3)}(x) = f'''g + 3f''g' + 3f'g'' + fg'''$.

What is the pattern?

Hint: think about Pascal's triangle.

Chain Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then the chain rule states that

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x).$$

We will prove this next Monday.

Q: What is $\frac{d}{dx}\left[(3x)^2\right]$?

Method 1 (algebra):

$= \frac{d}{dx}\left[9x^2\right]$

$= 9\frac{d}{dx}\left[x^2\right]$

$= 18x$ ✓

Method 2 (chain rule):

$= 2(3x)^1 \cdot \frac{d}{dx}\left[3x\right]$

$= 2 \cdot 3x \cdot 3$

$= 18x$ ✓

Q: What is $\frac{d}{d\theta}\left[\sin(2\theta)\right]$? (using chain rule!)

$$= \cos(2\theta) \cdot \frac{d}{d\theta}(2\theta)$$

$$= 2\cos(2\theta).$$

Q: What is $\frac{d}{dx}\left[e^{cx}\right]$? $\frac{d}{dx}\left[e^{x^2}\right]$?

A: $\frac{d}{dx}\left[e^{cx}\right] = e^{cx} \cdot \frac{d}{dx}\left[cx\right] = ce^{cx}.$

$\frac{d}{dx}\left[e^{x^2}\right] = e^{x^2} \cdot \frac{d}{dx}\left[x^2\right] = 2xe^{x^2}.$

Q: What is $\frac{d}{dx}\left[(3x^2+2x+1)^4\right]$?

A:

$$= 4(3x^2+2x+1)^3 \cdot \frac{d}{dx}\left[3x^2+2x+1\right]$$

$$= 4(3x^2+2x+1)^3 \cdot (6x+2).$$

Quotient Rule

What is $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$? (Quotient rule)

$$= \frac{d}{dx}\left[f(x)(g(x))^{-1}\right] = (g(x))^{-1}\left[\frac{d}{dx}f(x)\right] + f(x)\frac{d}{dx}\left[(g(x))^{-1}\right]$$

$$= f'(x)(g(x))^{-1} + f(x)\cdot -(g(x))^{-2}\cdot\frac{d}{dx}g(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

<u>Q:</u> What is $\frac{d}{dx}\left[x^2 e^{-x}\right]$?

$$= \frac{d}{dx}\left[\frac{x^2}{e^x}\right]$$

$$= \frac{\left[\frac{d}{dx}x^2\right]e^x - x^2\left[\frac{d}{dx}e^x\right]}{\left[e^x\right]^2}$$

$$= \frac{2xe^x - x^2e^x}{\left[e^x\right]^2}$$

$$= \frac{2x - x^2}{e^x} = (2x - x^2)e^{-x}.$$

Q: What is $\dfrac{d}{dx}\left[\dfrac{3x^2-7}{5x^5-10x^4+x-1}\right]$?

A:

$$= \frac{\frac{d}{dx}\left[3x^2-7\right]\cdot\left(5x^5-10x^4+x-1\right)-\left(3x^2-7\right)\frac{d}{dx}\left[5x^5-10x^4+x-1\right]}{\left[5x^5-10x^4+x-1\right]^2}$$

$$= \frac{(6x-7)\left(5x^5-10x^4+x-1\right)-\left(3x^2-7\right)\left(25x^4-40x^3+1\right)}{\left[5x^5-10x^4+x-1\right]^2}.$$

Q: Find equations of the tangent lines to the curve $y = \frac{x-1}{x+1}$ that are parallel to the line $x - 2y = 2$. ($\Rightarrow y = \frac{1}{2}x - 1$)

A: Parallel lines have the same slope!

$$\frac{dy}{dx} = \frac{1\cdot(x+1) - 1\cdot(x-1)}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

When does $\frac{dy}{dx} = \frac{1}{2}$? When $\frac{2}{(x+1)^2} = \frac{1}{2}$

$$\Rightarrow 4 = (x+1)^2 \Rightarrow x+1 = \pm 2 \Rightarrow \begin{cases} x = 1 \\ x = -3 \end{cases}$$

So the two points whose tangent lines have slope $= \frac{1}{2}$ are $(1,0)$ and $(-3,2)$.

What are the eqns of the two tangent lines?.

$$y - 0 = \frac{1}{2}(x-1)$$

$$\Rightarrow \boxed{y = \frac{1}{2}x - \frac{1}{2}}$$

and

$$y - 2 = \frac{1}{2}(x - (-3))$$

$$\Rightarrow y = \frac{1}{2}x + \frac{3}{2} + 2$$

$$\Rightarrow \boxed{y = \frac{1}{2}x + \frac{7}{2}}$$

Summary of 3 new rules:

Product Rule: $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$

Chain Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\,g'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$

MATH 3

Lecture #9

9/29/23

Jonathan Lindbloom

Warmup

Q: How many times does the curve $y = x$ intersect the curve $y = \cos(x)$? How could you find the point(s) of intersection?

A: Exactly once. One way would be to apply the bisection method to find a root of $g(x) = \cos(x) - x$.

Interestingly, another method is just to start at any $x_0 \in \mathbb{R}$ and compute x_n given by the recursion $x_n = \cos(x_{n-1})$ for large n! But that is a story for another time.

Q: On the curve $y = x^3$, a tangent is drawn from the point (a, a^3), $a > 0$ and a normal is drawn from the point $(-a, -a^3)$. The tangent and the normal meet on the y-axis. Find a.

A: ① We need a formula for the tangent line in terms of a. The slope at point a is $\left[\frac{d}{dx} f(x)\right]_{x=a} = \left[3x^2\right]_{x=a} = 3a^2$. Using point slope form, we see $y - a^3 = 3a^2(x-a)$

$$\Rightarrow \quad y = 3a^2 x - 3a^3 + a^3$$

$$\Rightarrow \quad y = 3a^2 x - 2a^3$$ is the tangent.

(2) Next we need an equation for the normal line in terms of a. The slope of the normal is the negation of the reciprocal of the slope of the tangent. So the normal's slope is

$$-\left[\frac{d}{dx} f(x)\right]_{x=-a}^{-1} = -\left[3(-a)^2\right]^{-1} = -\frac{1}{3a^2}.$$

Using point slope form, we see that

$$y-(-a^3) = -\frac{\cancel{3}1}{3a^2}\left(x-(-a)\right)$$

$$\Rightarrow y+a^3 = -\frac{1}{3a^2}x - \frac{1}{3a}$$

$$\Rightarrow y = -\frac{1}{3a^2}x - \frac{1}{3a} - a^3$$

is the equation of the normal at $(-a, -a^3)$.

(3) We are told that the tangent and normal meet on the y-axis.

$\Rightarrow$ the y-values of both curves are equal when $x=0$. Pluggings in, we see

$$-2a^3 = -\frac{1}{3a} - a^3$$

$$\Rightarrow \quad -a^3 = -\frac{1}{3a}$$

$$\Rightarrow \quad a^3 = \frac{1}{3a}$$

$$\Rightarrow \quad a^4 = \frac{1}{3}$$

$$\Rightarrow \quad a = \sqrt[4]{\frac{1}{3}}.$$

Derivatives of Trig Functions

What is $\frac{d}{dx}\tan(x)$? We will use $\tan(a+b) = \dfrac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$.

$$= \lim_{h\to 0} \frac{\tan(x+h)-\tan(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)} - \tan(x)}{h}$$

$$= \lim_{h\to 0} \frac{\tan(x)+\tan(h) - \tan(x)\left[1-\tan(x)\tan(h)\right]}{h\left[1-\tan(x)\tan(h)\right]}$$

$$= \lim_{h\to 0} \frac{\tan(h) + \tan^2(x)\tan(h)}{h\left[1-\tan(x)\tan(h)\right]}$$

$$= \lim_{h \to 0} \frac{\tan(h)\left[1 + \tan^2(x)\right]}{h\left[1 - \tan(x)\tan(h)\right]}$$

$$= \left(\lim_{h \to 0} \frac{\tan(h)}{h}\right)^{\to 1} \left(\lim_{h \to 0} \frac{1 + \tan^2(x)}{1 - \tan(x)\tan(h)}\right)$$

$$= \lim_{h \to 0} \frac{1 + \tan^2(x)}{1 - \tan(x)\tan(h)}$$

$$= \lim_{h \to 0} 1 + \tan^2(x)$$

$$= \sec^2(x).$$

Trig identity:

$1 + \tan^2(x) = \sec^2(x).$

Q: What is $\frac{d}{dt} te^t\cos(t)$?

A: $= \frac{d}{dt}\left[(te^t)\cos(t)\right]$

$= \left[\frac{d}{dt}(te^t)\right]\cos(t) + \left[te^t\right]\frac{d}{dt}\cos(t)$

$= \left[1\cdot e^t + te^t\right]\cos(t) + te^t\cdot(-\sin(t))$

$= \left[e^t + te^t\right]\cos(t) - te^t\sin(t).$

Special trig limits to know:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0$$

You should also be able to manipulate more complicated limits to this form.

Q: What is $\lim_{x \to 0} \frac{\sin(x) - \sin(x)\cos(x)}{x^2}$?

$$= \left(\lim_{x \to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x \to 0} \frac{1-\cos(x)}{x}\right)$$

$$= 1 \cdot 0$$

$$= 0.$$

Q: What is

$$\lim_{\theta \to 0} \theta \csc(\theta) ?$$

A: 1!

Q: What is $\lim_{x \to 0} \frac{\sin(x^2)}{x}$?

A: $= \lim_{x \to 0} x \cdot \frac{\sin(x^2)}{x^2}$

$= \left(\lim_{x \to 0} x\right) \cdot \left(\lim_{x \to 0} \frac{\sin(x^2)}{x^2}\right)$

$= 0 \cdot 1$

$= 0$.

Q: What is $\lim_{x \to 0} \frac{\sin(17x)}{3x}$?

A! $= \lim_{x \to 0} \frac{17}{3} \cdot \frac{3}{17} \cdot \frac{\sin(17x)}{3x}$

$= \lim_{x \to 0} \frac{17}{3} \frac{\sin(17x)}{17x}$

$= \frac{17}{3} \lim_{x \to 0} \frac{\sin(17x)}{17x}$

$= \frac{17}{3} \cdot 1$

$= \frac{17}{3}$

Summary of trig functions:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dv}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

Q: Show that $\frac{d}{dx}\cot(x) = -\csc^2(x)$.

A: $= \frac{d}{dx}\frac{\cos(x)}{\sin(x)} = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)}$

$$= -\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} = -\frac{1}{\sin^2(x)}$$

$$= -\csc^2(x).$$

A: (using chain rule)

$$\frac{d}{dx}[\tan(x)]^{-1} = -1[\tan(x)]^{-2} \cdot \sec^2(x)$$

$$= -\frac{\sec^2(x)}{\tan^2(x)} = -\frac{1}{\cancel{\cos^2(x)}} \cdot \frac{\cancel{\cos^2(x)}}{\sin^2(x)} = -\csc^2(x).$$

Q: Differentiate both sides of $\sin(x)+\cos(x) = \dfrac{1+\cot(x)}{\csc(x)}$ and see what happens!

A: $\dfrac{d}{dx}\,LHS = \cos(x)-\sin(x).$

$$\frac{d}{dx}\,RHS = \frac{\left(\frac{d}{dx}\,1+\cot(x)\right)\csc(x) - \left(1+\cot(x)\right)\left(\frac{d}{dx}\csc(x)\right)}{[\csc(x)]^2}$$

$$= \frac{\left(-\csc^2(x)\right)\csc(x) - \left(1+\cot(x)\right)\left(-\csc(x)\cot(x)\right)}{[\csc(x)]^2}$$

$$= \frac{-\csc^3(x) + \left(1+\cot(x)\right)\left(\csc(x)\cot(x)\right)}{[\csc(x)]^2}$$

$$= \frac{-\csc^2(x) + (1+\cot(x))\cot(x)}{[\csc(x)]^{\cancel{2}\,1}}$$

$$= -\csc(x) + \frac{\cot(x) + \cot^2(x)}{\csc(x)}$$

$$= -\csc(x) + \frac{\frac{\cos(x)}{\sin(x)}}{\frac{1}{\sin(x)}} + \frac{\frac{\cos^2(x)}{\sin^2(x)}}{\frac{1}{\sin(x)}}$$

$$= -\csc(x) + \cos(x) + \frac{\cos^2(x)}{\sin(x)}$$

$$= -\frac{1}{\sin(x)} + \cos(x) + \frac{\cos^2(x)}{\sin(x)}$$

What does this mean?

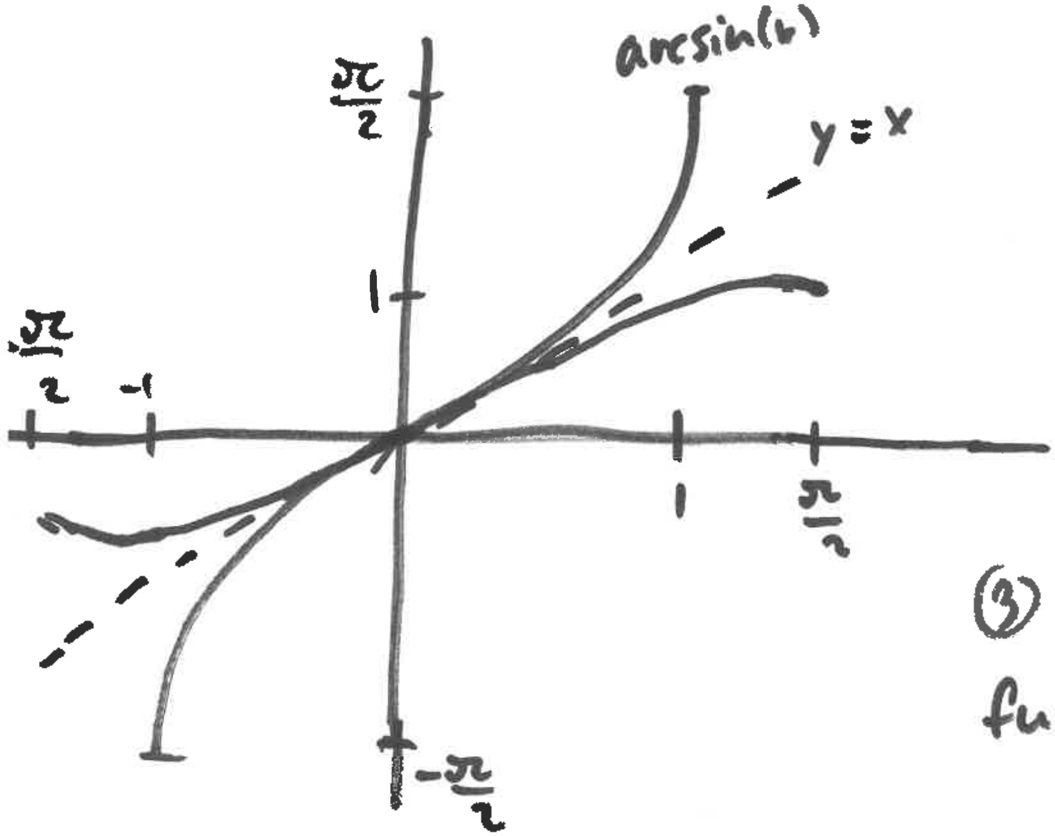
$$\cos(x) - \sin(x) = -\frac{1}{\sin(x)} + \cos(x) + \frac{\cos^2(x)}{\sin(x)}$$

$$\Rightarrow \sin(x)\cos(x) - \sin^2(x) = -1 + \sin(x)\cos(x) + \cos^2(x)$$

$$\Rightarrow -\sin^2(x) = -1 + \cos^2(x)$$

$$\Rightarrow \boxed{\sin^2(x) + \cos^2(x) = 1}$$

Q: What is $\frac{d}{dx}\arcsin(x)$?



① Say we want tangent at point x.

② Reflect $(x, \arcsin(x))$ about $y=x$ to obtain $(\arcsin(x), x)$.

③ Derivative of the inverse function $f^{-1}(x)$ is $\frac{d}{dx}\sin(x) = \cos(x)$.

④ So slope of tangent of the inverse function is $m = \cos(\arcsin(x))$.

⑤ $\Rightarrow$ slope of tangent of original function is $\hat{m} = \frac{1}{\cos(\arcsin(x))}$.

⑥ But can we do anything with $\frac{1}{\cos(\arcsin(x))}$?

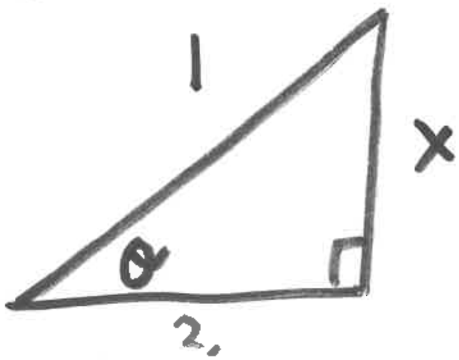
Yes!

With problems involving inverse trig, a good habbit to form is to draw a triangle.

Trig functions accept angles and return values, inverse trig functions accept values and return angles.

How to simplify $\dfrac{1}{\cos(\arcsin(x))}$?

$\uparrow$ value



Note, $x = \dfrac{x}{1}$ ← opposite, ← hypotenuse

$\Rightarrow \theta = \arcsin(x)$.

So what is $\cos(\arcsin(x))$?

It is $\dfrac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2} \Rightarrow \cos(\arcsin(x)) = \sqrt{1-x^2}$

$$\Rightarrow \frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \;!$$

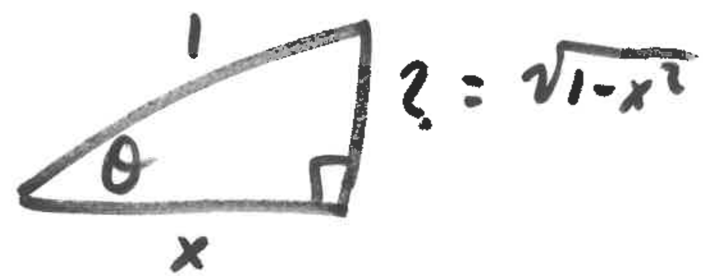
Q: What is $\frac{d}{dx}\arccos(x)$?

A: Same process.

① Reflect $(x, \arccos(x))$ about $y = x$ to obtain $(\arccos(x), x)$.

② Derivative of $f^{-1}(x)$ is $\frac{d}{dx}\cos(x) = -\sin(x)$.

③ Slope of inverse function's tangent at $(\arccos(x), x)$ is $m = -\sin(\arccos(x))$.



④ Can we simplify? Yes!

$\Rightarrow m = -\sqrt{1-x^2}$

⑤ So slope of tangent of original function $f(x) = \arccos(x)$ is $\tilde{m} = -\frac{1}{\sqrt{1-x^2}}$ $\Rightarrow \frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$.

Q: What is $\frac{d}{dx}\arctan(x)$?

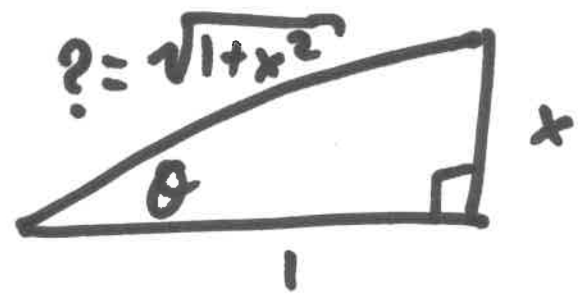
A: Same process.

① Reflect $(x, \arctan(x))$ about $y=x$ to obtain $(\arctan(x), x)$.

② Derivative of $f^{-1}(x)$ is $\frac{d}{dx}\tan(x) = \sec^2(x)$.

③ Slope of inverse function's tangent at $(\arctan(x), x)$ is $m = \sec^2(\arctan(x))$.

④ Can we simplify? Yes!



$$\Rightarrow \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{1}{\cos^2(\arctan(x))} = 1+x^2 = \sec^2(\arctan(x)).$$

5 So slope of tangent of original function $f(x) = \arctan(x)$ at point x is $\tilde{m} = \dfrac{1}{x^2+1}$.

$$\Rightarrow \frac{d}{dx} \arctan(x) = \frac{1}{x^2+1}.$$

MATH 3

Lecture #10

10/2/23

Jonathan Lindbloom

What is $\frac{d}{dx} f(g(x))$? (Chain Rule)

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{h} \times \frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right]$$

$$= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h} \right]$$

$$= \left(\lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right] \right) \cdot \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right)$$

$$= g'(x) \cdot \left(\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right)$$

Let $z = g(x)$. Since $g(x)$ is continuous, $g(x+h) \to g(x)$ as $h \to 0$, and setting $z' = g(x+h)$ we know that $z' \to z$ as $h \to 0$. Inserting, we get

$$= g'(x) \cdot \lim_{z' \to z} \frac{f(z') - f(z)}{z' - z}$$

$$= g'(x) f'(z)$$

$$= g'(x) f'(g(x)).$$

Topics:

1. Limits + Limit Laws
2. Precise Def. of Limits
3. Continuity
4. Trigonometric Functions
5. Limits at Infinity, Horizontal/Vertical Asymptotes
6. The Derivative + Limit Definition
7. Polymials + Exponentials (their derivatives)
8. Product and quotient Rules
9. Trig Derivatives

Some Practice Questions

Limits.

We say that $\lim_{x \to a} f(x) = L$ if

$\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$|x-a| < \delta \implies |f(x) - L| < \epsilon.$$

Continuity:

We say that $f(x)$ is continuous at $x=a$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$|x-a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

Q: What is $\lim_{x \to 7^+} \frac{x+3}{x-7}$?

What about $\lim_{x \to 7^-} \frac{x+3}{x-7}$?

A:

$$\lim_{x \to 7^+} \frac{x+3}{x-7} = +\infty$$

$$\lim_{x \to 7^-} \frac{x+3}{x-7} = -\infty$$

Q: Find the value of a that makes

$$f(x) = \begin{cases} x^3 - x^2 - 4x + 4, & x \le a \\ 4x - 8, & x > a \end{cases}$$ differentiable on $\mathbb{R}$.

A: First check continuity? Would require that

$a^3 - a^2 - 4a + 4 = 4a - 8$

$\Rightarrow a^3 - a^2 - 8a + 12 = 0.$

Hard to solve.

Derivative? Would require that $3x^2 - 2x - 4 = 4$ at $x = a$,

$\Rightarrow 3a^2 - 2a - 8 = 0$

$\Rightarrow (a + \frac{4}{3})(a - 2) = 0$

$\Rightarrow a = -\frac{4}{3}, \; a = 2$

Do either choices give us continuity?

$$(2)^3 - (2)^2 - 4(2) + 4 = 8 - 4 - 8 + 4 = 0,$$

$$4(2) - 8 = 0,$$

So yes! $a = 2$ gives both continuity & differentiability.

Q: What is $\frac{d}{dx} \frac{\ln(x)}{x}$?

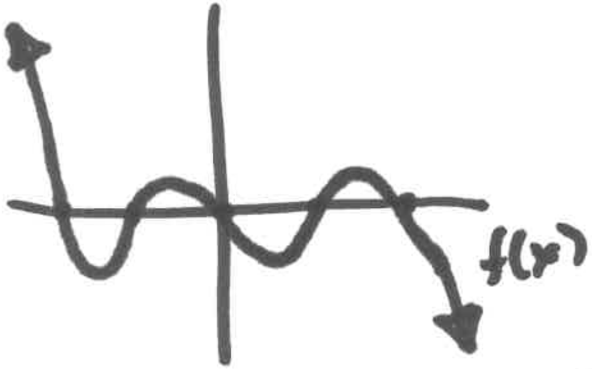
A:

$$= \frac{\left[\frac{d}{dx}\ln(x)\right]x - \ln(x)\left[\frac{d}{dx}x\right]}{x^2}$$

$$= \frac{\frac{1}{x}\cdot x - \ln(x)\cdot 1}{x^2}$$

$$= \frac{1 - \ln(x)}{x^2}.$$

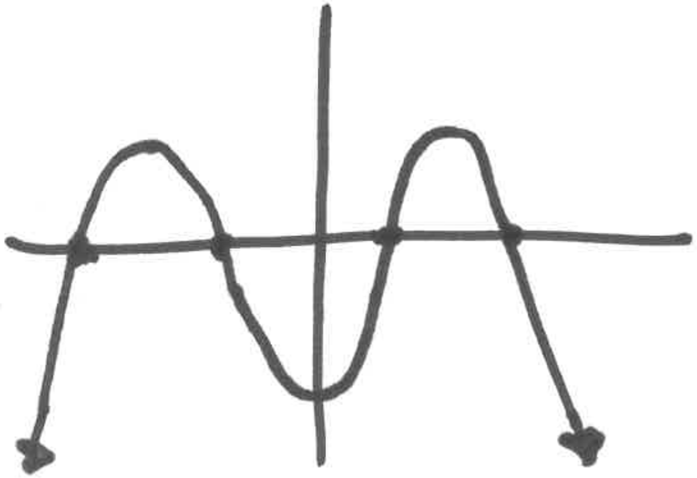
Q: Which graph below is closest to that of $f'(x)$, when the graph of $f(x)$ is



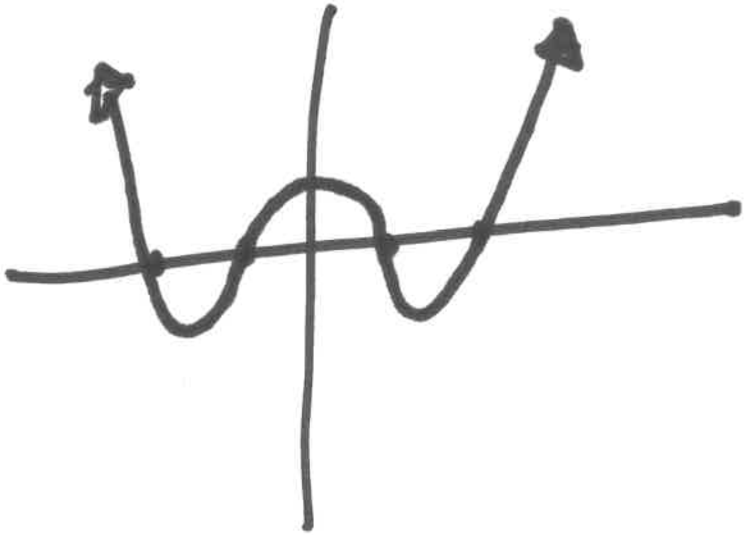
?

Options:

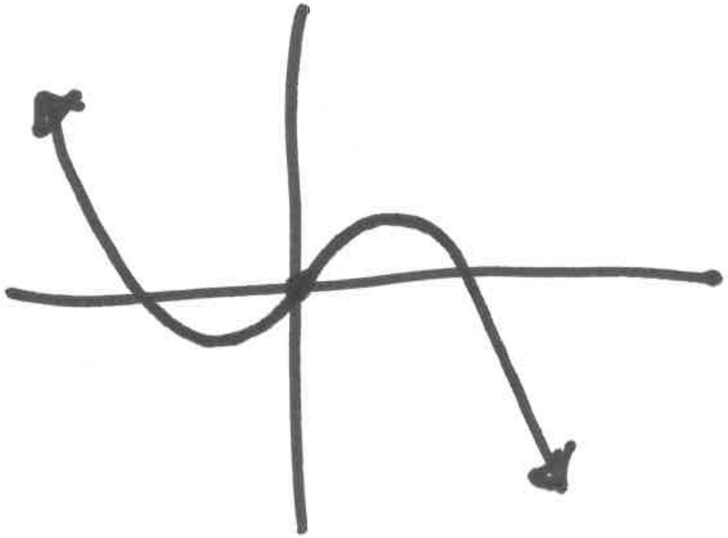
① (correct)



②



③



Q: What is $\frac{d}{dx} e^x \sin(2x)$?

A1:

$$= \left[\frac{d}{dx} e^x\right] \sin(2x) + e^x \left[\frac{d}{dx}(2\cos(x)\sin(x))\right]$$

$$= e^x \sin(2x) + 2e^x \left[\frac{d}{dx}(\cos(x)\sin(x))\right]$$

$$= e^x \sin(2x) + 2e^x \left[\left[\frac{d}{dx}\cos(x)\right]\sin(x) + \cos(x)\left[\frac{d}{dx}\sin(x)\right]\right]$$

$$= e^x \sin(2x) + 2e^x \left[-\sin^2(x) + \cos^2(x)\right]$$

$$= e^x \sin(2x) + 2e^x \cos(2x)$$

A2. (using chain rule)

$$= \left[\frac{d}{dx}e^x\right]\sin(2x) + e^x\left[\frac{d}{dx}\sin(2x)\right]$$

$$= e^x\sin(2x) + e^x\left(\cos(2x)\cdot\frac{d}{dx}2x\right)$$

$$= e^x\sin(2x) + 2e^x\cos(2x).$$

Q: What is $\dfrac{d}{dx}\dfrac{\tan(x)}{2x^2}$?

A:

$$= \frac{\left[\frac{d}{dx}\tan(x)\right](2x^2) - \tan(x)\left[\frac{d}{dx}2x^2\right]}{4x^4}$$

$$= \frac{2x^2\sec^2(x) - 4x\tan(x)}{4x^4}$$

$$= \frac{x\sec^2(x) - 2\tan(x)}{2x^3}.$$

Q: Find where the tangent to $y = x^3 + 2x + 1$ at $x = -1$ meets the curve again.

A: $y'(x) = 3x^2 + 2$ Point is $(-1, -2)$

Slope is $m = y'(-1) = 5$.

Tangent line: $y - (-2) = 5(x - (-1))$

$\Rightarrow \quad y + 2 = 5x + 5$

$\Rightarrow \quad y = 5x + 3$

So we want to find solutions to

$$x^3 + 2x + 1 = 5x + 3.$$

$$\Rightarrow \quad x^3 - 3x - 2 = 0$$

$$\Rightarrow \quad (x+1)\cdot(\underbrace{\qquad\qquad}_{\text{some quadratic}}) = 0$$

Turns out, it is $(x+1)(x+1)(x-2) = 0$

$$\Rightarrow \quad x = -1, \quad x = 2.$$

So the tangent meets the curve again at $x = 2$, or the point $(2, 13)$.

We have proven the power rule for:

- → positive integers n
- → $\frac{1}{n}$, n a positive integer

But what about:

- → negative integers n?
- → positive fractional (rational) exponents n?
- → negative rational exponents?
- → arbitrary real numbers n?

Q: What is $\frac{d}{dx} x^n$, when n a positive rational?

A: Let $n = \frac{p}{q}$ for positive integers p & q.

Then

$$\frac{d}{dx} x^n = \frac{d}{dx} x^{\frac{p}{q}} = \left(\frac{d}{dx}\left(x^{\frac{1}{q}}\right)^p\right)$$

$$= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{d}{dx} x^{\frac{1}{q}} = p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q} x^{\frac{1}{q}-1}$$

$$= \frac{p}{q} X^{\left(\frac{p-1}{q} + \frac{1}{q} - 1\right)}$$

$$= \frac{p}{q} X^{\left(\frac{p-1+1}{q} - 1\right)}$$

$$= \frac{p}{q} X^{\left(\frac{p}{q} - 1\right)} = n x^{n-1}.$$

⚠ Used the Chain rule!

Q: What is $\frac{d}{dx}\frac{1}{x^n}$, n a positive integer?

A:

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{x^n - (x+h)^n}{h x^n (x+h)^n}$$

$$= \lim_{h\to 0} \frac{x^n - \sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k}{h x^n (x+h)^n}$$

$$= \lim_{h\to 0} \frac{x^n - x^n - \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k}{h x^n (x+h)^n}$$

$$= \lim_{h\to 0} - \frac{\frac{1}{h} \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k}{x^n (x+h)^n}$$

$$= \lim_{h\to 0} - \frac{\sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^{k-1}}{x^n (x+h)^n}$$

As $h \to 0$, all terms is sum vanish except for 1st:

$$= -\frac{\binom{n}{1} x^{n-1}}{x^n (x+0)^n}$$

$$= -\frac{n x^{n-1}}{x^{2n}}$$

$$= -n \cdot \frac{1}{x^{2n-(n-1)}}$$

$$= -n \cdot \frac{1}{x^{n+1}}$$

$$= -n x^{-n-1}$$

Q: What is $\frac{d}{dx} \frac{1}{x^4}$?

A:

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^4} - \frac{1}{x^4}}{h}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$= \lim_{h \to 0} \frac{x^4 - (x+h)^4}{h x^4 (x+h)^4}$$

$$= \lim_{h \to 0} \frac{x^4 - [x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4]}{h x^4 (x+h)^4}$$

$$= \lim_{h \to 0} \frac{-4x^3h - 6x^2h^2 - 4xh^3 - h^4}{h x^4 (x+h)^4}$$

$$= \lim_{h \to 0} - \frac{4x^3 + 6x^2h + 4xh^2 + h^3}{x^4(x+h)^4}$$

$$= - \frac{4x^3}{x^4 x^4}$$

$$= - \frac{4x^3}{x^8}$$

$$= -4x^{-5}$$

$$\Rightarrow \frac{d}{dx} x^{-4} = -4x^{-5}.$$

MATH 3

Lecture #11

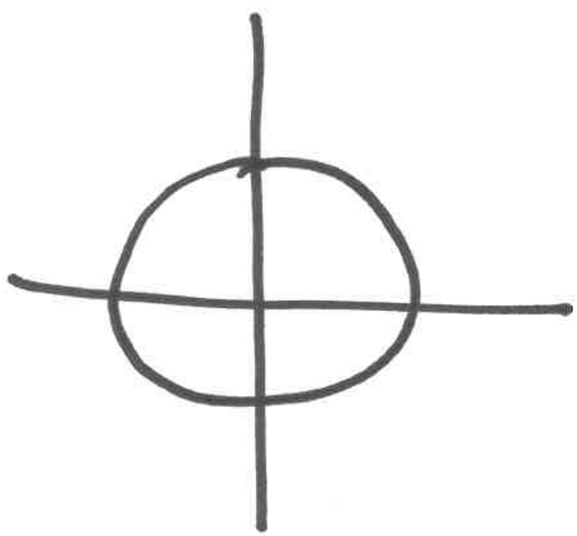
10/4/23

Jonathan Lindbloom

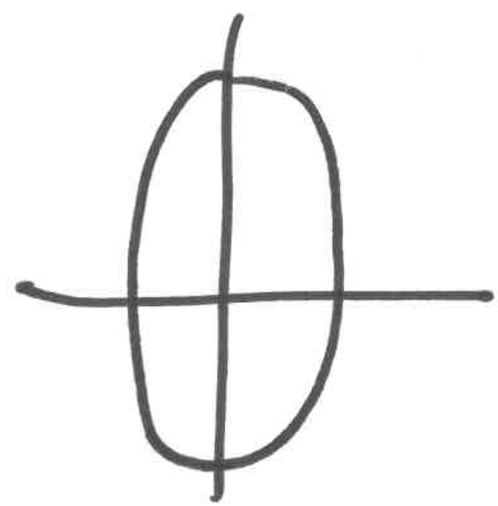
Implicit Differentiation

For some equations, it is not easy to solve for y in terms of x.

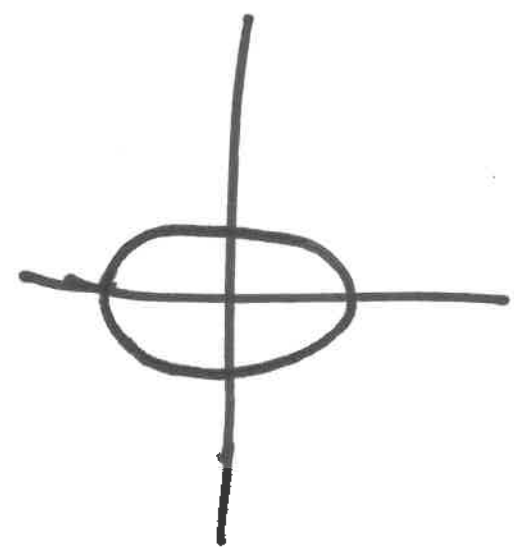
E.g.,



$x^2+y^2=r^2$, $r\in\mathbb{R}$.



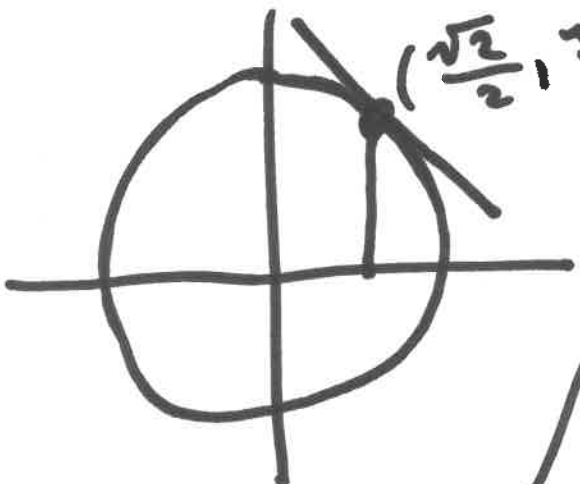
$x^2+\frac{1}{3}y^2=1$



$\frac{1}{2}x^2+y^2=1$

These are not functions, but rather curves.

Nonetheless, it still makes sense to talk about tangent lines to points on such curves, and thus also derivatives.



What is the eqn. of the tangent line?

$$x^2+y^2=1$$

<u>Method 1</u>: $x^2+y^2=1 \Rightarrow y^2 = 1-x^2$

$\Rightarrow y = \pm\sqrt{1-x^2}$.

Pick positive one to get top half of the circle.

So slope is

$$\left(\frac{d}{dx}\sqrt{1-x^2}\right)\Bigg|_{x=\frac{\sqrt{2}}{2}}$$

$$\Vert$$

$$\left[\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x\right]_{x=\frac{\sqrt{2}}{2}}$$

$\cdots = -1$. Point on the curve is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, so tangent line is

$$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \quad y = -x + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= -x + \sqrt{2}.$$

Method 2? Instead of solving for y and needing to make any $\pm$ choices, we can instead compute the derivative implicitly. What does this mean?

① Just take $\frac{d}{dx}$ of both sides of the implicit equation defining the curve, making sure to treat y as a function of x.

② Solve resulting equation for $\frac{dy}{dx}$.

$x^2 + y^2 = 1$?

$\rightarrow \quad \frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[1]$

$\rightarrow \quad 2x + \frac{d}{dx} y^2 = 0$

$\rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$

$\rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$

At point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$? $\frac{dy}{dx} = -1$,

same as other method. Will get same eqn of tangent line.

Note! For these types of tangent line questions, will need to be given both (x, y).

Q: Find $\frac{dy}{dx}$ where $y(x)$ is given implicitly by $\cos(y+x) = x$.

A: $$\frac{d}{dx}\left[\cos(y+x)\right] = \frac{d}{dx}[x]$$

$$\Rightarrow \quad -\sin(y+x)\cdot\frac{d}{dx}[y+x] = 1$$

$$\Rightarrow \quad -\sin(y+x)\left(\frac{dy}{dx}+1\right) = 1$$

$$\Rightarrow \quad \frac{dy}{dx}+1 = -\frac{1}{\sin(y+x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{\sin(y+x)} - 1.$$

Q: Find $\frac{dy}{dx}$ where $y(x)$ is given implicitly by $x^3y^2 + y\cos(x) = 7$.

A: $\frac{d}{dx}\left[x^3y^2 + y\cos(x)\right] = \frac{d}{dx}[7]$

$\Rightarrow \quad \frac{d}{dx}\left[x^3y^2\right] + \frac{d}{dx}\left[y\cos(x)\right] = 0$

$\Rightarrow \quad 3x^2y^2 + 2x^3y\frac{dy}{dx} + \frac{dy}{dx}\cos(x) - y\sin(x) = 0$

$\Rightarrow \quad \frac{dy}{dx}\left[2x^3y + \cos(x)\right] = y\sin(x) - 3x^2y^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin(x) - 3x^2y^2}{2x^3y + \cos(x)}$$

Why bother with implicit differentiation? Why not always solve for y in terms of x, then differentiate?

Sometimes this is <u>impossible</u>, at least to do this nicely in closed form.

<u>Ex:</u> Solving $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0$ for x is in general impossible to do in "closed form" whenever $n \geq 5$.

There <u>is</u> a closed form when $n = 3$, but you will not like it!

Q: Find the eqn. of the tangent line to the curve $y^2 = x^3 + 3x^2$ at point $(1, -2)$. (Tschirnhausen cubic).

A: $\frac{d}{dx}[y^2] = \frac{d}{dx}[x^3 + 3x^2]$

$\Rightarrow \quad 2y\frac{dy}{dx} = 3x^2 + 6x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2+6x}{2y}$.

At $x=1$, $y=-2$: $\frac{dy}{dx} = \frac{3+6}{2(-2)} = \frac{9}{-4} = -\frac{9}{4}$.

Using point-slope form:

$$y - (-2) = -\frac{9}{4}(x-1)$$

$$\Rightarrow \quad y + 2 = -\frac{9}{4}x + \frac{9}{4}$$

$$\Rightarrow \quad y = -\frac{9}{4}x + \frac{1}{4}.$$

Q: Find the eqn. of the tangent line to the curve $x^2+y^2=(2x^2+2y^2-x)^2$ at $(0,\frac{1}{2})$.

(cardioid)

A: $\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}\left[(2x^2+2y^2-x)^2\right]$

$\Rightarrow \quad 2x+2y\frac{dy}{dx} = 2(2x^2+2y^2-x)\cdot\frac{d}{dx}[2x^2+2y^2-x]$

$\Rightarrow \quad 2x+2y\frac{dy}{dx} = 2(2x^2+2y^2-x)(4x-1+4y\frac{dy}{dx})$

$\Rightarrow$ Stop! First plug in $x=0$, $y=\frac{1}{2}$.

We get

$$2\left(\frac{1}{2}\right)\frac{dy}{dx} = 2\left(2\left(\frac{1}{2}\right)^2\right)\left(-1+4\left(\frac{1}{2}\right)\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\left(\frac{1}{2}\right)\left(-1+2\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -1+2\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 1$$

So the tangent line is

$$y - \frac{1}{2} = 1 \cdot (x - 0)$$

$$\Rightarrow y = x + \frac{1}{2}.$$

Q: Find the eqn. of the tangent line to the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ at $(3,1)$. (lemniscate)

A: $\frac{d}{dx}\left[2(x^2+y^2)^2\right] = \frac{d}{dx}\left[25(x^2-y^2)\right]$

$\Rightarrow 4(x^2+y^2)\cdot\frac{d}{dx}\left[x^2+y^2\right] = 50x - 50y\frac{dy}{dx}$

$\Rightarrow 4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 50x - 50y\frac{dy}{dx}$

$\Rightarrow$ Stop! Just plug in $x=3$, $y=1$ before continuing.

We get

$$4(3^2+1^2)\left(2(3)+2(1)\frac{dy}{dx}\right) = 50(3) - 50(1)\frac{dy}{dx}$$

$$\Rightarrow \quad 40\left(6+2\frac{dy}{dx}\right) = 150 - 50\frac{dy}{dx}$$

$$\Rightarrow \quad 240 + 80\frac{dy}{dx} = 150 - 50\frac{dy}{dx}$$

$$\Rightarrow \quad 130\frac{dy}{dx} = -90$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{9}{13}$$

Now use point-slope form:

$$y - 1 = -\frac{9}{13}(x-3)$$

$$\Rightarrow \quad y = -\frac{9}{13}x + \frac{27}{13} + 1$$

$$\Rightarrow \quad y = -\frac{9}{13}x + \frac{40}{13}$$

Q: Find the eqn. of the tangent line to the curve $y^2(y^2-4) = x^2(x^2-5)$ at the point $(0,-2)$. (devil's curve)

A: $\frac{d}{dx}\left[y^2(y^2-4)\right] = \frac{d}{dx}\left[x^2(x^2-5)\right]$

$\Rightarrow \; 2y \cdot \frac{dy}{dx} \cdot (y^2-4) + y^2 \cdot 2y\frac{dy}{dx} = 2x(x^2-5) + x^2 \cdot 2x$

$\Rightarrow \; 2y(y^2-4)\frac{dy}{dx} + 2y^3\frac{dy}{dx} = 2x(x^2-5) + 2x^3$

$\Rightarrow$ Stop! First plug in $x=0$, $y=-2$.

We get

$$2(-2)\left[(-2)^2-4\right]\frac{dy}{dx} + 2(-2)^3\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -4\left[\cancelto{0}{4-4}\right]\frac{dy}{dx} - 16\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = 0.$$

So the eqn. of the tangent line is

$$y-(-2) = 0\cdot(x-0)$$

$$\Rightarrow \quad y+2 = 0$$

$$\Rightarrow \quad y = -2.$$

What is $\frac{d}{dx} \ln(x)$?

Let $y = \ln(x)$.

$\Rightarrow \quad e^y = x$

$\Rightarrow \quad \frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[x\right]$

$\Rightarrow \quad e^y \frac{dy}{dx} = 1$

$\Rightarrow \quad \frac{dy}{dx} = e^{-y}$

$\Rightarrow \quad \frac{dy}{dx} = e^{-\ln(x)} = \frac{1}{x}$.

What is $\frac{d}{dx}\arcsin(x)$?

Let $y = \arcsin(x)$.

$\Rightarrow \quad x = \sin(y)$

$\Rightarrow \quad \frac{d}{dx}[x] = \frac{d}{dx}[\sin(y)]$

$\Rightarrow \quad 1 = \cos(y)\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin(x))}$.

Let $\theta = \arcsin(x)$.

$\Rightarrow \cos(\theta) = \sqrt{1-x^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

What is $\dfrac{d}{dx} \arccos(x)$?

Let $y = \arccos(x)$. $\Rightarrow x = \cos(y)$

$$\Rightarrow \frac{d}{dx}[x] = \frac{d}{dx}[\cos(y)] \Rightarrow 1 = -\sin(y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin(y)} = -\frac{1}{\sin(\arccos(x))}.$$

Let $\theta = \arccos(x)$.

$$\sin(\arccos(x)) = \sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

What is $\frac{d}{dx}\arctan(x)$?

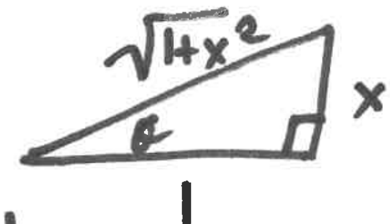
Let $y = \arctan(x) \implies x = \tan(y)$

$$\implies \frac{d}{dx}[x] = \frac{d}{dx}[\tan(y)]$$

$$\implies 1 = \sec^2(y)\frac{dy}{dx} \implies \frac{dy}{dx} = \cos^2(y).$$

$$= \cos^2(\arctan(x)).$$

Let $\theta = \arctan(x)$.



$$\implies \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}} \implies \cos^2(\arctan(x)) = \frac{1}{1+x^2}.$$

$$\implies \frac{dy}{dx} = \frac{1}{1+x^2}.$$

MATH 3

Lecture #12

10/6/23

Jonathan Lindbloom

Derivatives of Logarithmic Functions

Q: What is $\frac{d}{dx} \ln(x)$?

A: We have already discussed that $\frac{d}{dx} \ln(x) = \frac{1}{x}$ using an argument about the derivative of e^x. Let's review this.

Q: Find $\frac{d}{dx}\ln(kx)$ from first principles.

A: Let $f(x) = \ln(kx)$. Then we know that $f^{-1}(x) = \frac{1}{k}e^x$. Recall that the slope of $f(x)$ at x is the reciprocal of the slope of $f^{-1}(x)$ at $f(x)$.

So

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx}\left[\frac{1}{k}e^x\right] = \frac{1}{k}e^x$$

and

$$\left[\frac{d}{dx} f^{-1}(x)\right]_{x=f(x)} = \frac{1}{k}e^{\ln(kx)} = \frac{1}{k}kx = x$$

So we conclude that $\frac{d}{dx}\ln(kx) = \frac{1}{x}$.

What about using the chain rule?

$$\frac{d}{dx}\ln(kx) = \frac{1}{kx} \cdot \frac{d}{dx}[kx]$$

$$= \frac{1}{kx} \cdot k$$

$$= \frac{1}{x}.$$

Logarithmic Differentiation

Now that we know implicit differentiation, we can use a technique called "logarithmic differentiation" that sometimes helps when finding derivatives of complicated expressions.

$$y = f(x), \quad f(x) \text{ complicated?}$$

What is $\frac{dy}{dx}$?

* Idea: take log of both sides, then differentiate.

Q: What is $\dfrac{d}{dx} \dfrac{x^2+7}{\sqrt{x-3}}$?

A: Let $f(x) = \dfrac{x^2+7}{\sqrt{x-3}}$.

$$\underset{y}{\overset{\shortparallel}{f(x)}}$$

$$\Rightarrow \quad \ln(y) = \ln(x^2+7) - \ln(\sqrt{x-3})$$

$$\Rightarrow \quad \ln(y) = \ln(x^2+7) - \frac{1}{2}\ln(x-3).$$

Now differentiate:

$$\Rightarrow \quad \frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\ln(x^2+7) - \frac{1}{2}\ln(x-3)\right]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{\frac{d}{dx}\left[x^2+7\right]}{x^2+7} - \frac{1}{2}\,\frac{\frac{d}{dx}\left[x-3\right]}{x-3}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+7} - \frac{1}{2}\cdot\frac{1}{x-3}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{2x}{x^2+7} - \frac{1}{2}\cdot\frac{1}{x-3}\right].$$

Is this right?.

$$\Rightarrow \frac{dy}{dx} = \left[\frac{x^2+7}{\sqrt{x-3}}\right]\left[\frac{2x}{x^2+7} - \frac{1}{2}\frac{1}{x-3}\right)$$

$$= \frac{2x}{\sqrt{x-3}} - \frac{1}{2}\frac{x^2+7}{(x-3)^{\frac{3}{2}}}$$

Using quotient rule:

$$\frac{d}{dx}\left[\frac{x^2+7}{\sqrt{(x-3)}}\right] = \frac{2x\cdot\sqrt{x-3} - (x^2+7)\cdot\frac{1}{2}(x-3)^{-\frac{1}{2}}}{x-3}$$

$$= \frac{2x}{\sqrt{x-3}} - \frac{1}{2}\,\frac{x^2+7}{(x-3)^{\frac{3}{2}}} \checkmark .$$

Let's do a harder one.

Q: $$\frac{d}{dx}\,\frac{x^{\frac{5}{7}}\sqrt[10]{x+7}}{(x-1)^{\frac{9}{13}}\sqrt[8]{3x^2+1}} = \,?$$

A: Let $y =$.

$$\Rightarrow \ln(y) = \ln\left(x^{\frac{5}{7}}\right) + \ln\left(\sqrt[10]{x+7}\right) - \left[\ln\left((x-1)^{\frac{9}{13}}\right) + \ln\left(\sqrt[8]{3x^2+1}\right)\right]$$

$$\Rightarrow \ln(y) = \frac{5}{7}\ln(x) + \frac{1}{10}\ln(x+7) - \frac{9}{13}\ln(x-1) - \frac{1}{8}\ln(3x^2+1)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{5}{7}\frac{1}{x} + \frac{1}{10}\frac{1}{x+7} - \frac{9}{13}\frac{1}{x-1} - \frac{1}{8}\frac{1}{3x^2+1}\cdot 6x$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left[\qquad\qquad\qquad\qquad\right].$$

The power rule (in general form): (assuming $y>0$)

what is $\frac{d}{dx} x^n$, $n \in \mathbb{R}$?

Let $y = x^n$. $\Rightarrow \ln(y) = \ln(x^n)$

$$\Rightarrow \ln(y) = n \ln(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = n \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{n}{x} y$$

$$\Rightarrow \frac{dy}{dx} = \frac{n}{x} x^n = n x^{n-1}.$$

Q: What is $\frac{d}{dx}\left(x^{x^{\frac{3}{2}}}\right)$?

A: Let $y = x^{x^{\frac{3}{2}}}$.

$$\Rightarrow \ln(y) = \ln(\downarrow) = x^{\frac{3}{2}}\ln(x).$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}\ln(x) + x^{\frac{3}{2}}x^{-1}$$

$$\Rightarrow \frac{dy}{dx} = x^{x^{\frac{3}{2}}}\left[\frac{3}{2}x^{\frac{1}{2}}\ln(x) + x^{\frac{1}{2}}\right].$$

Q: What is $\frac{d}{dx} x^{\sin(x)}$?

A: Let $y = x^{\sin(x)}$.

$$\Rightarrow \ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\sin(x)\ln(x)\right)$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \left(\cos(x)\ln(x) + \frac{1}{x}\sin(x)\right)$$

$$= x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{1}{x}\sin(x)\right).$$

Q: What is $\frac{d}{dx} x^{x^x}$?

A: Let $y = x^{x^x}$.

$\Rightarrow \ln(y) = \ln(x^{x^x}) = x \ln(x^x) = x^2 \ln(x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[x^2 \ln(x)\right]$

$\Rightarrow \frac{dy}{dx} = x^{x^x}\left(2x\ln(x) + x^2 \cdot \frac{1}{x}\right)$

$= x^{x^x}\left(2x\ln(x) + x\right)$.

NO!!! THIS IS WRONG!!!

Q: (again) What is $\frac{d}{dx} x^{x^x}$?

A: Let $y = x^{x^x}$. * In general, $a^{b^c} \neq (a^b)^c$!

$$\Rightarrow \ln(y) = \ln\left(x^{x^x}\right) = \ln\left(x^{(x^x)}\right) = x^x \ln(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[x^x \ln(x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \ln(x) \cdot \frac{d}{dx}\left[x^x\right] + y x^x \frac{1}{x}$$

???

Let $z = x^x$.

$$\Rightarrow \ln z = \ln(x^x) = x\ln(x)$$

$$\Rightarrow \frac{1}{z}\frac{dz}{dx} = \frac{d}{dx}\left[x\ln(x)\right]$$

$$\Rightarrow \frac{dz}{dx} = z\cdot\left(\ln(x) + x\cdot\frac{1}{x}\right)$$

$$\Rightarrow \frac{dz}{dx} = x^x\left(\ln(x)+1\right).$$

So $$\frac{dy}{dx} = x^{x^x}\left(\ln(x)\left[x^x(\ln(x)+1)\right] + x^x \frac{1}{x} \right)$$

$$= x^{x^x}\left(x^x[\ln(x)]^2 + x^x \ln(x) + \frac{1}{x} x^x \right)$$

$$= x^{(x^x+x)}\left([\ln(x)]^2 + \ln(x) + \frac{1}{x} \right).$$

Derivatives of Inverse Trig Functions

Derivatives of Inverse Trig Functions:

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}.$$

Activity?

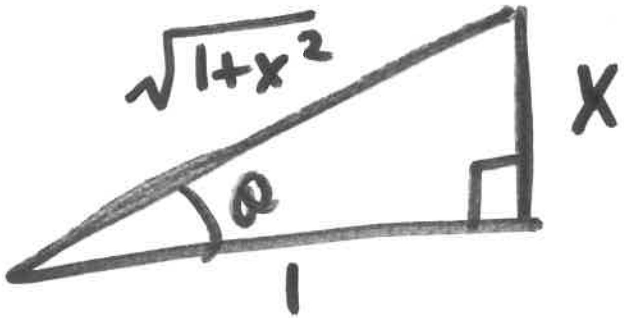
Q: What is $\frac{d}{dx}\arctan(x)$?

A: Let $y = \tan^{-1}(x)$. $\Rightarrow x = \tan(y)$

$$\Rightarrow 1 = \sec^2(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \cos^2(y)$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{\cos^2(\tan^{-1}(x))}_{\text{Simplify?}}.$$

Let $\theta = \tan^{-1}(x)$. $\Rightarrow$ $x = \tan(\theta)$.



$$\Rightarrow \cos(\theta) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \cos^2(\theta) = \frac{1}{1+x^2}$$

$$\Rightarrow \cos^2(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}.$$

Q: What is $\frac{d}{dx}\csc^{-1}(x)$?

A: Let $y = \csc^{-1}(x)$. $\Rightarrow \csc(y) = x$.

$$\Rightarrow -\cot(y)\csc(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\cot(y)\csc(y)}$$

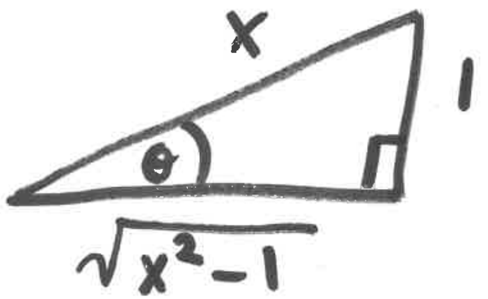
$$= -\tan(y)\sin(y)$$

$$\Rightarrow \frac{dy}{dx} = -\underbrace{\tan(\csc^{-1}(x))}_{\text{simplify?}}\ \underbrace{\sin(\csc^{-1}(x))}_{\text{simplify?}}$$

What is $\tan(\csc^{-1}(x))$?

Let $\theta = \csc^{-1}(x) \Rightarrow x = \csc(\theta)$.

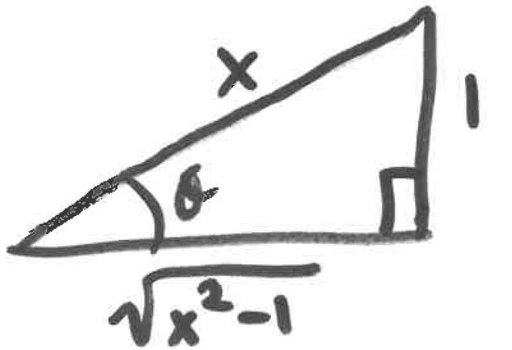
$$\Rightarrow x = \frac{1}{\sin(\theta)} \Rightarrow \sin(\theta) = \frac{1}{x}.$$



$$\Rightarrow \tan(\theta) = \frac{1}{\sqrt{x^2-1}} \Rightarrow \tan(\csc^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}.$$

What is $\sin(\csc^{-1}(x))$?

Re-use the same triangle.



$$\Rightarrow \sin(\theta) = \frac{1}{x}.$$

$$\Rightarrow \sin\left(\csc^{-1}(x)\right) = \frac{1}{x}.$$

Putting together, we get

$$\frac{d}{dx}\csc^{-1}(x) = -\tan(\csc^{-1}(x))\sin(\csc^{-1}(x))$$

$$= -\frac{1}{\sqrt{x^2-1}}\cdot\frac{1}{x}$$

$$= -\frac{1}{x\sqrt{x^2-1}}.$$

Q: What is $\frac{d}{dx} \cot^{-1}(x)$?

A: Let $y = \cot^{-1}(x) \Longrightarrow \cot(y) = x$.

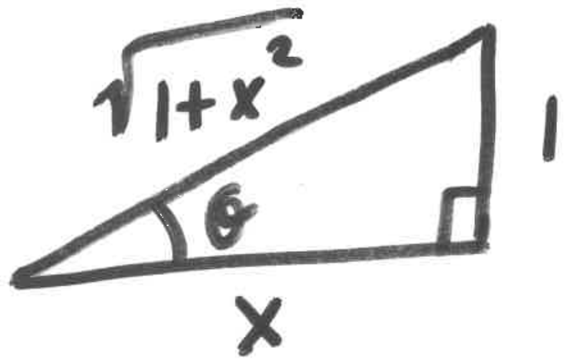
$$\Longrightarrow -\csc^2(y) \frac{dy}{dx} = 1$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{1}{\csc^2(y)}$$

$$\Longrightarrow \frac{dy}{dx} = -\sin^2(y)$$

$$= -\underbrace{\sin^2(\cot^{-1}(x))}_{???}.$$

Let $\theta = \cot^{-1}(x)$. $\Rightarrow$ $x = \cot(\theta)$.



$\Rightarrow \sin(\theta) = \dfrac{1}{\sqrt{1+x^2}}$

$$\Rightarrow \sin^2(\cot^{-1}(x)) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}.$$

Basic Applications

G: Gravitational constant

$G \simeq 9.8 \frac{m}{s^2}$, derived from Law of Universal Gravitation

$\Rightarrow$ Acceleration due to gravity is G.

Let $h(t)$ denote the height of a ball as a function of time. Say you stand at height $h=0$, and throw the ball straight upwards with velocity V_0. How does the ball move in time? i.e., what is $h(t)$?

Acceleration = 2nd derivative.

$$\Rightarrow h''(t) = \underbrace{-G}_{\text{constant}}.$$

What functions have constants as derivatives?

Linear!

$$\Rightarrow h'(t) = -Gt + C_1 \quad \text{for some } C_1 \in \mathbb{R}.$$

Similar reasoning.

$$\Rightarrow h(t) = -\frac{G}{2}t^2 + C_1 t + C_2,$$

for some $C_2 \in \mathbb{R}$.

What are C_1 + C_2 ?

Since $h(0) = C_2$ + $h(0) = 0$, $\Rightarrow C_2 = 0$.

Since we threw the ball with velocity V_0,

$\Rightarrow h'(0) = V_0 \Rightarrow h'(0) = C_1 = V_0$.

Conclusion: the ball moves like

$$h(t) = V_0 t - \frac{1}{2} G t^2.$$

This is the parabolic motion equation!

Why is calculus so important to physics?

<u>Many</u> physical laws that describe how quanties are related / evolve over time are given as differential equations (find function given derivatives + other conditions).

E.g. $$\frac{\partial}{\partial t} u(x,t) = D \frac{\partial^2}{\partial x^2} u(x,t)$$

Diffusion Equation

MATH 3

Lecture #13

10/9/23

Jonathan Lindbloom

Exponential Growth

+ Decay

We say that a quantity $y(t)$ exhibits exponential growth/decay if it obeys the differential equation

$$\frac{dy}{dt} = k\,y(t)$$

for some $k \in \mathbb{R}$.

$$\Rightarrow \quad y(t) = C e^{kt}$$

for some $C \in \mathbb{R}$.

From our definition of exponential growth/decay, does $y(t) = a^{kt}$ exhibit exponential growth/decay?

$$\frac{dy}{dt} = \ln(a) a^{kt} \cdot \frac{d}{dt}(kt)$$

$$= k \ln(a) a^{kt}$$

$$= \underbrace{k \ln(a)}_{\text{constant}} y(t),$$

so yes!

Interpretation?

$$\frac{dy}{dt} = k\,y(t) \Rightarrow \frac{\frac{dy}{dt}}{y} = k$$

The relative growth (or decay) rate is constant (k) in time.

This models things such as:

→ Population growth / decline

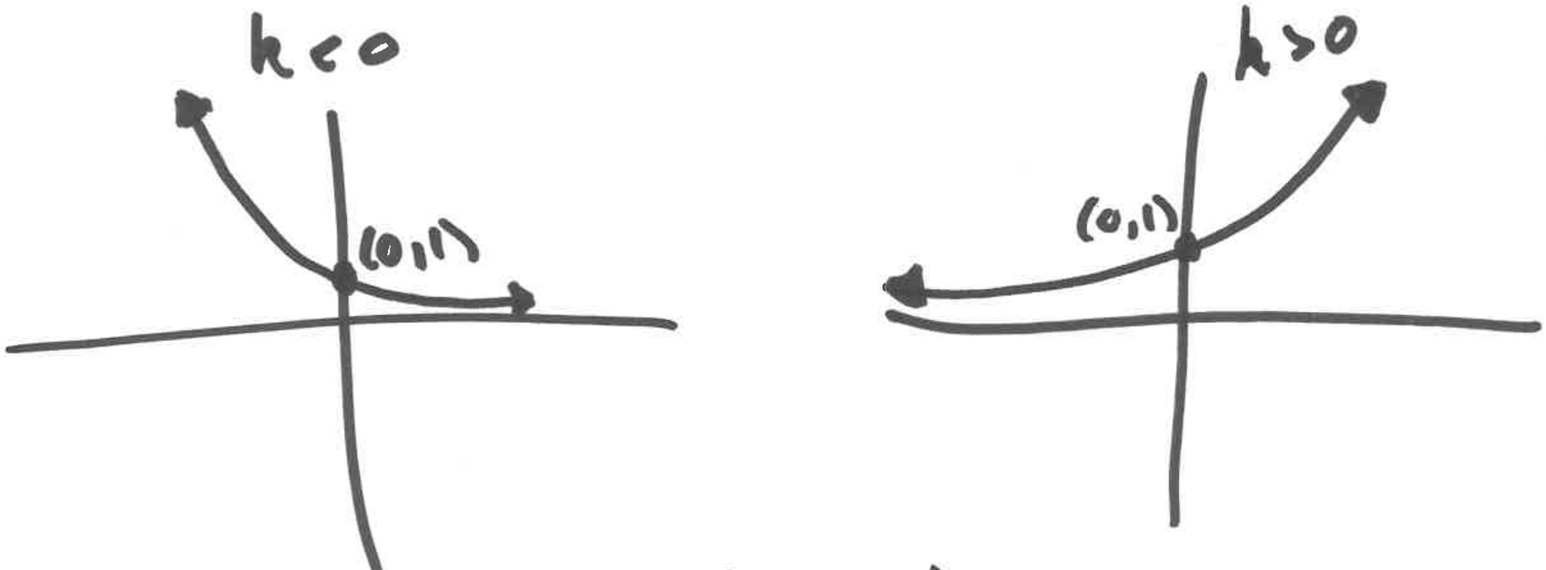
→ Radioactive decay

→ Compound interest

→ Thermodynamics (cooling)

<u>Thm</u>: The <u>only</u> solutions of $\frac{dy}{dt} = ky(t)$ are the functions $y(t) = y(0)e^{kt}$.

- The case $k > 0$ gives exponential growth.
- The case $k < 0$ gives exponential decay



What happens when $k = 0$?

Say you make an initial investment of P_0 dollars into a security that yields r percent interest, compounded annually.

How much money do you have at the end of the first year?

$$P_1 = P_0(1+r).$$

End of nth year?

$$P_n = P_0(1+r)^n.$$

What if we compounded m times per year, instead of just once? Say every 6 months?

$$P_{\frac{1}{2}} = P_0\left(1+\frac{r}{2}\right)$$

$$P_1 = P_0\left(1+\frac{r}{2}\right)^2$$

$$\Rightarrow P_n = P_0\left(1+\frac{r}{2}\right)^{2n}$$

If we compound m times per year?

$$P_n = P_0\left(1+\frac{r}{m}\right)^{mn}$$

Ex: $P_0 = \$100$, $r = 5\%$.

Compounding times per year	P_1 (approx)
$m = 1$	105
$m = 2$	105.062
$m = 12$	105.116
$m = 24$	105.122
$m = 365$	105.127
$m = 3.154 \times 10^7$	105.127

So we can't make infinite money by compounding more frequently. What is

$$\lim_{m\to\infty} P_1 = \lim_{m\to\infty} P_0\left(1+\frac{r}{m}\right)^m ?$$

$$= \lim_{m\to\infty} P_0\left(\left[1+\frac{r}{m}\right]^{\frac{m}{r}}\right)^r = P_0 \cdot \lim_{s\to\infty}\left(\left(1+\frac{1}{s}\right)^s\right)^r$$

$$= P_0 \cdot \left(\lim_{s\to\infty}\left(1+\frac{1}{s}\right)^s\right)^r$$

$$= P_0 e^r.$$

Why does $\displaystyle\lim_{h\to\infty}\left(1+\frac{1}{n}\right)^n = e$?

Consider $f(x) = \ln(x)$. $\Rightarrow f'(x) = \dfrac{1}{x}$, and $f'(1) = 1$.

But

$$f'(1) = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\ln(1+h)$$

$$= \lim_{h\to 0}\ln\left((1+h)^{\frac{1}{h}}\right),$$

$$\Rightarrow 1 = \lim_{h\to 0}\ln\left((1+h)^{\frac{1}{h}}\right) \Rightarrow e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

$$\Rightarrow e = \lim_{h\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Q: Consider two investments, A & B. A gives a 3% annualized interest rate compounded yearly, B gives a 3% interest rate but is compounded continuously. Which should you choose?

A

$$P_1 = P_0(1+0.03)$$
$$= P_0 \cdot 1.03$$
$$= 1.03 P_0$$

B

$$P_1 = Pe^{0.03}$$
$$\approx 1.03045 P_0$$

Pick B!

Q: Now you have two options, A and B. A gives 3% interest compounded monthly, B gives r percent interest but is compounded continuously. What value of r would make you indifferent to both choices?

$$\left(1+\frac{0.03}{12}\right)^{12} = e^{r}$$

$$\Rightarrow \ln(e^{r}) = \ln\left[\left(1+\frac{0.03}{12}\right)^{12}\right]$$

$$\Rightarrow \quad r = 12\ln\left(1+\frac{0.03}{12}\right) \simeq 0.0299626$$

Q: Say $P_0 = 100$, $r = 4\%$, and we compound continuously. How long will it take until you have \$1000?

A:

$$1000 = 100\left(e^{0.04t}\right)$$

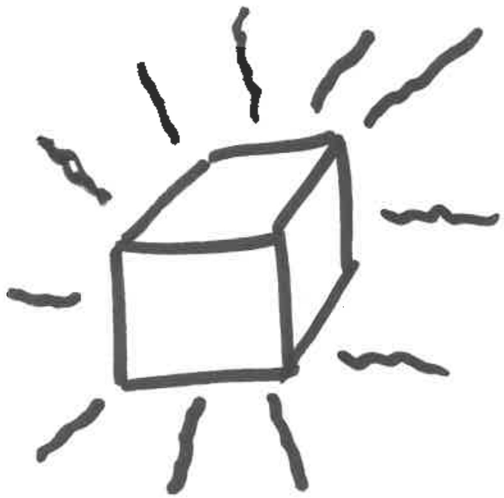
$$\Rightarrow 10 = e^{0.04t}$$

$$\Rightarrow \ln(10) = 0.04t$$

$$\Rightarrow t = \frac{\ln(10)}{0.04} \approx 57.5646$$

years.

Radioactive Decay:



Let m_0 denote the initial mass of a radioactive substance, and $m(t)$ the mass at time t.

Experimentally, it has been found that

$$\frac{dm}{dt} = km \, , \quad k \in \mathbb{R}_{<0} \, .$$

This gives exponential decay. However, we often report the half-life instead of the constant k.

How to find half-life given k and vice versa?

We have $m(0) = m_0$, and want t s.t. $m(t) = \frac{1}{2} m_0$.

But we also know that $m(t) = m(0)e^{kt}$.

$$\Rightarrow \frac{1}{2} m_0 = m_0 e^{kt}$$

$$\Rightarrow \frac{1}{2} = e^{kt}$$

$$\Rightarrow -\ln(2) = kt$$

$$\Rightarrow k = -\frac{\ln(2)}{t}.$$

exponential decay rate

half-life

Q: A sample of einsteinium-252 decayed to 64.3% of its original mass after 300 days.

(1) What is the half-life?

(2) How long would it take the original sample to decay to $\frac{1}{3}$ of its original mass?

A: We know that $m(t) = m(0)e^{kt}$, and $m(300) = 0.643\, m(0)$.

$$\Rightarrow m(300) = m(0)e^{300k} = 0.643\, m(0)$$

$$\Rightarrow e^{300k} = 0.643$$

$$\Rightarrow 300k = \ln(0.643)$$

$$\Rightarrow k = \frac{1}{300}\ln(0.643) \approx -0.001472$$

So the half-life is

$$t = -\frac{\ln(2)}{k}$$

$$\approx 470.89 \text{ days.}$$

For the original sample to decay to $\frac{1}{3}$ of its initial mass, it would take

$$m(t) = m(0)e^{-0.001472t} = \frac{1}{3}m(0)$$

$$\Rightarrow e^{-0.001472t} = \frac{1}{3}$$

$$\Rightarrow \quad t = \frac{1}{-0.001472}\ln\left(\frac{1}{3}\right)$$

$$\approx 746.34 \quad \text{days.}$$

Newton's Law of Cooling:

Rate of cooling of an object is proportional to the temperature difference between an object and its surroundings.

$$\frac{dT}{dt} = k\,(T - T_s)$$

ambient temperature

Is $T(t)$ an exponentially growing or decaying function? NO!

But if we define a new function $h(t) = T(t) - T_s$, then we see that $\frac{dh}{dt} = \frac{d}{dt}[T(t) - T_s] = \frac{dT}{dt}$ and $k(T - T_s) = k\,h(t)$.

$$\Rightarrow \quad \frac{dh}{dt} = k\,h(t) \Rightarrow h(t) = Ce^{kt}$$

$$\Rightarrow T(t) - T_s = Ce^{kt} \Rightarrow T(t) = T_s + Ce^{kt}.$$

If $k<0$, what happens as $t \to \infty$?

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty} \left[T_s + Ce^{kt}\right] = T_s.$$

Say that a hot iron rod (500°F) is suspended in cold air (5°F). After 5 minutes, the rod is 300°F. How hot will the rod be after 1 hour (from start)? Assume Newton's Law of Cooling.

<u>A:</u> We know that $T(t) = T_s + Ce^{kt}$. ($T_s = 5°F$)

Also, $T(0) = T_s + C = 500 \Rightarrow C = 500 - T_s = 495$.

So $T(t) = 5 + 495e^{kt}$. What is k? We also know that $T(5) = 5 + 495e^{5k} = 300 \Rightarrow e^{5k} = \frac{59}{99}$

$\Rightarrow 5k = \ln\left(\frac{59}{99}\right) \Rightarrow k = \frac{1}{5}\ln\left(\frac{59}{99}\right)$

≈ -0.1035

So after 1 hour from the start, the rod's temperature will be

$$T(60) = 5 + 495e^{60\cdot(-0.1035)}$$

$$\approx 5.9946\ ^\circ F.$$

__Fibonacci Numbers:__ Let $F_0 = 0$, $F_1 = 1$,

and define a sequence by the recursion

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n = 2, 3, \ldots.$$

Does this sequence grow subexponentially, superexponentially, or exponentially?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

We can show that $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

Note that $\lim_{n \to \infty} \left(\frac{1-\sqrt{5}}{2} \right)^n = 0$, so for large n,

$$F_n \approx \frac{1}{\sqrt{5}} \Phi^n$$

where $\Phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

So we see that for large n, F_n grows (approximately) exponentially.

Q: Approximate n such that $F_n > 2 \times 10^{422}$.

A: $F_n \approx \frac{1}{\sqrt{5}} \Phi^n$, so

$$\frac{1}{\sqrt{5}} \Phi^n = 2 \times 10^{422}$$

$$\Rightarrow n \ln(\Phi) - \frac{1}{2}\ln(5) = \ln(2) + 422 \ln(10)$$

$$\Rightarrow n \approx \frac{\ln(2) + 422 \ln(10) + \frac{1}{2}\ln(5)}{\ln(\Phi)}$$

$$\Rightarrow n \approx 2022.46,$$

So we can pick $n = 2023$.

MATH 3

Lecture #14

10/11/23

Jonathan Lindbloom

Related Rates

Sometimes we would like to know the rate of change of some quantity, but we only have access to the rate of change of a related/derived quantity. It turns out that it is often possible to relate the desired rate of change to that of the related quantity, hence "related rates".

Q: Suppose we increase the side length of a cube s at a constant rate of 3 cm/s. What is the rate of change of the cube's volume?

A: Volume is related to the side length by

$$V = s^3.$$

This holds for any t, so

$$V(t) = s(t)^3.$$

Differentiating w.r.t. t, we obtain

$$\frac{dV}{dt} = 3[s(t)]^2 \cdot \frac{ds}{dt}. \quad \left(\text{for any } t!\right)$$

Note the rate of change of the volume is not constant.

Given	Want	Relation?
$\frac{d}{dt}$(side length s of a square)	$\frac{d}{dt}$(Volume)	$V = s^3$
$\frac{d}{dt}$(radius r of a circle)	$\frac{d}{dt}$(Area)	$A = \pi r^2$
$\frac{d}{dt}$(radius r of a sphere)	$\frac{d}{dt}$(Volume)	$V = \frac{4}{3}\pi r^3$
$\frac{d}{dt}$(radius r of a sphere)	$\frac{d}{dt}$(surface Area)	$SA = 4\pi r^2$
$\frac{d}{dt}$(hypotenuse h of a triangle)	$\frac{d}{dt}$(length of some side)	several, e.g., $a^2+b^2=c^2$ $c^2=a^2+b^2-2ab\cos(c)$

Q: What is the rate of change of the surface area of a sphere with radius 10 cm, given that rate of change of the sphere's radius is constant and equal to 2 cm/second?

Know?	Want?	Relations?
$\frac{dr}{dt} = 2$ cm/second	$\frac{dA}{dt}$	$A = 4\pi r^2$

1 First step is to write down information in mathematical notation.

2 Second step is usually to differentiate a relation w.r.t. time.

(2)

$$A = 4\pi r^2$$

$$\Rightarrow \frac{d}{dt}[A] = \frac{d}{dt}[4\pi r^2]$$

$$\Rightarrow \frac{dA}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\Rightarrow \boxed{\frac{dA}{dt} = 8\pi r \frac{dr}{dt}}$$

(3) Once you have an equation relating rates, plug in what you know and solve for desired quantity!

$$\Rightarrow \frac{dA}{dt} = 8\pi(10\text{ cm})(2\text{ cm/second})$$

$$= 160\pi\text{ cm}^2\text{/second}.$$

Q: If a snowball melts so that its surface area decreases at a rate of $1\ cm^2/min$, find the rate at which the diameter decreases when the diameter is 10 cm.

Know?	Want?	Relations?
$\frac{dA}{dt} = -1 \text{ cm}^2/\text{min}$	$\frac{dD}{dt}$	$A = 4\pi r^2$
$D = 10 \text{ cm}$		$D\cancel{d} = 2r$

Equation relating D and A?

$$D = 2r \Rightarrow r = \frac{D}{2}$$

$$A = 4\pi r^2 \Rightarrow A = 4\pi\left(\frac{D}{2}\right)^2$$

$$\Rightarrow A = \pi D^2.$$

Differentiating:

$$\frac{d}{dt}[A] = \frac{d}{dt}\{\pi D^2]$$

$$\Rightarrow \frac{dA}{dt} = 2\pi D \frac{dD}{dt}$$

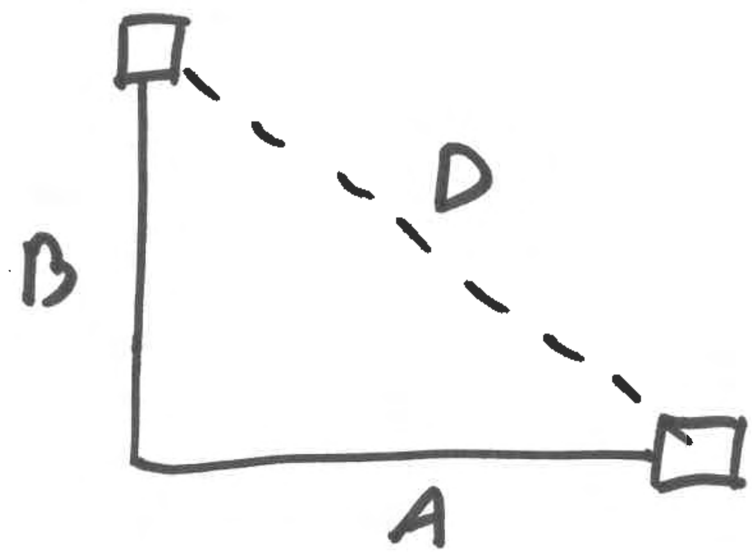
Now plug in:

$$-1 \frac{cm^2}{min} = 2\pi \cdot (10\,cm) \cdot \frac{dD}{dt}$$

$$\Rightarrow \frac{dD}{dt} = -\frac{1}{20\pi} \frac{cm}{min}$$

Q: Two cars meet at an intersection. The first car continues north at 60 mph, while the second car continues east at 80 mph. After 20 minutes, what is the rate of change of the distance between the two vehicles?

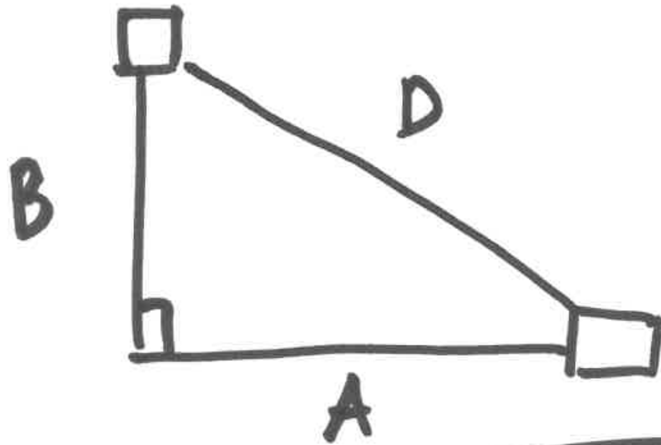
Diagram:



Know?	Want?	Relations?
$\frac{dA}{dt} = 80$	$\frac{dD}{dt}$	$D^2 = A^2 + B^2$
$\frac{dB}{dt} = 60$		
20 minutes have passed		

20 minutes $= \frac{1}{3}$ hour.

After $\frac{1}{3}$ hour,



$\Rightarrow A = \frac{1}{3} \cdot 80 = \frac{80}{3}$

$\Rightarrow B = \frac{1}{3} \cdot 60 = \frac{60}{3} = 20$

$\Rightarrow D = \sqrt{20^2 + \left(\frac{80}{3}\right)^2}$.

Differentiating:

$$\frac{d}{dt}\left[D^2\right] = \frac{d}{dt}\left[A^2 + B^2\right]$$

$$\Rightarrow \quad 2D\frac{dD}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt}$$

$$\Rightarrow \quad D\frac{dD}{dt} = A\frac{dA}{dt} + B\frac{dB}{dt}$$

$$\Rightarrow \quad \frac{dD}{dt} = \frac{1}{D}\left[A\frac{dA}{dt} + B\frac{dB}{dt}\right].$$

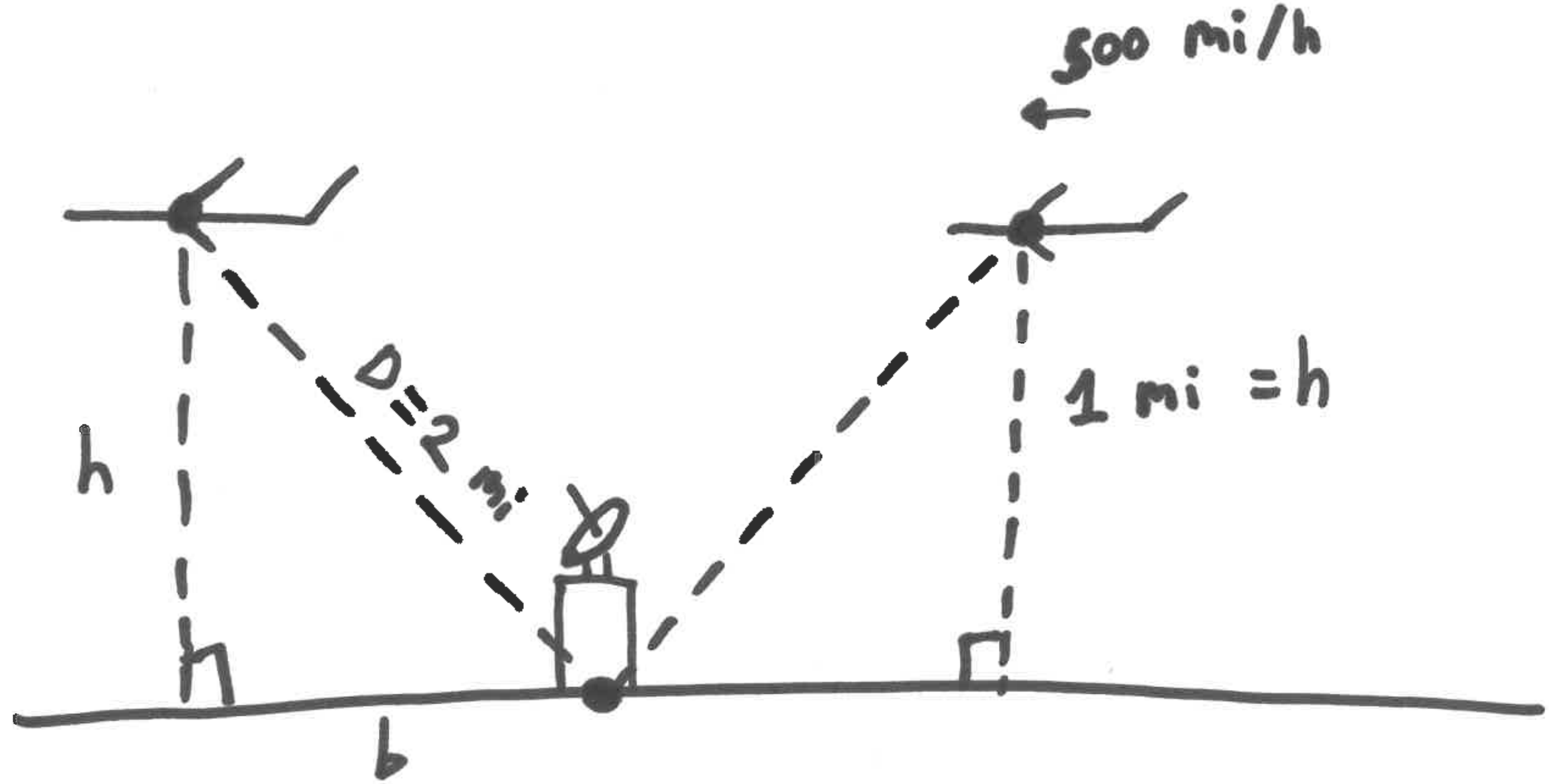
Plugging in:

$$\frac{dD}{dt} = \frac{1}{D}\left[\frac{80}{3}\cdot 80 + 20\cdot 60\right]$$

$$\simeq 100 \text{ mph}.$$

Q: A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when the plane is 2 mi away from the station.

Diagram:



Know?	Want?	Relations?
$D = 2$ mi	$\frac{dD}{dt}$	$h^2 + b^2 = D^2$
$h = 1$ mi		
$\frac{db}{dt} = 500$ mi/h		

Differentiate!

$$\frac{d}{dt}\left[h^2+b^2\right]=\frac{d}{dt}\left[D^2\right]$$

$$\Rightarrow\quad 2h\frac{dh}{dt}+2b\frac{db}{dt}=2D\frac{dD}{dt}$$

$$\Rightarrow\quad h\frac{dh}{dt}+b\frac{db}{dt}=D\frac{dD}{dt}.$$

$$\Rightarrow\quad \frac{dD}{dt}=\frac{1}{D}\left[h\frac{dh}{dt}+b\frac{db}{dt}\right].$$

Analysis steps?

→ Altitude is constant, so $\frac{dh}{dt} = 0$

→ $\frac{db}{dt} = 500$ mi/h

→ $D = 2$ mi

→ $h = 1$ mi

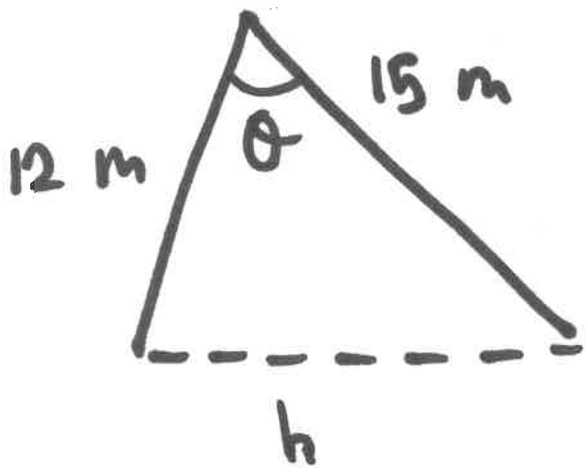
→ $D^2 = b^2 + h^2 \Rightarrow b = \sqrt{D^2 - h^2} = \sqrt{3}$ mi

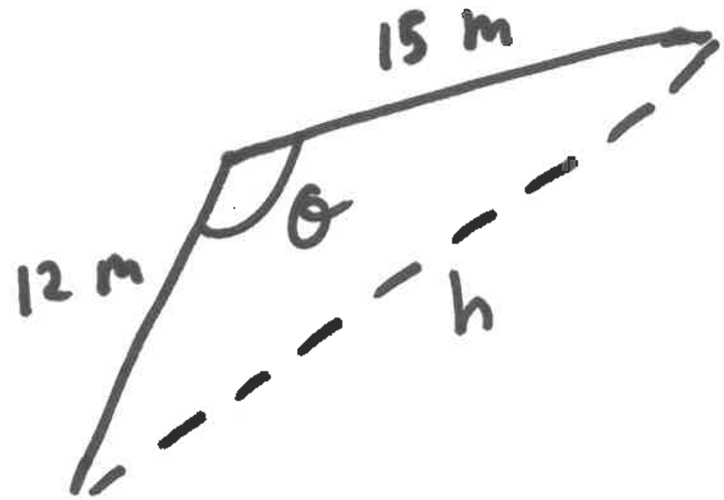
So, $$\frac{dD}{dt} = \frac{1}{2\text{ mi}}\left[(\sqrt{3}\text{ mi})(500\text{ mi/h})\right]$$

$$= 250\sqrt{3}\text{ mi/h}.$$

Q: Two sides of a triangle have lengths 12 m and 15 m. The angle between them is increasing at a rate of $2°/\text{min}$. How fast is the length of the third side increasing when the angle between the sides of fixed length is $60°$?

Diagram:





Know?	Want?	Relations?
two fixed sides of 12 m and 15 m $\theta = 60^\circ$	$\frac{dh}{dt}$	$h^2 = 12^2 + 15^2 - 2(12)(15)\cos(\theta)$ $\Rightarrow h^2 = 369 - 360\cos(\theta)$

Differentiate!

$$\frac{d}{dt}\left[h^2\right] = \frac{d}{dt}\left[369 - 360\cos(\theta)\right]$$

$$\Rightarrow \quad 2h\frac{dh}{dt} = 360\sin(\theta)\frac{d\theta}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{180}{h}\sin(\theta)\frac{d\theta}{dt}$$

Analysis step:

→ $\theta = 60^\circ = \frac{\pi}{3}$ radians

→ $\frac{d\theta}{dt} = 2^\circ / \text{min} = \frac{\pi}{90}$ radians / min

→ But what is h? Use the relation again!

$$h^2 = 12^2 + 15^2 - 2(12)(15)\cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow h^2 = 369 - 360\cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow h = \sqrt{369 - 360 \cdot \frac{1}{2}} = \sqrt{369 - 180}$$

$$\approx 13.7477$$

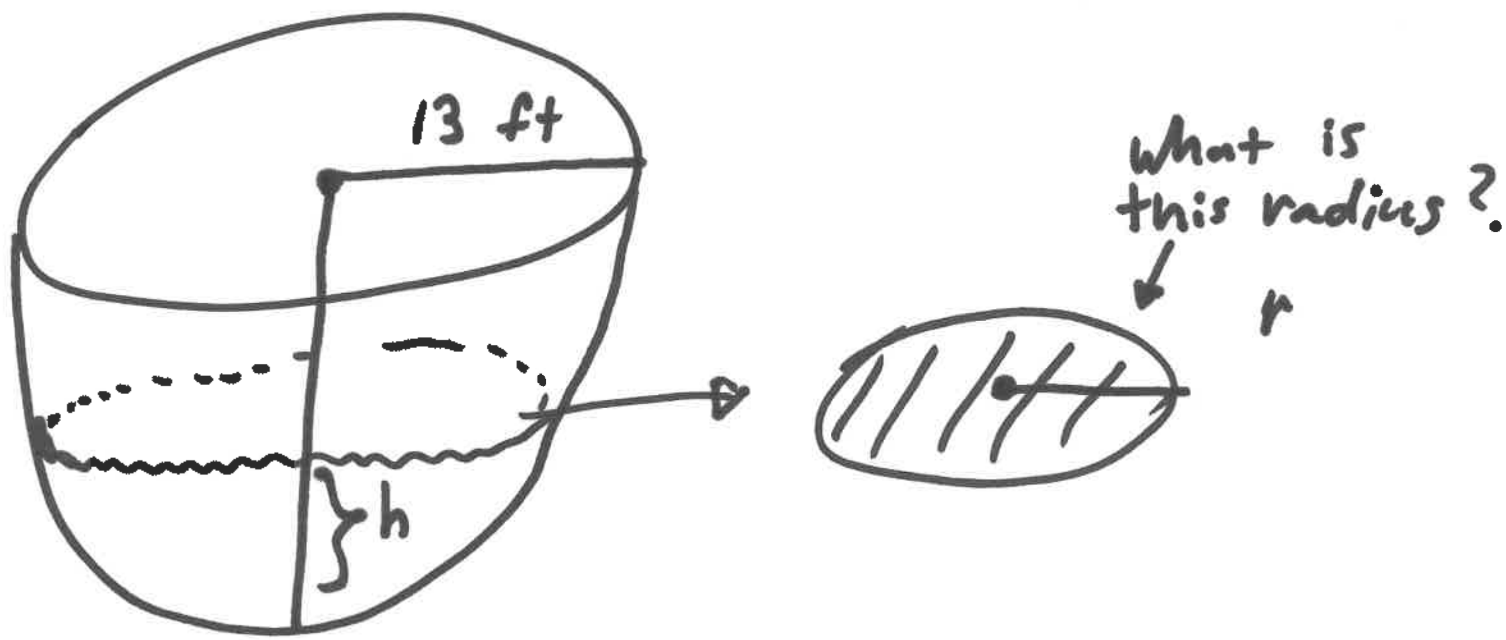
Plugging in:

$$\frac{dh}{dt} = \frac{180}{\sqrt{369-180}} \cdot \sin\left(\frac{\pi}{3}\right) \cdot \frac{\pi}{90}$$

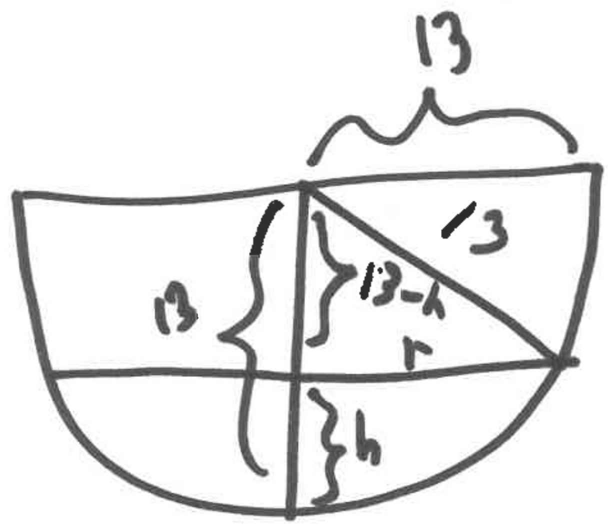
$$\approx 0.3958$$

Q: An open hemispherical tank has radius 13 ft. Water begins flowing into the tank in such a way that the depth of water is increasing at a rate of 3 ft/hr. At what rate is the top circular surface area of the water changing when the depth of water is 8 ft. ?

Diagram:



Know?	Want?	Relation?
$h = 8$ ft	$\frac{dA}{dt}$	$A = \pi r^2$
radius of bowl is 13 ft.		$r^2 + (13-h)^2 = 13^2$
$\frac{dh}{dt} = 3$ ft/hr		



Pythagorean thm:

$$\Rightarrow r^2 + (13-h)^2 = 13^2$$

Differentiating:

$$\frac{d}{dt}\left[r^2 + (13-h)^2\right] = \frac{d}{dt}\left[13^2\right]$$

$$\Rightarrow 2r\frac{dr}{dt} + 2(13-h)\cdot -1 \cdot \frac{dh}{dt} = 0$$

$$\Rightarrow 2r\frac{dr}{dt} = \left[26-2h\right]\frac{dh}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \left[13-h\right]\frac{dh}{dt}.$$

But we want $\frac{dA}{dt}$! Since $A = \pi r^2$,

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2]$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Combining with previous rate equation,

$\frac{dr}{dt} = \frac{1}{r}[13-h]\frac{dh}{dt}$, we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi[13-h]\frac{dh}{dt}.$$

Plugging in:

$$\frac{dA}{dt} = 2\pi\left[13-h\right]\frac{dh}{dt}$$

$$= 2\pi\left[13-8\right]\cdot 3$$

$$= 6\pi \cdot 5$$

$$= 30\pi.$$

MATH 3

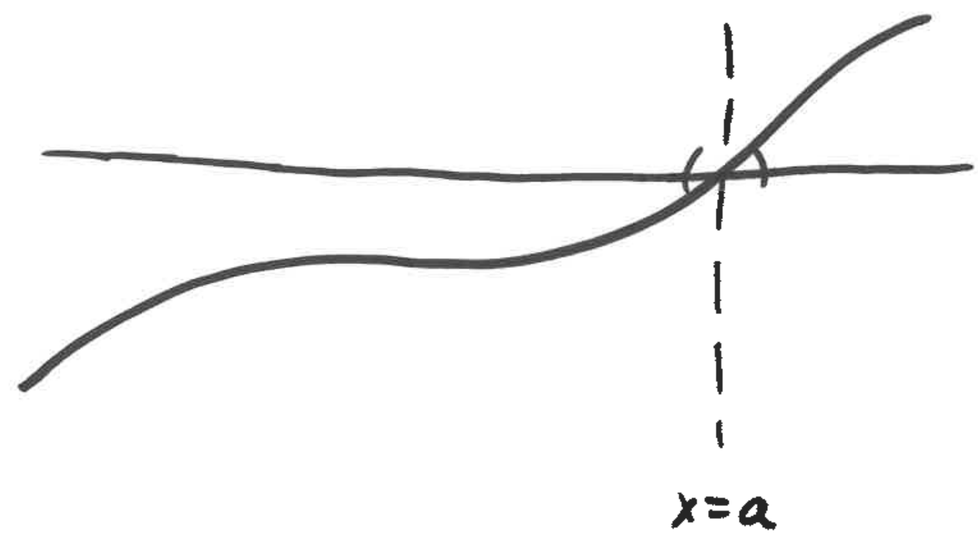
Lecture #15

10/13/23

Jonathan Lindbloom

Q: Let P_n denote the set of all polynomials of degree $\leq n$. Given a function $f(x)$, what is the best approximation $p_0(x) \in P_0$ for $x \approx a$?

☆ P_0 is comprised of the constant functions



☆ Intuitively, the best approximation should be $p_0(x) = f(a)$!

We can encode this with the condition

$$\text{(i)} \quad \lim_{x \to a} p_0(x) = f(a).$$

Is $p_0(x) = f(a)$ the only function in P_0 satisfying (i)?

Yes!

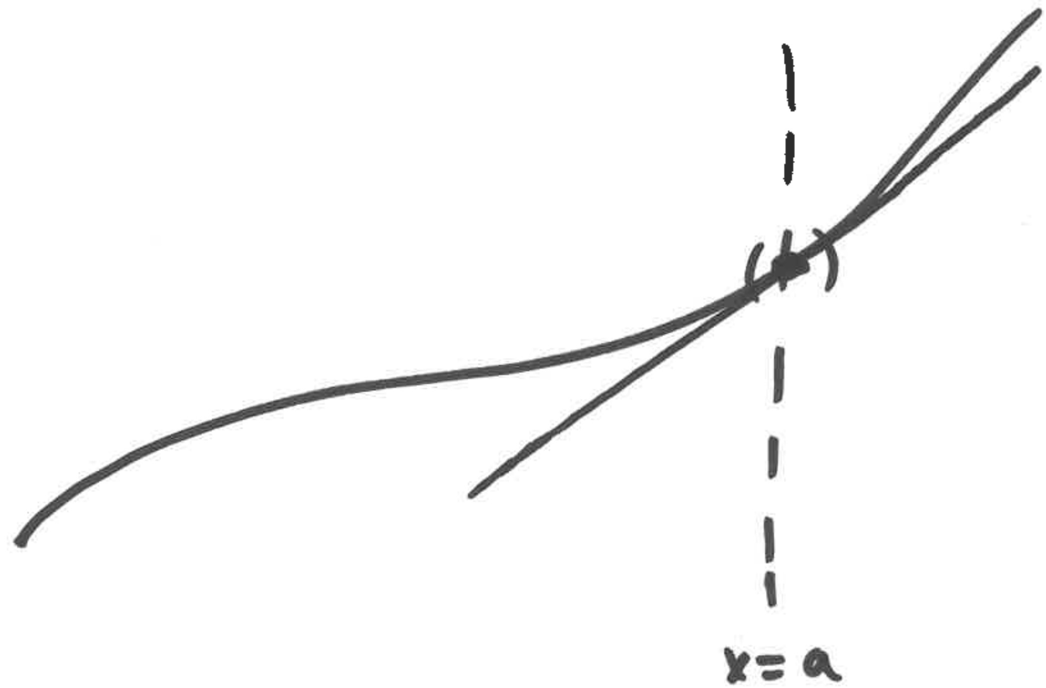
<u>proof</u>: Let $g(x) \in P_0$. Then by definition, $g(x) = c$ for some $c \in \mathbb{R}$. Condition (i) implies that

$$\lim_{x \to a} g(x) = \underbrace{\lim_{x \to a} c}_{= c} = f(a),$$

$$\Rightarrow g(x) = f(a).$$

Q: Given a function $f(x)$, what is the best approximation $p_1(x) \in P_1$ for $x \approx a$?

☆ P_1 is comprised of linear functions



☆ Intuitively, the best approximation should be $p_1(x)$ = eqn of tangent line!

We can encode this with the condition

$$\text{(ii)} \quad p_1'(a) = f'(a).$$

(coupled with condition (i)).

<u>Q</u>: What functions $p_1 \in P_1$ satisfy both (i) and (ii)?

Claim: Only $p_1(x) = f(a) + f'(a)(x-a)$.

Proof?: Let $p_1 \in P_1$. Then $p_1(x) = c_0 + c_1 x$ for some $c_0, c_1 \in \mathbb{R}$.

Condition (i) requires that

$$f(a) = \lim_{x \to a} p_1(x) = \lim_{x \to a} \left[c_0 + c_1 x \right]$$
$$= c_0 + c_1 a$$

$$\Rightarrow \quad c_0 + c_1 a = f(a).$$

Condition (ii) requires that

$$f'(a) = p_1'(a) = c_1 .$$

$$\Rightarrow \quad c_1 = f'(a) .$$

So $\quad p_1(x) = c_0 + f'(a) .$

Using $\quad c_0 + c_1 a = f(a)$, we see that

$$c_0 + f'(a)\, a = f(a)$$

$$\Rightarrow \quad c_0 = f(a) - f'(a)\, a .$$

So

$$p_1(x) = f(a) - f'(a)\, a + f'(a)\, x$$

$$= f(a) + f'(a)(x-a) .$$

This is exactly the equation of the tangent!

Higher-order approximations:

You may be interested in what happens if we look for a function $p_n \in P_n$ such that

$$p_n^{(k)}(a) = f^{(k)}(a)$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

for $0 \le k \le n$. After some algebra, we would find that

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

This is the n'th degree Taylor polynomial centered at $x=a$

For this reason, we say that the tangent line to $f(x)$ at point $x=a$ is the best linear approximation to $f(x)$ near $x=a$.

We sometimes call this the linearization of $f(x)$ at $x=a$, given by

$$L(x) = f(a) + f'(a)(x-a)$$
$$\approx f(x).$$

Q: Find the linearization of $f(x) = \sin(x)$ at $x = 0$.

A: We know that $L(x) = f(a) + f'(a)(x-a)$, so with $f = \sin(x) \Rightarrow f'(x) = \cos(x)$ and $a = 0$ we have

$$L(x) = \sin(0) + \cos(0)(x-0)$$
$$= 0 + 1 \cdot (x-0)$$
$$= x.$$

This justifies the small-angle approximation $\sin(x) \approx x$ for x near zero.

Q: Find the linearization of $f(x) = \sqrt[3]{x}$ at $x = 8$.

$f' = \frac{1}{3}x^{-\frac{2}{3}}$

A: We have

$$L(x) = f(a) + f'(a)(x-a)$$
$$= f(8) + f'(8)(x-8)$$
$$= 2 + \frac{1}{12}(x-8).$$

Q: Estimate $\sqrt{1.01}$ using a linearization.

A: Let $f(x) = \sqrt{x}$, and $a = 1.0$. Then the linearization of $f(x)$ at $a = 1.0$ is

$$L(x) = f(1) + f'(1)(x-1)$$
$$= 1 + \frac{1}{2}(1)^{-\frac{1}{2}}(x-1)$$
$$= 1 + \frac{1}{2}(x-1).$$

$$\Rightarrow f(1.01) \approx L(1.01) = 1 + \frac{1}{2}(1.01-1)$$
$$= 1 + \frac{1}{2} \cdot \frac{1}{100}$$
$$= 1 + \frac{1}{200}$$
$$= \frac{201}{200} \approx 1.005.$$

Exact is $\sqrt{1.01} \approx 1.00499\ldots$

Q: We measure the side of a cube to have length $s = 5 \pm 0.1$ cm. Estimate an error bound for the volume of the cube using a linearization.

A: We know that $V(s) = s^3$. Linearizing about $s = 5$, we obtain

$$V(s) \approx L(s) = V(5) + V'(5)(s-5)$$
$$= 125 + 75(s-5)$$

Using the linearization, we see that

$$
\begin{aligned}
V(5 \pm 0.1) &\approx L(5 \pm 0.1) \\
&= 125 + 75\big((5 \pm 0.1) - 5\big) \\
&= 125 + (75 \cdot \pm 0.1) \\
&= 125 \pm 7.5 .
\end{aligned}
$$

So the absolute error is about ± 7.5.

What about relative error?

$$\text{rel err} = \frac{(125 \pm 7.5) - (125)}{(125)}$$

$$= \pm \frac{7.5}{125}$$

$$= \pm\, 0.06$$

$$= \pm\, 6\,\% .$$

But these are just approximations! What are the exact absolute/relative errors?

For absolute error, in worst case we have $V \approx 117.649$ and in best $V = 132.651$, which gives $E_L = -7.351$ and $E_U = 7.651$

Our estimated error was ± 7.5!

For relative error, we have $E_L^{(rel)} \approx -0.062 = -6.2\%$ and $E_U^{(rel)} \approx 0.058 = 5.8\%$.

Our estimated relative error was 6%!

Note on Differentials:

Sometimes, we write

$$\frac{dy}{dx} = f'(x)$$

as

$$dy = f'(x)\,dx.$$

We call dy and dx 'differentials'.

Although you must be very careful when working with these.

Preferred treatment of differentials:

Instead of writing $dy = f'(x)\,dx$, instead think of

$$\Delta y = f'(x)\,\Delta x,$$

where here $\Delta x = h$ and $\Delta y = y(x+h) - y(x)$. Think of dy and dx as approximately representing Δy and Δx, for small h.

Q: What is the differential dV for $V(s) = s^3$ in terms of ds?

A: $\frac{dV}{ds} = 3s^2 \Rightarrow dV \approx 3s^2\, ds$.

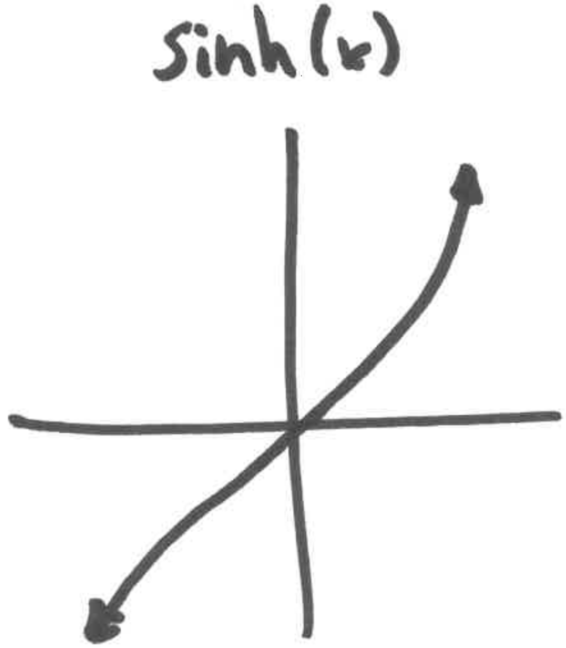
Hyperbolic Trig Functions

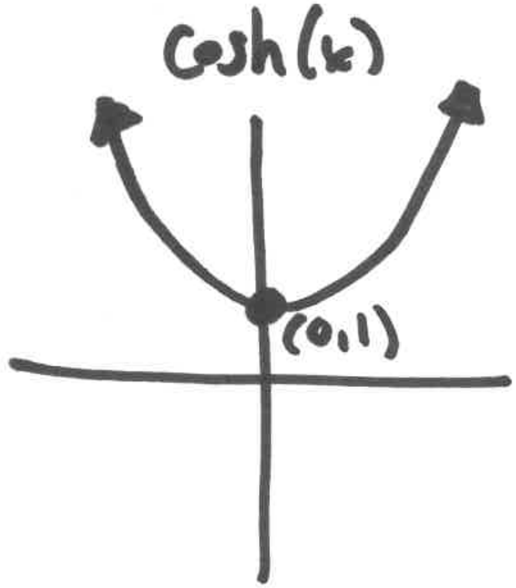
Definitions:

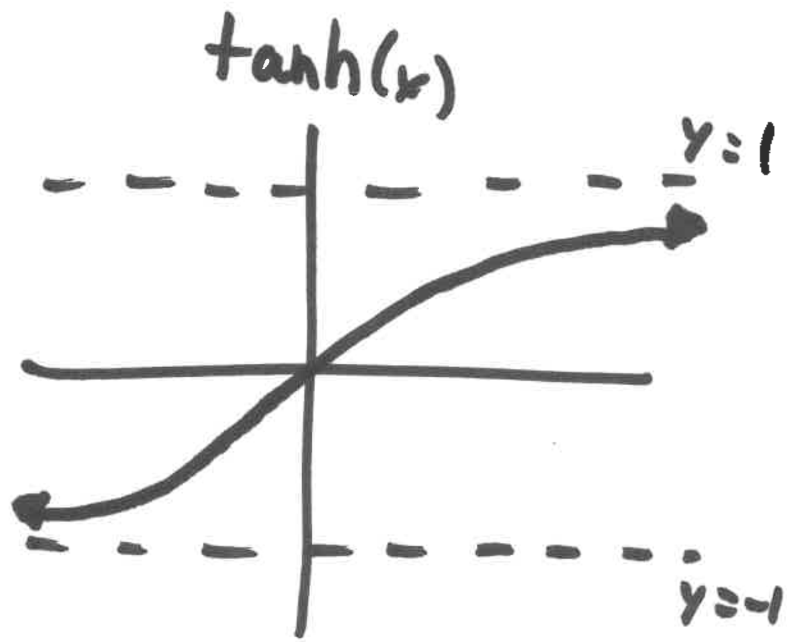
$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}$$


sinh(x)


cosh(x)
(0,1)


tanh(x)
y=1
y=-1

Some Hyperbolic Trig Identities:

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad 1 - \tanh^2(x) = \operatorname{sech}^2(x)$$

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

Q: Prove that $\cosh^2(x) - \sinh^2(x) = 1$.

A: $\cosh^2(x) - \sinh^2(x)$

$$= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \left[\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})\right]\left[\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})\right]$$

$$= e^x \cdot e^{-x}$$

$$= e^0$$

$$= 1.$$

Derivatives of Hyperbolic Functions:

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = \sinh(x)$$

$$\frac{d}{dx}\operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x) \qquad \frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$$

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x) \qquad \frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

Q: Prove that $\frac{d}{dx}\sinh(x) = \cosh(x)$.

A: $\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left[\frac{1}{2}\left(e^{x} - e^{-x}\right)\right]$

$= \frac{1}{2}\left[\left(\frac{d}{dx}e^{x}\right) - \left(\frac{d}{dx}e^{-x}\right)\right]$

$= \frac{1}{2}\left[e^{x} + e^{-x}\right]$

$= \cosh(x)$

by definition.

Inverse hyperbolic functions:

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in \mathbb{R}$$

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right), \quad x \geq 1$$

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad -1 < x < 1$$

Q: Prove that $\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$, for $x \geq 1$.

A: Let $y = \cosh^{-1}(x)$. $\Rightarrow \cosh(y) = x$.

$$\Rightarrow \frac{1}{2}\left[e^{y}+e^{-y}\right] = x$$

$$\Rightarrow e^{y}+e^{-y} = 2x$$

$$\Rightarrow e^{y}e^{y}+e^{y}e^{-y} = e^{y}\cdot 2x$$

$$\Rightarrow e^{2y}+e^{0}-2xe^{y} = 0$$

$$\Rightarrow e^{2y}-2xe^{y}+1 = 0$$

Solving for y in terms of x?

Let $z \equiv e^y$. Then:

$$z^2 - 2xz + 1 = 0$$

$$\Rightarrow z = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2-1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2-1}}{2}$$

$$= x \pm \sqrt{x^2-1}$$

So $\quad z = x \pm \sqrt{x^2-1}$.

$$\Rightarrow \quad e^y = x \pm \sqrt{x^2-1}$$

Note that $y \in \mathbb{R}$, but that if we pick $x - \sqrt{x^2-1}$, since $x \geq 1$, this function decreases monotonically from $x=1$ for increasing x, so we must pick

$$e^y = x + \sqrt{x^2-1},$$

$$\Rightarrow \quad y = \ln\left(x + \sqrt{x^2-1}\right).$$

Derivatives of inverse hyperbolic functions:

$$\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\operatorname{csch}^{-1}(x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\operatorname{sech}^{-1}(x) = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2}$$

Q: Prove that $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$.

A: Since $\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1<x<1$,

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right]$$

$$= \frac{1}{2}\cdot\frac{d}{dx}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{d}{dx}\left[\frac{1+x}{1-x}\right]$$

$$= \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{1\cdot(1-x)-(-1)(1+x)}{(1-x)^2}$$

$$= \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2} = \frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2}.$$

Q: Prove that $\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$.

A: $\frac{d}{dx}\cosh^{-1}(x) = \frac{d}{dx}\ln\left(x+\sqrt{x^2-1}\right)$

$$= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{d}{dx}\left[x + (x^2-1)^{\frac{1}{2}}\right]$$

$$= \frac{1}{x+\sqrt{x^2-1}} \cdot \left[1 + \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \cdot 2x\right]$$

$$= \frac{1}{x+\sqrt{x^2-1}} + \frac{x}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}$$

$$= \frac{x+\sqrt{x^2-1}}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}.$$

MATH 3

Lecture #16

10/16/23

Jonathan Lindbloom

Maxima

+

Minima

Def: We say that $c \in D$ is the absolute maximizer (minimizer) of $f(x)$ on D if $\forall x \in D$, $f(c) \geq f(x)$ $\left(\forall x \in D, f(c) \leq f(x) \right)$

The corresponding value $f(c)$ is the absolute maximum value (minimum value) of $f(x)$ on D.

Def: We say that c is a local maximizer (minimizer) of $f(x)$ if $f(c) \geq f(x)$ ($f(c) \leq f(x)$) for x near c.

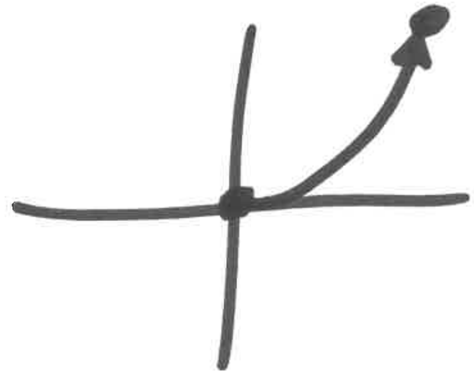
The corresponding value $f(c)$ is local maximum value (minimum value) of $f(x)$.

Q: Given a function $f: D \to \mathbb{R}$, how can we find the local/global maximum/minimum values of $f(x)$?

Q: How do we know whether/when these exist?

Extreme Value Theorem: If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum and absolute minimum at some values in $[a,b]$.

__Q:__ Does $f(x) = x^2$ attain a maximum and minimum value on $[0,1]$?



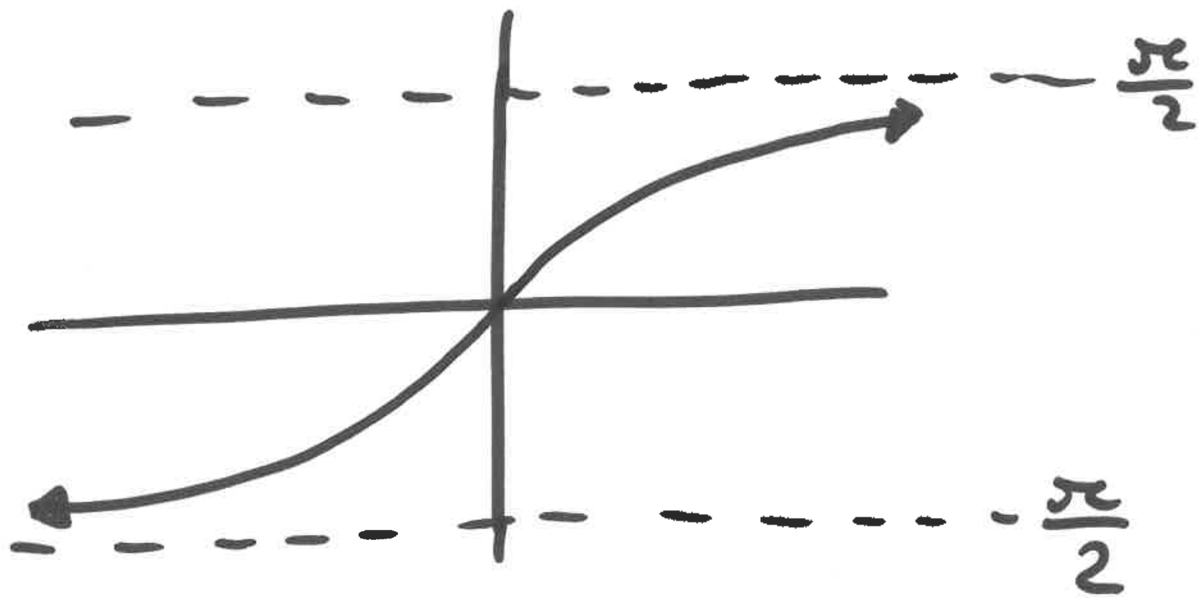
Yes!

__Q:__ Does $f(x) = \frac{1}{x}$ attain a maximum and minimum value on $(0,1]$?

Yes for minimum but __no__ for maximum!

Q: Does $f(x) = \arctan(x)$ attain a global maximum or minimum on $(-\infty, \infty) = \mathbb{R}$?

Neither!



Q: Does $f(x) = \begin{cases} -x^2+1, & x \neq 0 \\ 0, & x = 0 \end{cases}$ attain a global maximum or minimum on $[-1, 1]$?

Yes for minimum, no for maximum!

__Fermat's Theorem:__ If $f(x)$ has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

__Proof?__ (Case: local maximum) $f(x)$ having a local maximum at c $\Rightarrow$ $\exists$ an open interval (a,b) containing c such that $\forall x \in (a,b)$, $f(c) \geq f(x)$. Set $x = c+h$ for parameter h, then observe that

$$f(c+h) - f(c) \leq 0$$

$$\Rightarrow \quad \frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{for sufficiently small } h>0.$$

$$\Rightarrow \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\qquad\qquad\qquad\;\; \shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$\Rightarrow f'(c) \leq 0$$

If $h<0$ is sufficiently small, then we have

$$f(c+h) - f(c) \leq 0 \Rightarrow \frac{f(c+h) - f(c)}{h} \geq 0$$

$$\Rightarrow \lim_{h \to 0^+} \frac{f(c+h) - f(h)}{h} \geq 0$$

$$\shortparallel$$

$$f'(c)$$

$$\Rightarrow f'(c) \geq 0.$$

Together, these imply that $f'(c) = 0$.

CAUTION: Fermat's Theorem does <u>not</u> tell us that if $f'(c)=0$ that then c is either a local maximum or minimum of $f(x)$. It only tells us that

$$\left.\begin{array}{c} c \text{ is local} \\ \text{max/min} \\ + \\ f'(c) \text{ exists} \end{array}\right\} \Rightarrow f'(c)=0$$

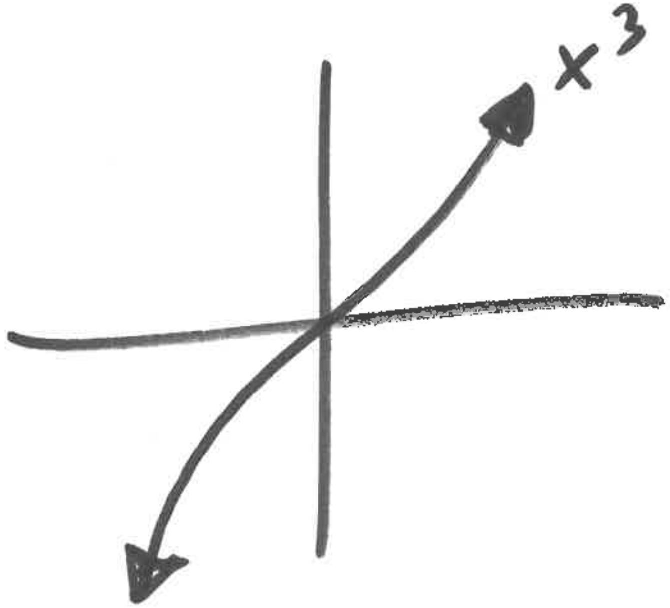
But if we are careful, we can still use it to help find local max/min.

Def: A critical number (point) of $f(x)$ is a number c in the domain of $f(x)$ such that either $f'(c) = 0$ or $f''(c)$ does not exist.

Fermat's Theorem says that if $f(x)$ has a local max/min at c, then c is a critical point of f.

How can $f'(c)=0$ but c not be a local max/min?

Consider $f(x)=x^3$. The only critical point is $c_1=0 \Rightarrow f'(c_1)=0$, but clearly this isn't a local max/min.



Q: Find all critical points of

$$f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2 + 7.$$

A: $f'(x) = x^3 - 2x^2 - 3x$.

$$f'(x) = 0 \Rightarrow x^3 - 2x^2 - 3x = 0$$

$$\Rightarrow x(x^2 - 2x - 3) = 0$$

$$\Rightarrow x(x-3)(x+1) = 0$$

$\Rightarrow$ $x = 0$, $x = -1$, $x = 3$

are all critical points.

Q: What are the critical points of

$f(x) = |7x - 9|$?

A: Thinking piece-wise, $f(x) = \begin{cases} 9-7x, & x \le \frac{9}{7}, \\ 7x-9, & x > \frac{9}{7}. \end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} -7, & x < \frac{9}{7} \\ \text{DNE}, & x = \frac{9}{7}, \\ 7, & x > \frac{9}{7}. \end{cases}$$

So $x = \frac{9}{7}$ is the only critical point of $f(x)$.

How to find global max/min?

If $f(x)$ is continuous on $[a,b]$, then we are lucky; there is a general procedure we can follow.

Closed Interval Method:

① Find all critical points $c_1, \ldots, c_n$.
Compute $f(c_1), \ldots, f(c_n)$.

② Find $f(a)$ and $f(b)$.

③ The global max/min are in the set $\{f(c_1), \ldots, f(c_n), f(a), f(b)\}$.

Q: What is the absolute max/min of $f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2 + 7$ on the interval $[-2, 4]$?

A: We already saw that the critical points of $f(x)$ were $c_1 = 0$, $c_2 = -1$, $c_3 = 3$.

The corresponding function values are $f(0) = 7$, $f(-1) = \frac{77}{12}$, $f(3) = -\frac{17}{4}$.

We also compute that $f(-2) = \frac{31}{3}$ and $f(4) = \frac{13}{3}$.

Let's make a table:

point	f(point)	
$a = -2$	$\frac{31}{3} \approx 10.33$	$\leftarrow$ absolute max
$c_1 = 0$	7	
$c_2 = -1$	$\frac{77}{12} \approx 6.42$	
$c_3 = 3$	$-\frac{17}{4} \approx -4.25$	$\leftarrow$ absolute min
$b = 4$	$\frac{13}{3} \approx 4.33$	

Q: Find the absolute max/min of $f(x) = x + \frac{1}{x}$ on $[0.1, 5]$.

A: First find critical points.

$$f'(x) = 1 + -x^{-2} = 1 - \frac{1}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 = \frac{1}{x^2} \Rightarrow x = \pm 1.$$

← only interested in $+1$

points	f(points)	
$a = 0.1$	10.1	← max
$c_1 = 1$	2	← min
$b = 5$	$\frac{26}{5} \approx 5.2$	

Q: Find the absolute max/min of $f(x) = e^x - 3x$ on $[0,2]$.

A: First find critical points.

$f'(x) = e^x - 3 \Rightarrow f'(x) = 0 \Leftrightarrow e^x - 3 = 0$

$\Rightarrow e^x = 3$

$\Rightarrow x = \ln(3)$.

points	f(point)	
$a = 0$	1	
$c_1 = \ln(3)$	$3 - 3\ln(3) \approx -0.30$	$\leftarrow$ min
$b = 2$	$e^2 - 6 \approx 1.39$	$\leftarrow$ max

Q: Find the absolute max/min of $f(x)=(x^2-4)^3$ on the interval $[-2,3]$.

A: First find the critical points.

$$f'(x) = 3(x^2-4)^2 \cdot 2x = 6x(x^2-4)^2$$
$$= 6x\left[(x+2)(x-2)\right]^2$$
$$= 6x(x+2)^2(x-2)^2.$$

So the critical points are

$c_1 = 0$, $c_2 = -2$, $c_3 = 2$.

We collect our candidates:

point	f(point)	
$a = -2$	0	
$c_1 = 0$	-64	← absolute min
$c_2 = -2$	0	
$c_3 = 2$	0	
$b = 3$	125	← absolute max

Q: Classify all critical points of $f(x) = \frac{1}{4}x^4 - x^3 + x^2 + 8$ as local max, local min, or neither.

A: $f'(x) = x^3 - 3x^2 + 2x = x(x^2 - 3x + 2)$
$= x(x-2)(x-1)$

$f'(x) = 0 \Rightarrow$ $x = 0$
$x = 1$
$x = 2$

How to check local max/min or neither?

$\underline{c_1 = 0}$

$f(-0.1) \approx 8.01$

$f(0.1) \approx 8.01$

$f(0) = 8$

$\Rightarrow c_1 = 0$ is a local min

$\underline{c_2 = 1}$

both ≈ 8.245

$f(0.9) \approx 8.25$

$f(1.1) \approx 8.25$

$f(1) = \frac{33}{4} \approx 8.25$

$\Rightarrow c_2 = 1$ is a local max

$\underline{c_3 = 2}$

$f(1.9) \approx 8.01$

$f(2.1) \approx 8.01$

$f(2) = 8$

$\Rightarrow c_3 = 2$ is a local min

Q: For $a, b > 0$, find the maximum value of $f(x) = x^a(1-x)^b$ on $[0,1]$.

A: First find critical points.

$$f'(x) = ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1}$$

$$f'(x) = 0 \Rightarrow \quad ax^{a-1}(1-x)^b = bx^a(1-x)^{b-1}$$

$$\Rightarrow ax^{-1}(1-x)^b = b(1-x)^{b-1}$$

$$\Rightarrow ax^{-1} = b(1-x)^{-1}$$

$$\Rightarrow \frac{x}{a} = \frac{1-x}{b}$$

$$\Rightarrow \frac{1}{a}x = \frac{1}{b} - \frac{1}{b}x$$

$$\Rightarrow \left(\frac{1}{a}+\frac{1}{b}\right)x = \frac{1}{b}$$

$$\Rightarrow x = \frac{\frac{1}{b}}{\left(\frac{1}{a}+\frac{1}{b}\right)} = \frac{a}{a+b}.$$

So $c_1 = \frac{a}{a+b}$ is the only critical point,

and $f(c_1) = \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b$.

Note that at the endpoints, $f(0)=0$ + $f(1)=0$.

$\therefore$ The global maximum of $f(x)$ on $[0,1]$

is $f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b$.

Q: Let $\sigma(x): \mathbb{R} \to \mathbb{R}$ denote the sigmoid function given by $\sigma(x) = \frac{1}{1+e^{-x}}$. What is the maximum value of $\sigma'(x)$ on $\mathbb{R}$? How can you justify this?

A: We compute that $\sigma'(x) = \dfrac{e^{-x}}{(1+e^{-x})^2}$,

$$\sigma''(x) = \frac{2e^{-2x}}{(1+e^{-x})^3} - \frac{e^{-x}}{(1+e^{-x})^2}.$$

$$\sigma''(x) = 0 \implies 2e^{-2x} - e^{-x}(1+e^{-x}) = 0$$

$$\implies 2e^{-2x} - e^{-2x} - e^{-x} = 0$$

$$\Rightarrow e^{-2x} - e^{-x} = 0 \quad \Rightarrow \quad e^{-2x} = e^{-x}$$

$$\Rightarrow -2x = x \quad \Rightarrow \quad x = 0.$$

So $c_1 = 0$ is the only critical point of $\sigma'(x)$.

But how to find absolute max on unbounded interval $\mathbb{R}$?

Where is $\sigma'(x)$ increasing? ($\sigma''(x) > 0$)

$$\frac{2e^{-2x}}{(1+e^{-x})^3} - \frac{e^{-x}}{(1+e^{-x})^2} > 0 \Rightarrow 2e^{-2x} - e^{-2x} - e^{-x} > 0$$

$$\Rightarrow e^{-2x} > e^{-x} \Rightarrow -2x > -x \Rightarrow -x > 0$$

$$\Rightarrow x < 0.$$

Similarly, where is $\sigma'(x)$ decreasing?

When $x > 0$.

Since $\sigma'(x)$ is increasing for $x<0$, $x=0$ is a critical point, and $\sigma'(x)$ decreases for $x>0$, $\sigma'(0) = \frac{1}{4}$ must be the global max.

Q: Prove that $f(x) = x^{101} + x^{51} + x + 1$ has no local max/min.

A: $f'(x) = 101x^{100} + 51x^{50} + 1$.

$f'(x) = 0$ has any solutions?

Let $z = x^{50}$.

$\rightarrow \quad 101z^2 + 51z + 1 = 0$.

Discriminant $\Delta = b^2 - 4ac = (51)^2 - 4(101)$

$= 2197 > 0$,

$\Rightarrow$ 2 real roots

$\Rightarrow$ $f(x)$ has two critical points.

But note also that $f'(x) \geq 0$.

So the two critical points cannot be local max/min, since otherwise we would have to have $f'(x) < 0$ somewhere.

$\therefore$ $f(x) = x^{101} + x^{51} + x + 1$

has no local max/min.

MATH 3

Lecture #17

10/18/23

Jonathan Lindbloom

Review of Theorems

Extreme Value Theorem: If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum and absolute minimum at some values in $[a,b]$.

Fermat's Theorem: If $f(x)$ has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Closed Interval Method: Suppose $f(x)$ is continuous on $[a,b]$. Then by the EVT the max/min of $f(x)$ on $[a,b]$ is achieved and we can find them by checking:

→ The endpoints

→ Points where $f'(x)$ exists and equals zero

→ Points where $f'(x)$ does not exist

critical points!

Q: Find the absolute max/min of the following function $f(x)$ on $[-5,5]$.

$$f(x) = \begin{cases} \frac{1}{20}x^3 + \frac{3}{10}x^2 + \frac{11}{20}x + \frac{3}{10}, & x < 3, \\ 9 - |x| & , x \geq 3. \end{cases}$$

A: First, let's find all points when $f'(x) = 0$. for $x<3$, $f'(x) = \frac{3}{20}x^2 + \frac{6}{10}x + \frac{11}{20} = 0$

$\Rightarrow 3x^2 + 12x + 11 = 0 \Rightarrow x = -2 \pm \frac{\sqrt{3}}{3}$.

For $x > 3$, $f'(x)$ is never zero!

Evaluate our candidates:

point	f(point)
$a = -5$	-1.2 ⟵ min
$c_1 = -2 - \frac{\sqrt{3}}{3}$	≈ 0.0192
$c_2 = -2 + \frac{\sqrt{3}}{3}$	≈ -0.0192
point where $f'(x)$ DNE! → $c_3 = 3$	6 ⟵ max
$b = 5$	4

Mean Value Theorem

Rolle's Theorem: Let $f(x)$ be a function satisfying:

(1) $f(x)$ is continuous on $[a,b]$.

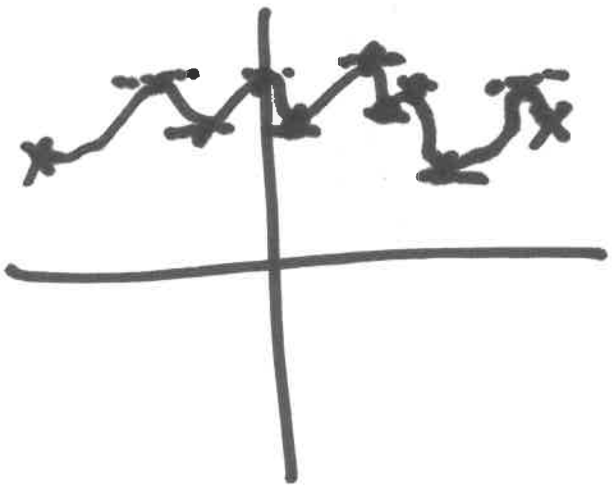
(2) $f'(x)$ exists on (a,b).

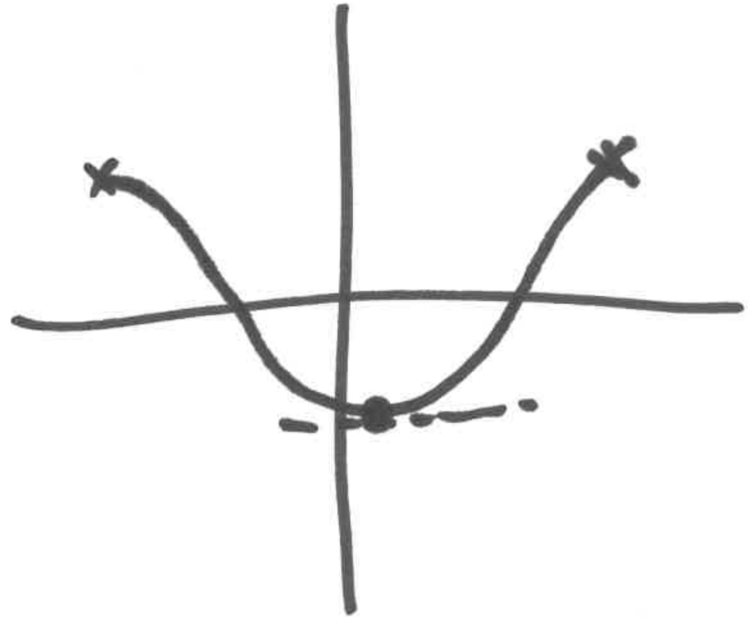
(3) $f(a)=f(b)$.

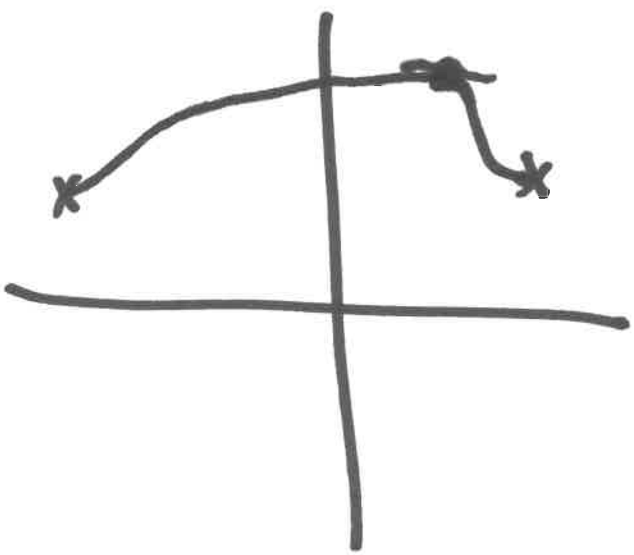
Then $\exists\, c \in (a,b)$ such that $f'(c)=0$.

In English? If $f(x)$ is continuous and differentiable, then $f(x)$ has to "roll" around if it ends where it begins.

Examples:







Proof? Break it into cases.

$\boxed{1}$ ($f(x)=$ constant) If $f(x)=k$ on (a,b), then pick any point $c\in(a,b)$. $\Rightarrow f'(c)=0$.

$\boxed{2}$ $\left(\begin{array}{l}\exists x\in(a,b)\\ \text{s.t. } f(x)>f(a)\end{array}\right)$ By the EVT, $f(x)$ achieves its maximum somewhere on $[a,b]$. Since $f(a)=f(b)$, the max must occur at some $c\in(a,b)$. $\Rightarrow$ By Fermat's Theorem that $f'(c)=0$.

$\boxed{3}$ $\left(\begin{array}{l}\exists x\in(a,b)\\ \text{s.t. } f(x)<f(a)\end{array}\right)$ By the EVT, $f(x)$ achieves its minimum somewhere on $[a,b]$. Since $f(a)=f(b)$, the min must occur at some $c\in(a,b)$.
$\Rightarrow$ By Fermat's Theorem that $f'(c)=0$.

Q: Prove that $\underbrace{x^3+x-1}_{f(x)}$ has only 1 real root.

A: First we show there is at least one real root. We note that $f(0)=-1$ and $f(1)=1+1-1=1$, so by the IVT there must be a real root r_1 of $f(x)$ in $(0,1)$.

Why is this the only real root? $f'(x)=3x^2+1>0$ for all x, so $f(x)$ is strictly increasing.

Suppose $\exists\, r_2$ s.t. $r_2 \neq r_1$ + $f(r_2)=0$. Then $f(r_1)=f(r_2)$, and by Rolle's Thm $\exists\, c \in (r_1, r_2)$ s.t. $f'(c)=0$. But $f'(x)>0$!

Mean Value Theorem: Let $f(x)$ be a function satisfying:

(1) $f(x)$ is continuous on $[a,b]$.

(2) $f'(x)$ exists on (a,b).

Then $\exists c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

or, equivalently,

$$f(b)-f(a) = f'(c)(b-a).$$

In English? MVT states

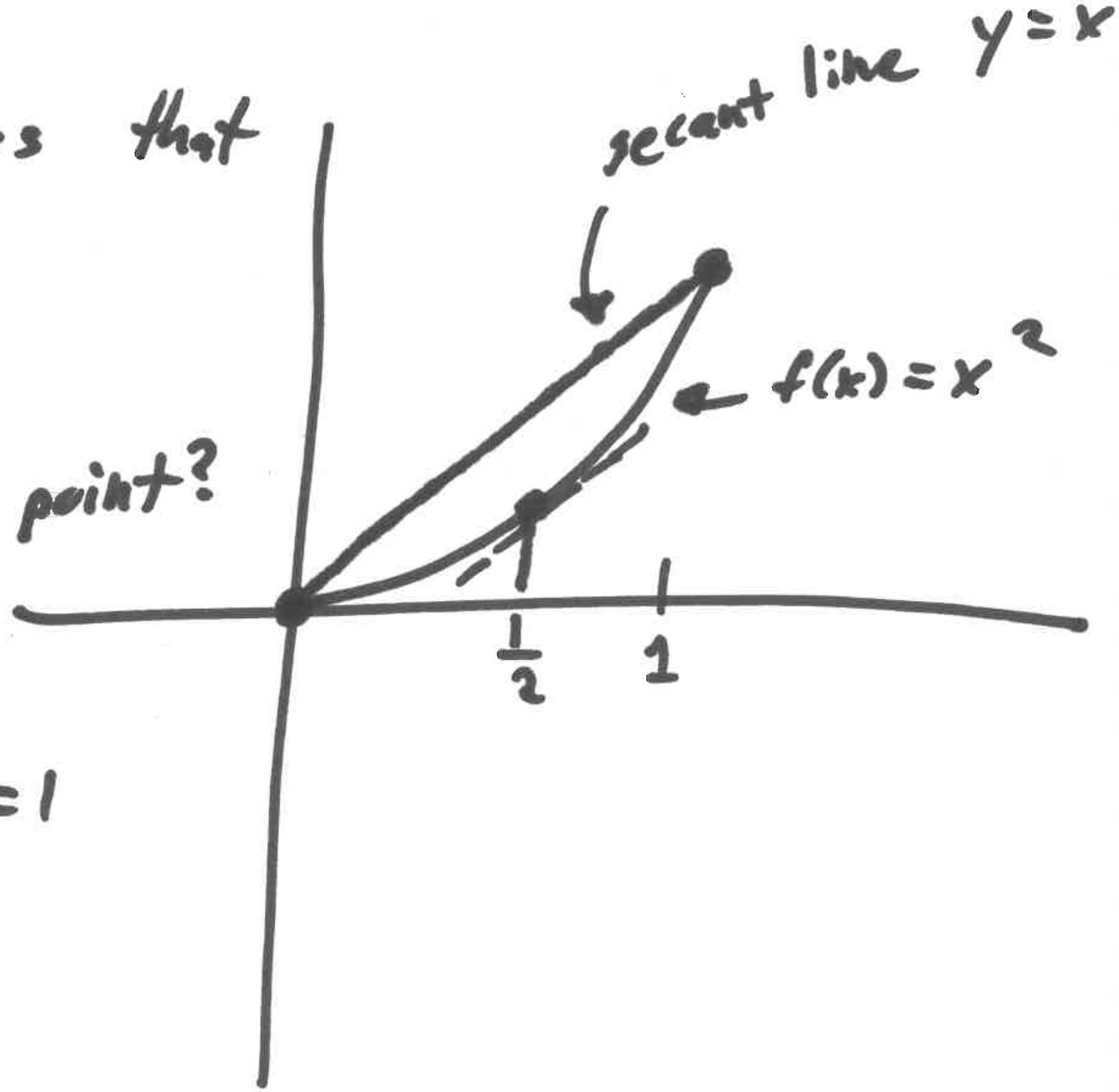
that $f(x)$ continuous + differentiable
$[a,b]$ (a,b)

$\Rightarrow$ the tangent line has slope
equal to the slope of the secant
line between endpoints at some
point $c \in (a,b)$.

Ex: $f(x) = x^2$ on $[0, 1]$.

MVT guarantees that $\exists\, c \in (0,1)$ s.t. $f'(c) = 1$.

Where is this point?

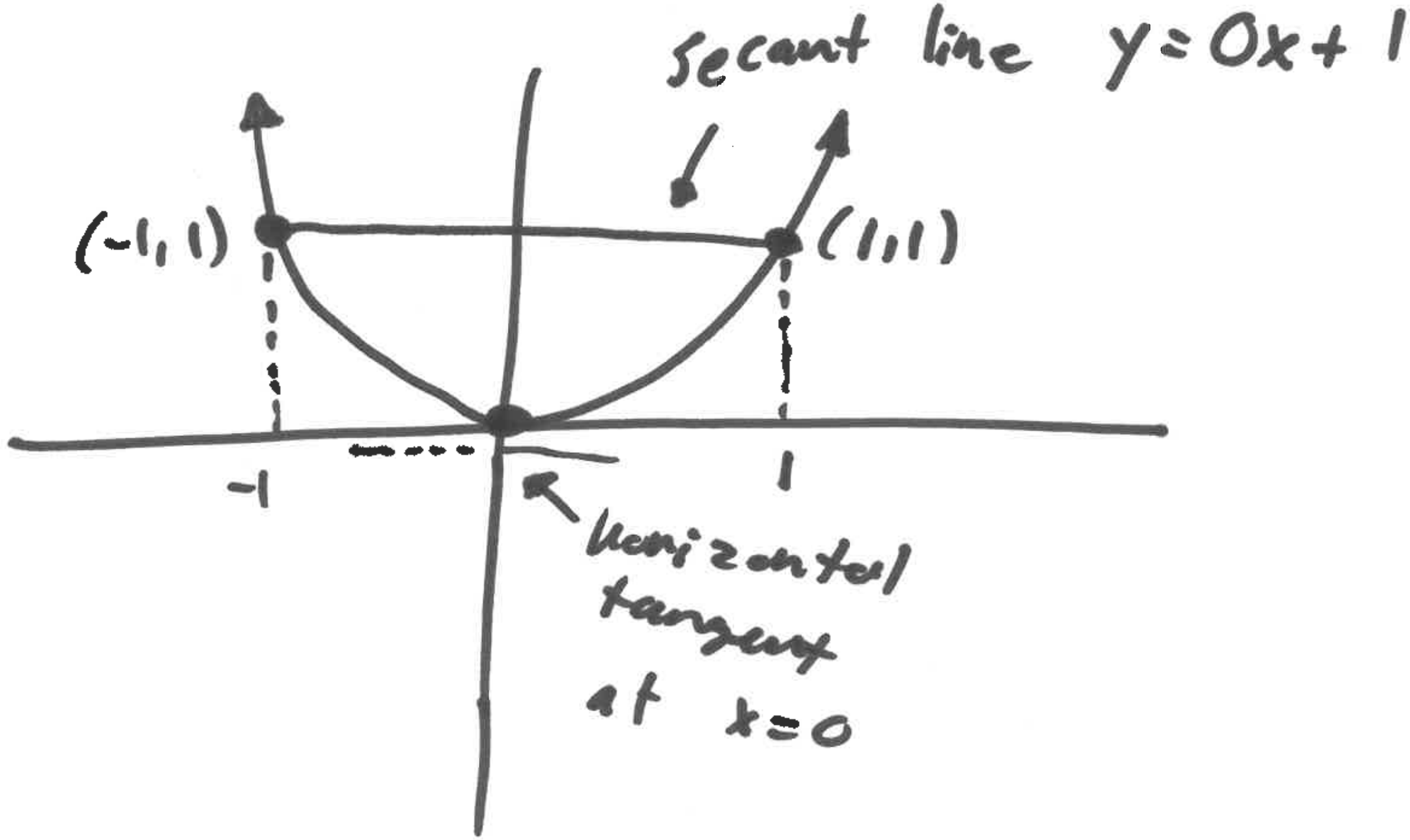


$f'(x) = 2x$,

$f'(x) = 1 \Rightarrow 2x = 1$

$\Rightarrow x = \frac{1}{2}$

Ex: $f(x) = x^2$ on $[-1, 1]$.



Some facts proven using the MVT:

$\boxed{1}$ If $f'(x)=0$ on (a,b), then $f(x)$ constant on (a,b).

$\boxed{2}$ If $f'(x)=g'(x)$ $\forall x \in (a,b)$, then $f-g$ is constant on (a,b).

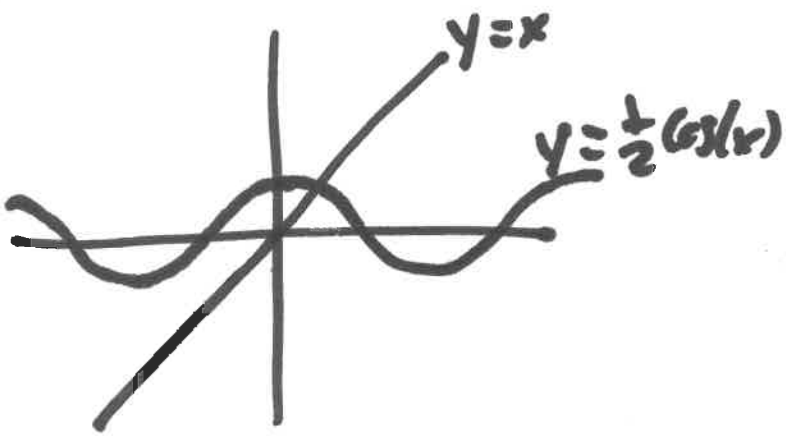
$$\Downarrow$$

$f(x)=g(x)+c$ on (a,b).

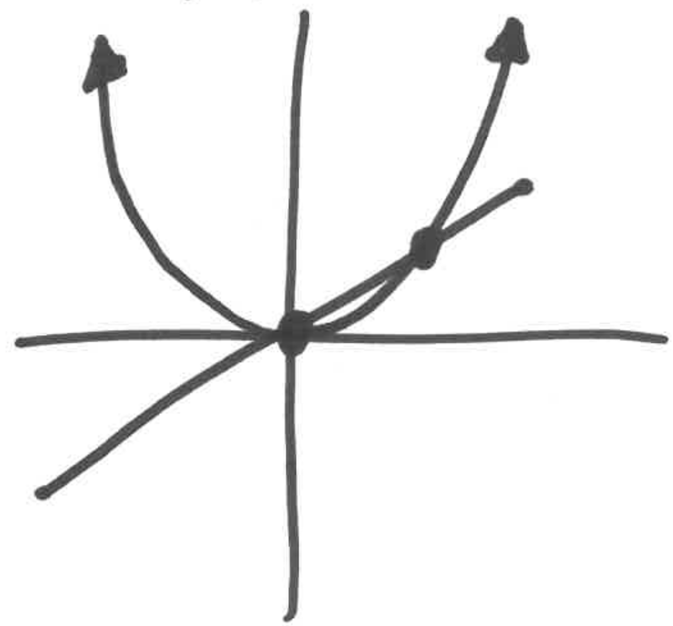
Q: a is a fixed point of $f(x)$ if $f(a)=a$. Prove that if $f'(x)\neq 1$ for all $x\in\mathbb{R}$, then $f(x)$ has at most one fixed point.

A: Let's think of some examples before proving this.

$$f(x)=\frac{1}{2}\cos(x) \rightarrow f' = -\frac{1}{2}\sin(x)$$



$$f(x)=x^2 \rightarrow f'(x)=2x$$



* Recall that $f(a)=a \iff \underbrace{f(a)-a}_{:=g(a)}=0$

So condition that $f'(x)\neq 1$ for any x is same as $g'(x)\neq 0$ for any x.

$= f'(x)-1$

Suppose that $g'(x)\neq 0$ for any x, and there are (at least) two distinct fixed points a_1 + a_2 of $f(x)$.

$\Rightarrow g(a_1)=g(a_2)=0$. But then by Rolle's Thm applied to (a_1, a_2), $\exists\, c\in(a_1,a_2)$ such that $g'(c)=0$, a contradiction.

$\therefore$ There can only be at most one fixed point of f.

Derivatives and the Shapes of Graphs

Question for this lecture:

What can you say about $f(x)$ given access to $f'(x)$?

Increasing / Decreasing Test:

Suppose $f(x)$ is continuous on $[a,b]$ and $f'(x)$ exists on (a,b). Then:

$\boxed{1}$ $\forall x \in (a,b)$ $f'(x) > 0 \Rightarrow f$ increasing on $[a,b]$.

$\boxed{2}$ $\forall x \in (a,b)$ $f'(x) < 0 \Rightarrow f$ decreasing on $[a,b]$.

Proof of $\boxed{1}$? We want to show that $f'(x)>0$ on (a,b) $\Rightarrow$ $f(x)$ is increasing on (a,b), i.e., that $x_1<x_2 \Rightarrow f(x_1)<f(x_2)$ for $x_1, x_2 \in [a,b]$.

Idea: apply MVT to the interval $[x_1, x_2]$.

By the MVT (note hypotheses are satisfied!), we know that $\exists c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1},$$

$$\Rightarrow f(x_2)-f(x_1) = f'(c)(x_2-x_1).$$

Finally, note that $f'(c)>0$ by assumption and $x_2-x_1>0$, so $f(x_2)-f(x_1)>0$!

Proof of $\boxed{2}$? Essentially mirrored from the previous. We want to show that $f'(x) < 0$ on $(a,b) \Rightarrow f(x)$ is decreasing on (a,b), i.e., that $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for $x_1, x_2 \in [a,b]$.

Applying MVT to interval $[x_1, x_2]$, $\exists c \in (x_1, x_2)$ such that

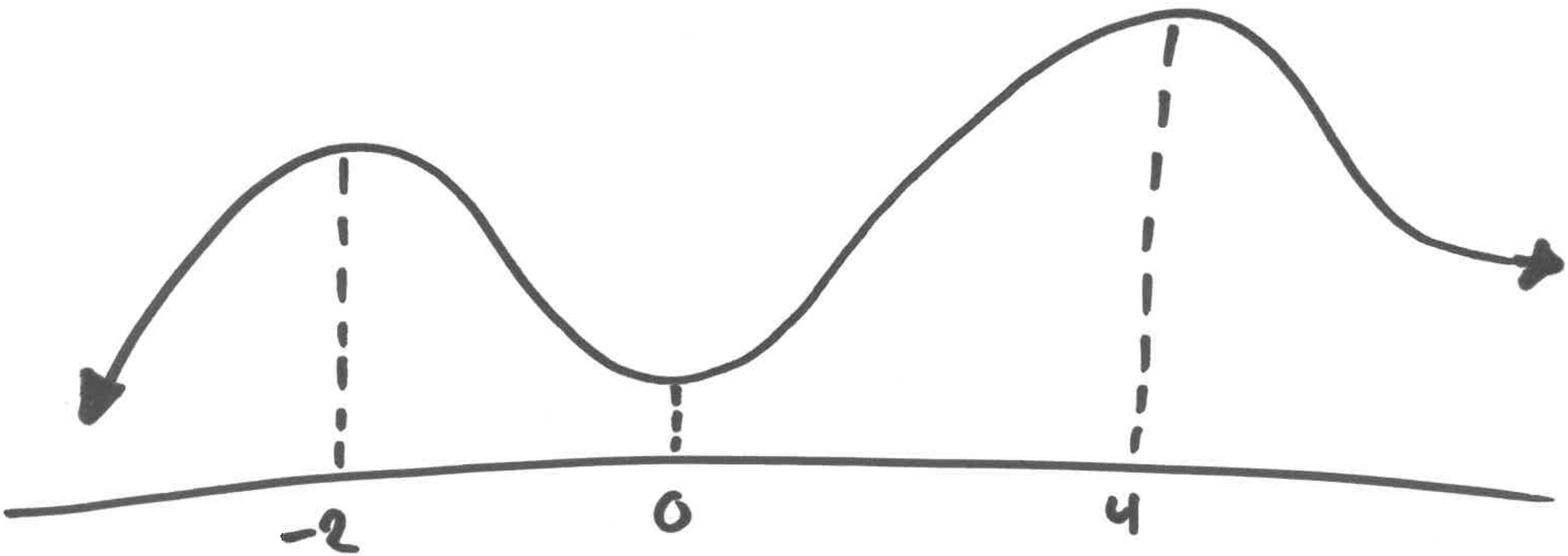
$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

$$\Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Note $f'(c) < 0$ + $x_2 - x_1 > 0$, so $f(x_2) - f(x_1)$ must be negative, i.e., $f(x_2) - f(x_1) < 0$.

$$\Rightarrow f(x_2) < f(x_1), \Rightarrow f(x_1) > f(x_2).$$

Where is this function increasing / decreasing?



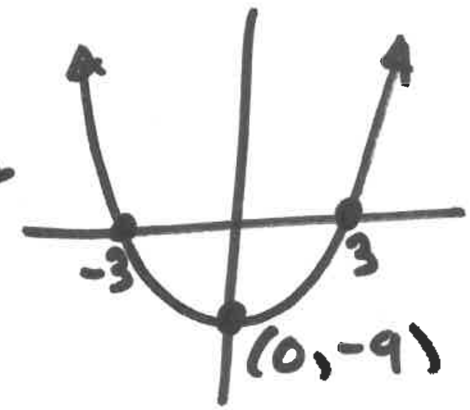
$$S_{inc} = (-\infty, -2] \cup [0, 4]$$

$$S_{dec} = [-2, 0] \cup [4, \infty)$$

Q: Find the sets on which $f(x) = \frac{1}{3}x^3 - 9x$ is increasing / decreasing.

A: Look at the derivative!

$f'(x) = x^2 - 9 = (x-3)(x+3)$. →



From the graph of $f'(x)$, we can see that $f(x)$ is increasing on

$$S_{inc} = (-\infty, -3] \cup [3, \infty)$$

and decreasing on

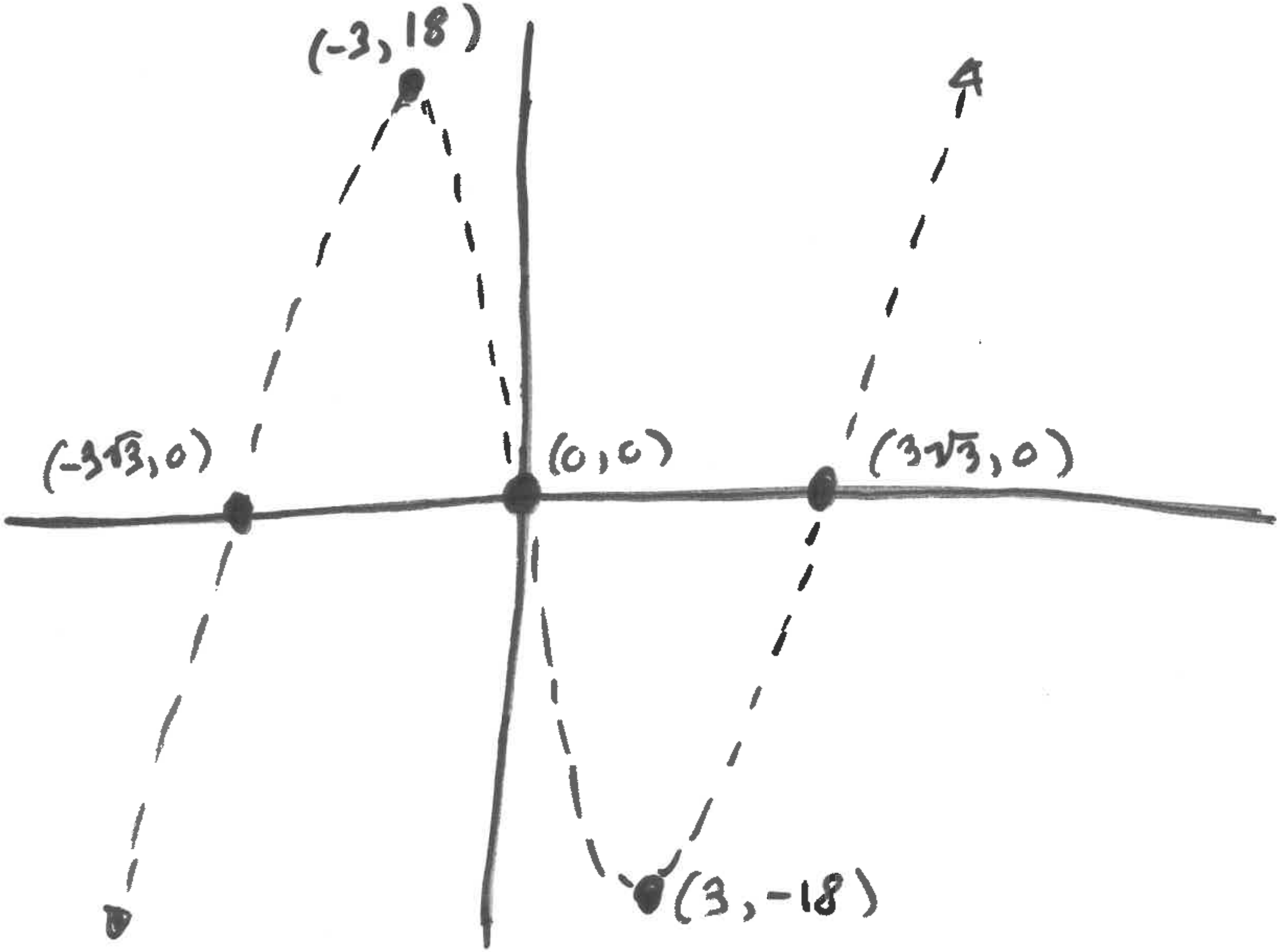
$$S_{dec} = [-3, 3].$$

Q: Can you try to graph $f(x) = \frac{1}{3}x^3 - 9x$ as accurately as possible, without using a calculator and without evaluating $f(x)$ at many points? What is your thought process?

Some helpful information to have:

- → roots of $f(x)$
- → behavior at $x \to \pm\infty$
- → x & y intercepts
- → local max & min
- → Increasing set and decreasing set!

• $f'(x) = x^2 - 9$, $f'(x) = 0 \Rightarrow x = \pm 3$. $f(-3) = 18$, $f(3) = -18$.



• roots are $x = 0$, $x = \pm 3\sqrt{3}$ • y-intercept at $(0,0)$

• $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to +\infty} f(x) = +\infty$

MATH 3

Lecture #18

10/20/23

Jonathan Lindbloom

Review of last lecture's theorems:

MVT: Let $f(x)$ satisfy

1. $f(x)$ is continuous on $[a,b]$.
2. $f'(x)$ exists on (a,b).

Then $\exists c \in (a,b)$ such that $f'(c) = 0$.

Rolle's Thm as special case? If also $f(a) = f(b)$,

then

Last lecture we ended with the increasing/decreasing test. Let's prove the first statement,

$\boxed{1}$ $\forall x \in (a,b)\ f'(x) > 0 \Rightarrow f(x)$ increasing on $[a,b]$.

<u>Proof?</u> We want to show that $f'(x) > 0$ on (a,b) $\Rightarrow f(x)$ increasing on $[a,b]$, i.e., that

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall x_1, x_2 \in [a,b]$.

☆ Ideas? What thm can we use?

Let's apply MVT to the interval $[x_1, x_2]$, where $x_1, x_2 \in [a,b]$ are arbitrary. $(x_2 > x_1)$

By the MVT (note hypotheses are satisfied!) we know that $\exists c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

$$\Rightarrow f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

But $x_2 > x_1 \Rightarrow x_2 - x_1 > 0$, and by assumption $f'(c) > 0$, so we must have $f(x_2) - f(x_1) > 0$.
$\Rightarrow f(x_2) > f(x_1)$. Thus $f(x)$ must be increasing on $[a,b]$.

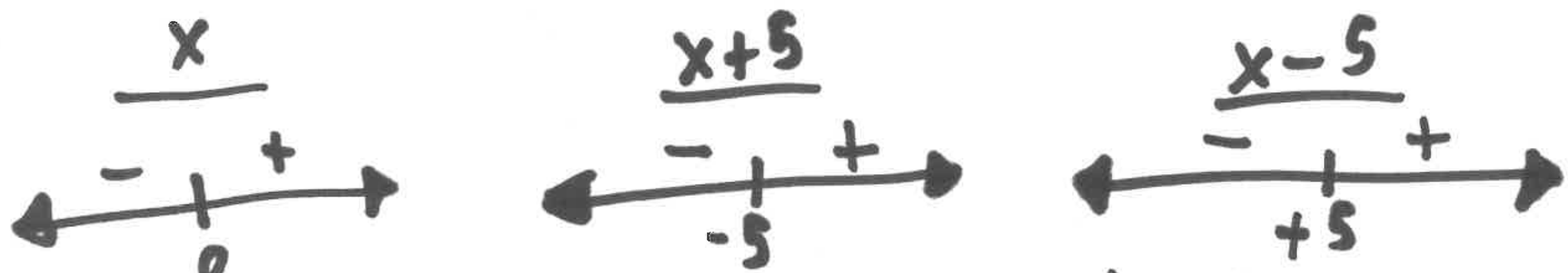
Q: Where is the function $f(x) = \frac{1}{4}x^4 - \frac{25}{2}x^2$ increasing? Decreasing?

A: $f'(x) = x^3 - 25x = x(x^2 - 25)$

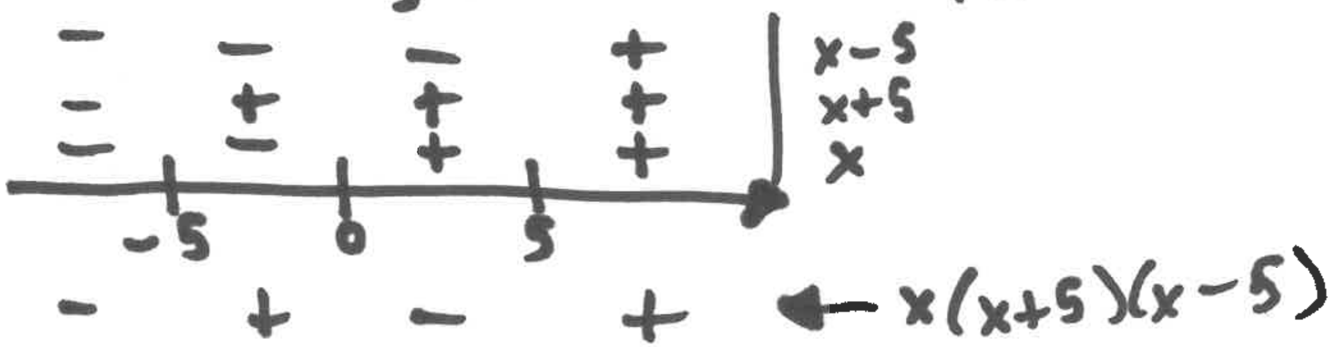
$= x(x+5)(x-5)$.

When is $f'(x)$ positive or negative?

Idea: draw some number lines.

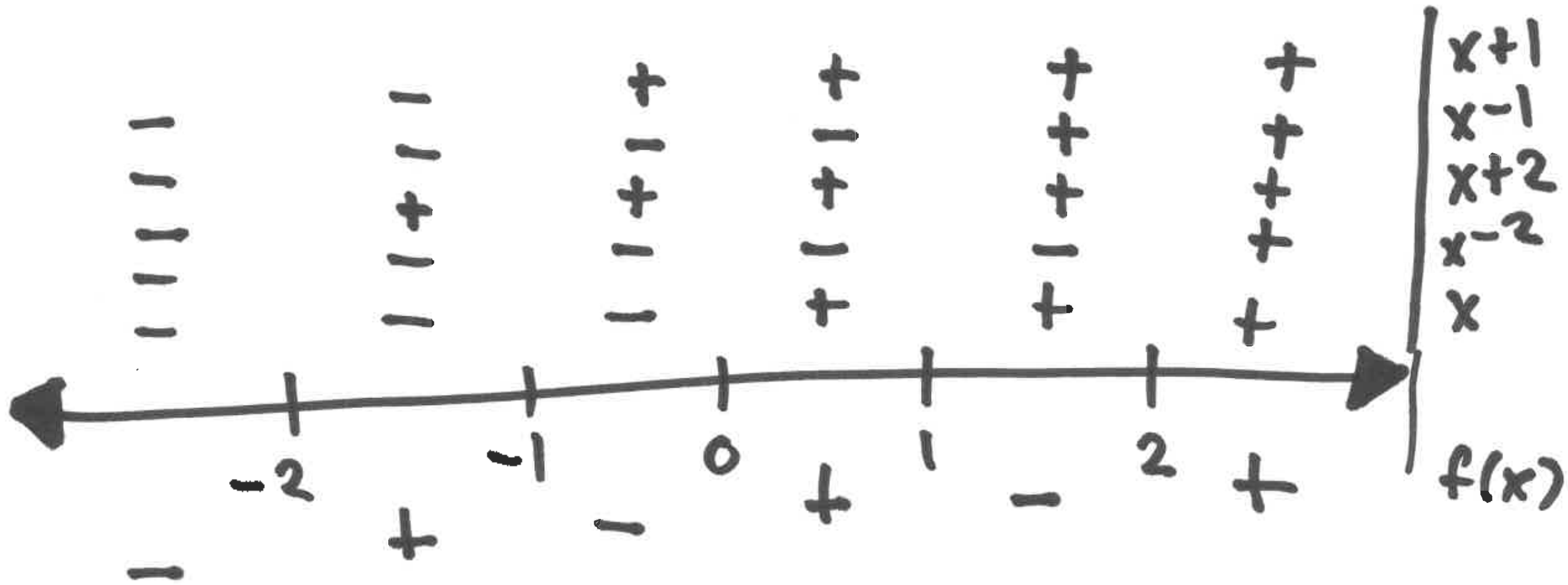


Overlay?



Q: Where is the function $f(x) = \frac{1}{6}x^6 - \frac{5}{4}x^4 + 2x^2$ increasing? Decreasing?

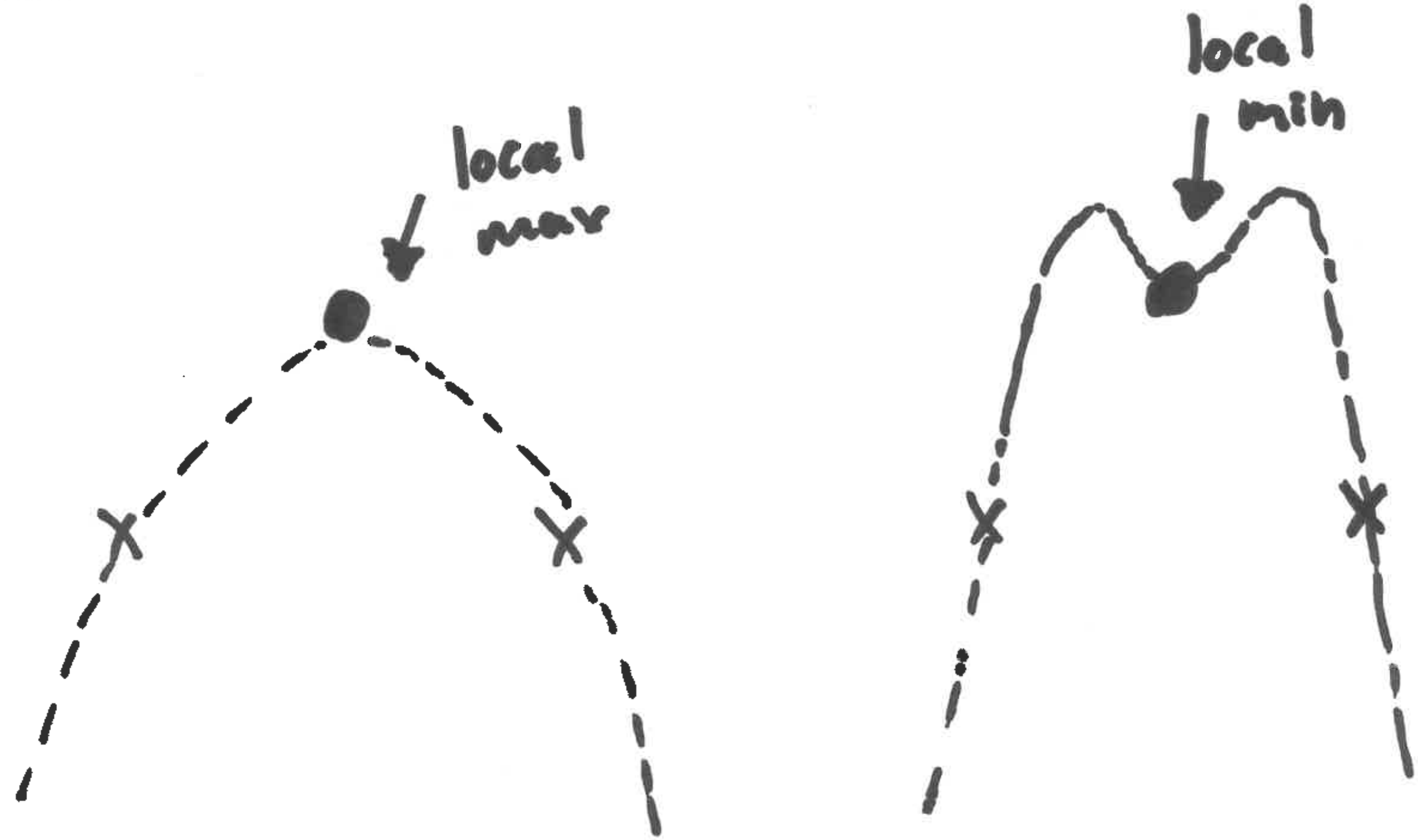
A:
$$
\begin{aligned}
f'(x) &= x^5 - 5x^3 + 4x \\
&= x(x^4 - 5x^2 + 4) \\
&= x(x^2-4)(x^2-1) \\
&= x(x-2)(x+2)(x-1)(x+1)
\end{aligned}
$$


x+1
x−1
x+2
x−2
x
−2
−1
0
1
2
f(x)

Guiding question for this lecture:

How can we more effectively determine if a critical point is a local max/min (or neither)? Compared to just plugging in points near the critical points?

Caution: What can go wrong with just plugging in points near the critical point?



We have no good way to distinguish these two cases!

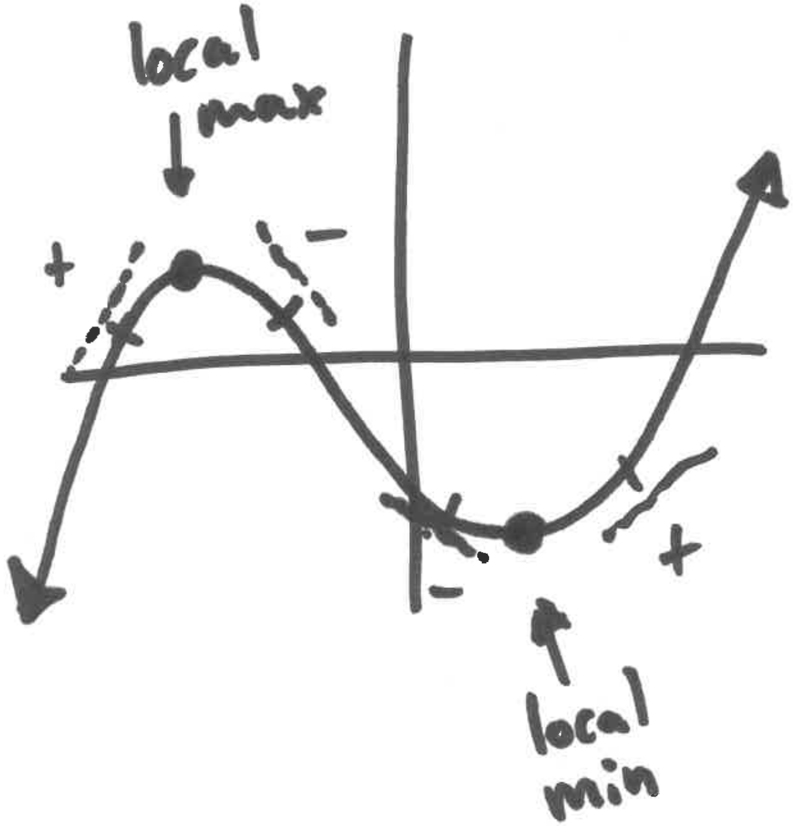
1st tool for determining local max/min

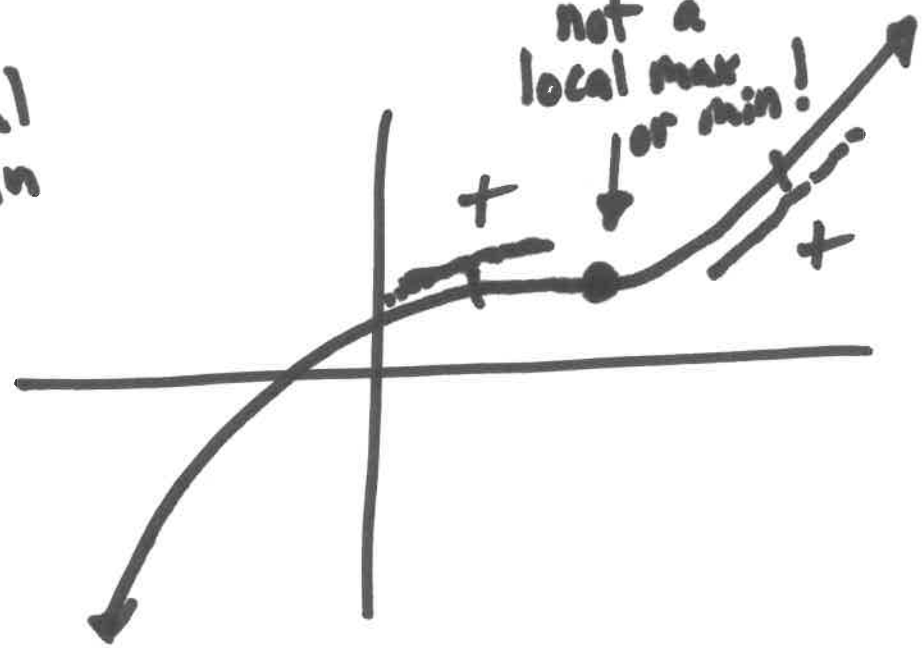
1st Derivative Test: Suppose c is a critical point of $f(x)$.

$\boxed{1}$ If f' goes $+ \to -$ as you cross c, then c is a local maximum

$\boxed{2}$ If f' goes $- \to +$ as you cross c, then c is a local minimum

$\boxed{3}$ If sign of f' doesn't change as you cross c, c is not a local max/min.


local
max
+
-
-
+
local
min


not a
local max
or min!
+
+

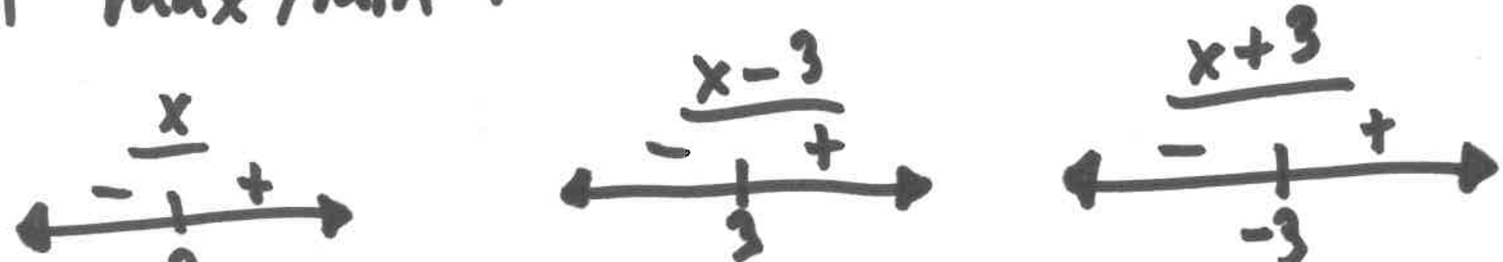
Q: Find the critical points of $f(x) = \frac{1}{4}x^4 - \frac{9}{2}x^2 + 7$ and classify them as local max/min.

A: First, find critical points. $f'(x) = x^3 - 9x$

$$= x(x^2 - 9)$$

$$= x(x-3)(x+3).$$

So the critical points are $c_1 = 0$, $c_2 = -3$, $c_3 = 3$. How to tell if local max/min?



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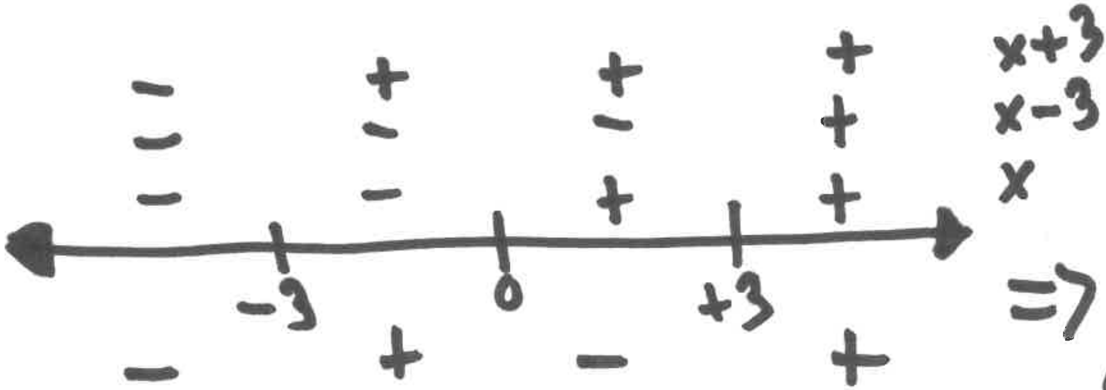


$\Rightarrow$ $c_1 = 0$ is max
$c_2 = -3$ is min
$c_3 = 3$ is min

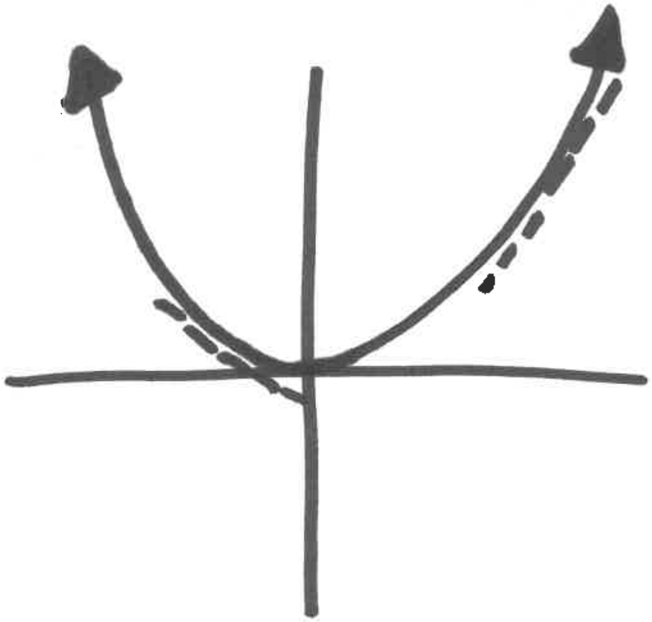
2nd tool for determining local max/min

Def:

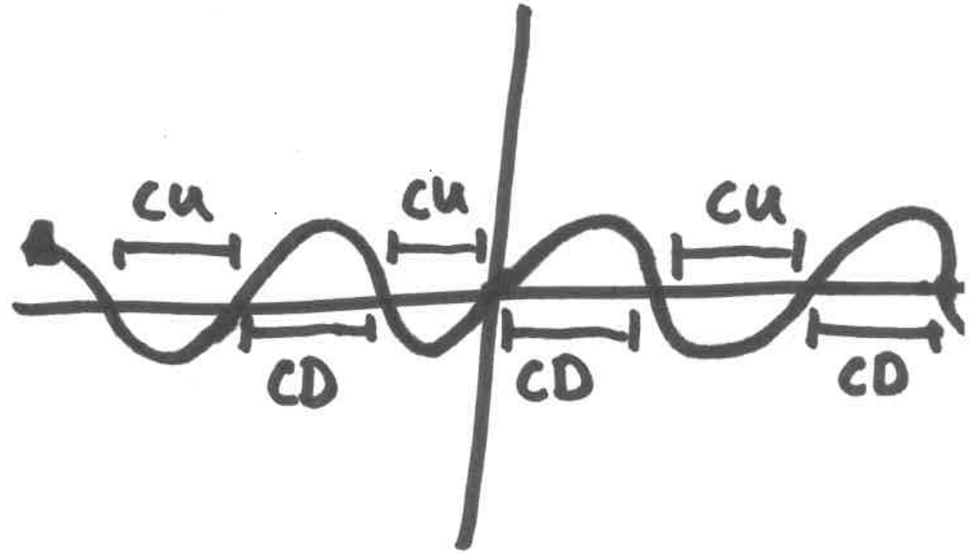
We say that $f(x)$ is concave upward on an interval I if the graph of $f(x)$ lies above all of its tangent lines on I. (also called convex)

We say that $f(x)$ is concave downward if its graph lies below all of its tangent lines on I.

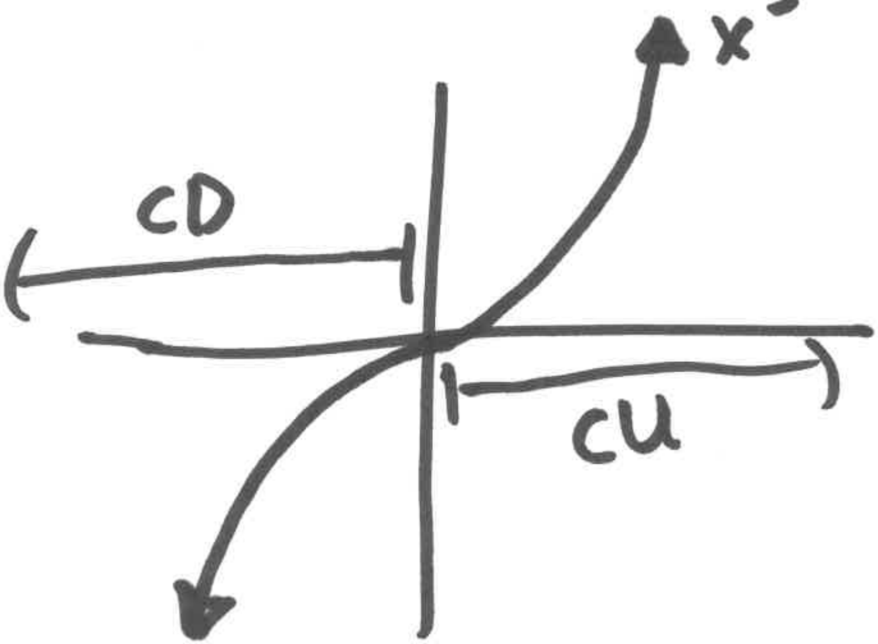
<u>Def</u>: A point P on a curve $y = f(x)$ is an inflection point if $f(x)$ is continuous at P and changes concavity there (CU $\rightarrow$ CD or CD $\rightarrow$ CU).

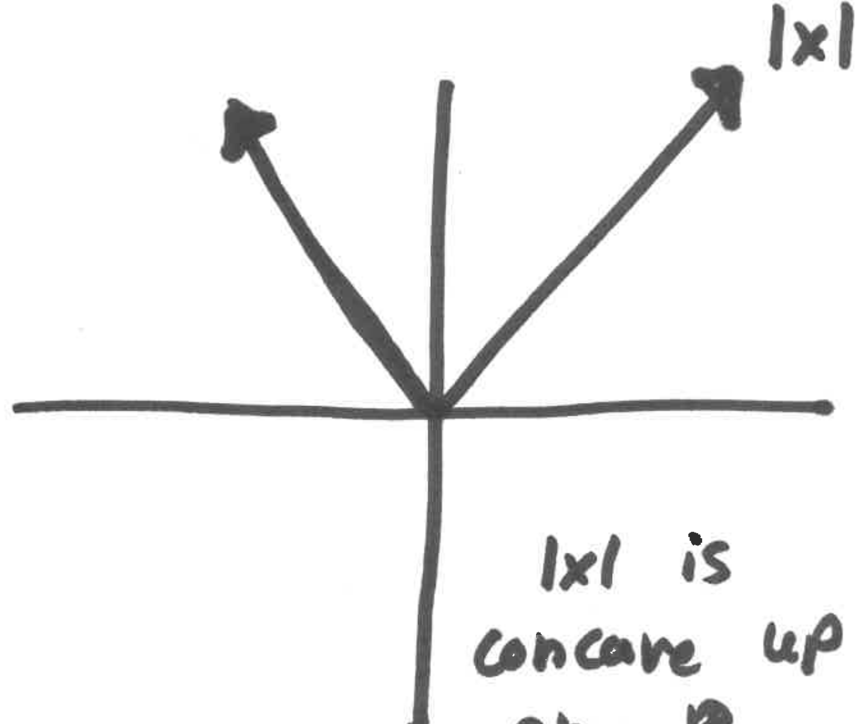


x^2 is concave up on $\mathbb{R}$



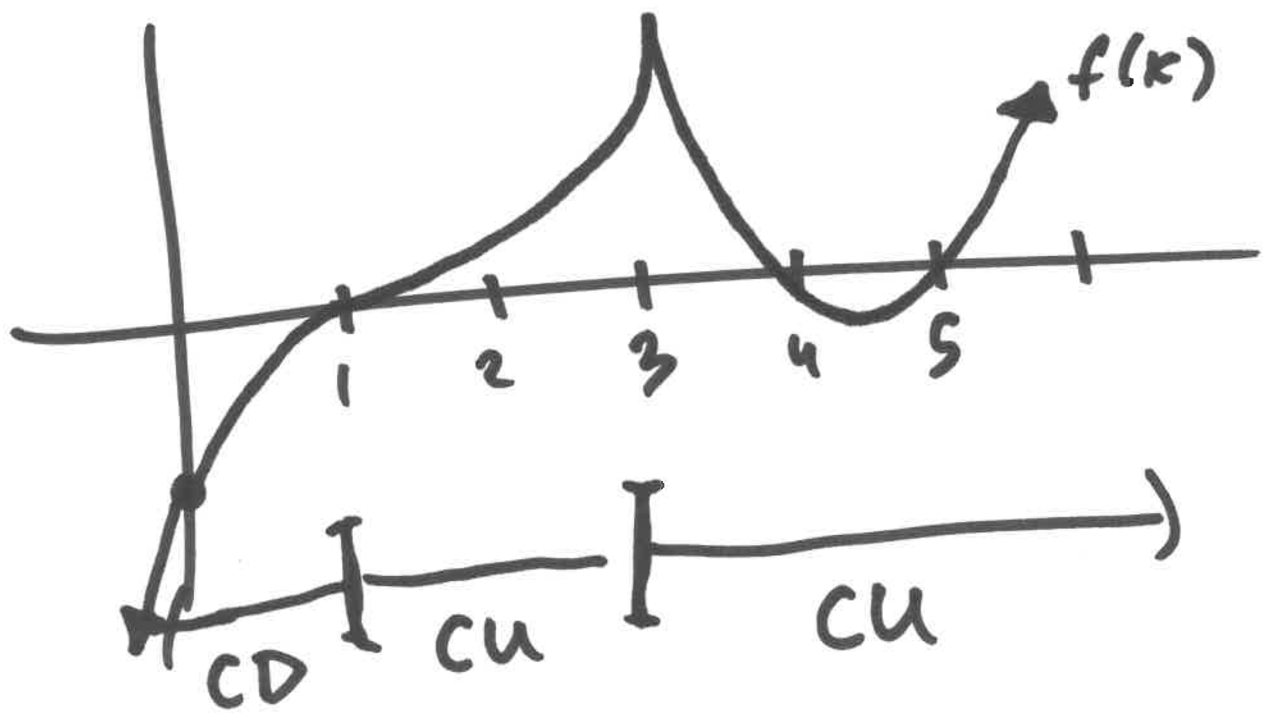
$\sin(x)$





$|x|$ is concave up on $\mathbb{R}$

Where is f(x) concave up? Concave down?
Are there any inflection points?



Concavity Test:

$\boxed{1}$ If $f''(x) > 0$ on an interval I,
then $f(x)$ is concave upward on I.

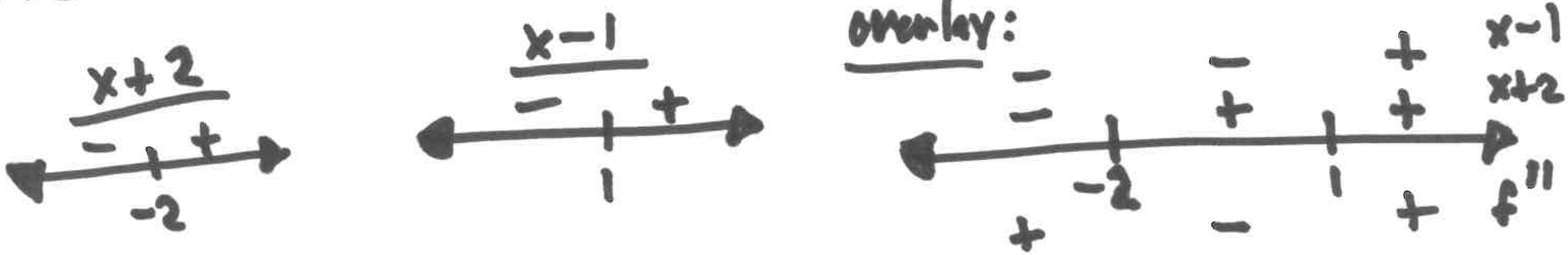
$\boxed{2}$ If $f''(x) < 0$ on an interval I,
then $f(x)$ is concave downward on I.

Q: When is $f(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 - x^2$ concave up? concave down? Are there any inflection points?

A: For concavity, we look at $f''(x)$.

$f'(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x$, $f''(x) = x^2 + x - 2$.

Note $f''(x) = x^2 + x - 2 = (x+2)(x-1)$.



So $f(x)$ is concave up on $(-\infty,-2] \cup (1,\infty)$
and concave down on $(-2,1)$.

$\Rightarrow$ Inflection points are at -2 and 1.

2nd Derivative test: Assume f'' continuous near c.

$\boxed{1}$ $f'(c)=0$ + $f''(c)>0 \Rightarrow$ c is a local minimum

$\boxed{2}$ $f'(c)=0$ + $f''(c)<0 \Rightarrow$ c is a local maximum

What can go wrong?

→ If $f''(c)=0$, can't conclude anything

→ $f''(c)$ may not exist

Q: Find the critical points of $f(x) = \frac{1}{4}x^4 - \frac{9}{2}x^2 + 7$ and classify them as local max/min.

A: We already found critical points in previous example; they were $c_1 = 0$, $c_2 = -3$, $c_3 = 3$.

Use 2nd derivative test:

$f'(x) = x^3 - 9x \Rightarrow f''(x) = 3x^2 - 9$.

$f''(0) = -9 < 0 \Rightarrow c_1 = 0$ is local max

$f''(-3) = 27 - 9 > 0 \Rightarrow c_2 = -3$ is local min

$f''(3) = 27 - 9 > 0 \Rightarrow c_3 = 3$ is local min.

☆ The 2nd derivative test is nice because if $f''(x)$ is easy to compute, then it might be easy to determine if a critical point is a local max / min. <u>But</u> it might be inconclusive. This is why we need the 1st derivative test.

Q: Consider $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3$.

(a) On what intervals is $f(x)$ increasing/decreasing?

(b) What are the critical points of $f(x)$?

(c) On what intervals is $f(x)$ concave up? Concave down?

(d) Does $f(x)$ have any inflection points?

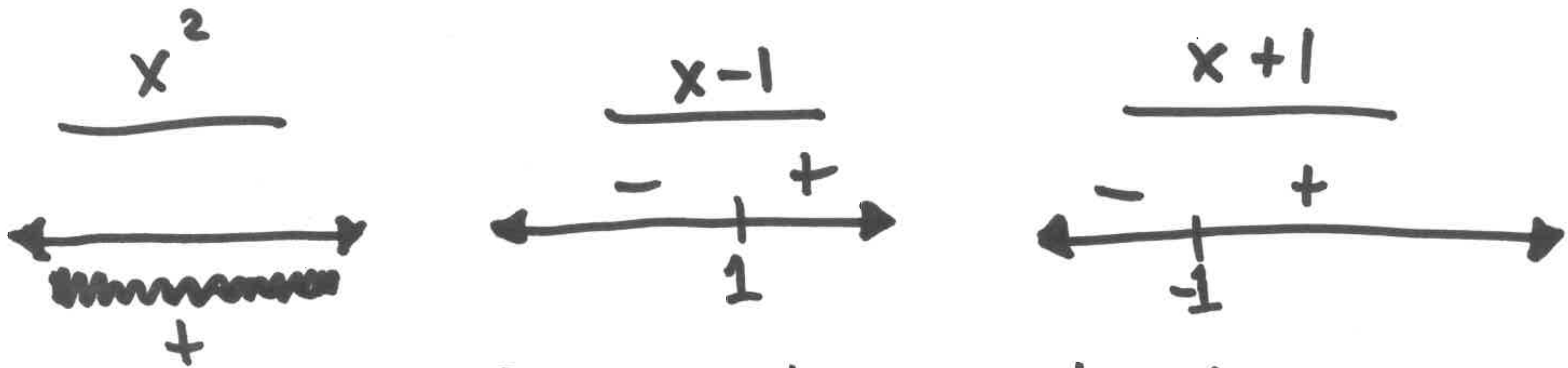
(a) $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3 \Rightarrow f'(x) = x^4 - x^2$

$$= x^2(x^2-1)$$

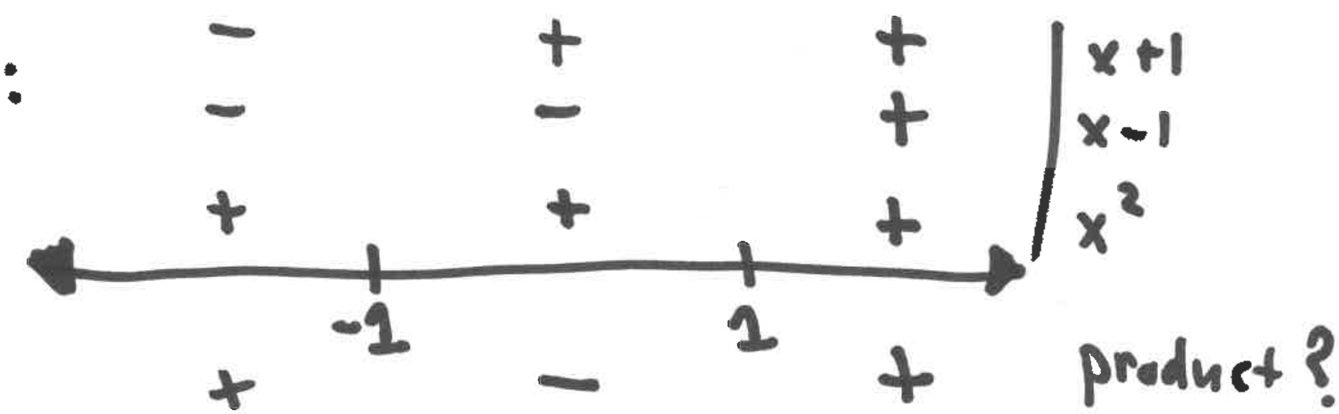
$$= x^2(x-1)(x+1).$$

Where is $f'(x)$ positive or negative?

Consider each term individually.



Then overlay:



So $f(x)$ is increasing on $(-\infty, -1]$ + $[1, \infty)$ and decreasing on $[-1, 1]$.

(b) What are the critical points?

Just $c_1 = 0$, $c_2 = -1$, $c_3 = +1$.

No points of non-differentiability.

(c) Where is $f(x)$ concave up/down?

$$f''(x) = \frac{d}{dx}\, x^2(x-1)(x+1)$$

$$= \frac{d}{dx}\, x^4 - x^2$$

$$= 4x^3 - 2x$$

$$= 2x(2x^2-1)$$

roots are $\pm \frac{\sqrt{2}}{2}$

$\underline{2x}$

$-$ 0 $+$

$\underline{2x^2-1}$

$+$ $-\frac{\sqrt{2}}{2}$ $-$ $\frac{\sqrt{2}}{2}$ $+$

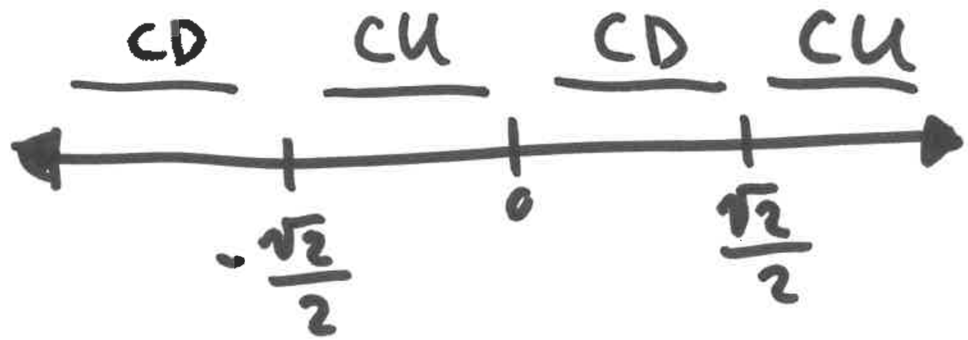
Overlay:

$+$	$-$	$-$	$+$	$2x^2-1$
$-$	$-$	$+$	$+$	$2x$

$-$ $-\frac{\sqrt{2}}{2}$ $+$ 0 $-$ $\frac{\sqrt{2}}{2}$ $+$

So $f(x)$ is concave downward on the intervals $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$ and $\left(0, \frac{\sqrt{2}}{2}\right)$, and concave upwards on the intervals $\left(-\frac{\sqrt{2}}{2}, 0\right)$ and $\left(\frac{\sqrt{2}}{2}, \infty\right)$.

(d)



So the two points of inflection are $\pm \frac{\sqrt{2}}{2}$.

MATH 3

Lecture #19

10/23/23

Jonathan Lindbloom

L'Hospital's Rule

Def: If $\lim_{x \to a} \frac{f(x)}{g(x)}$ with both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ may or may not exist and we say that the limit is an indeterminant form of type $\frac{0}{0}$.

Def: If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ with both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$), then $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ may or may not exist and we say that the limit is an indeterminant form of type $\frac{\infty}{\infty}$.

L'Hospital's Rule: Suppose $f(x)$ & $g(x)$ are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose also that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0,$$

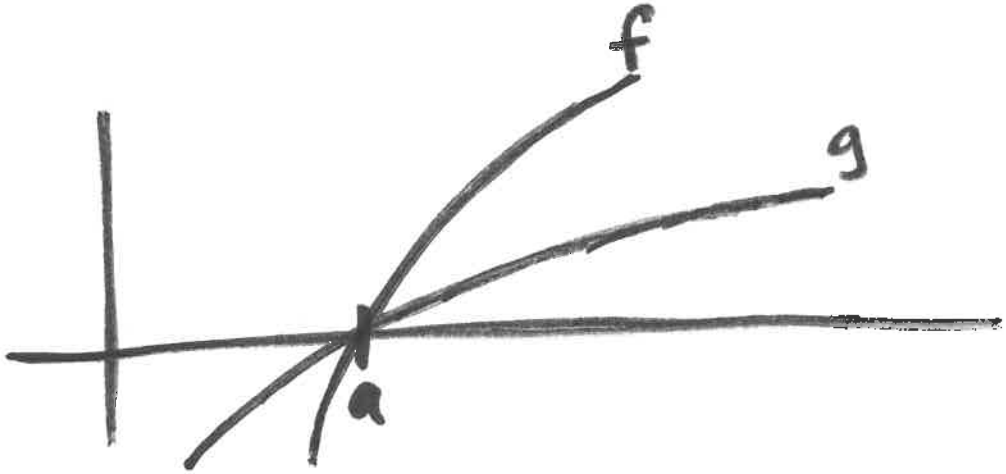
or that

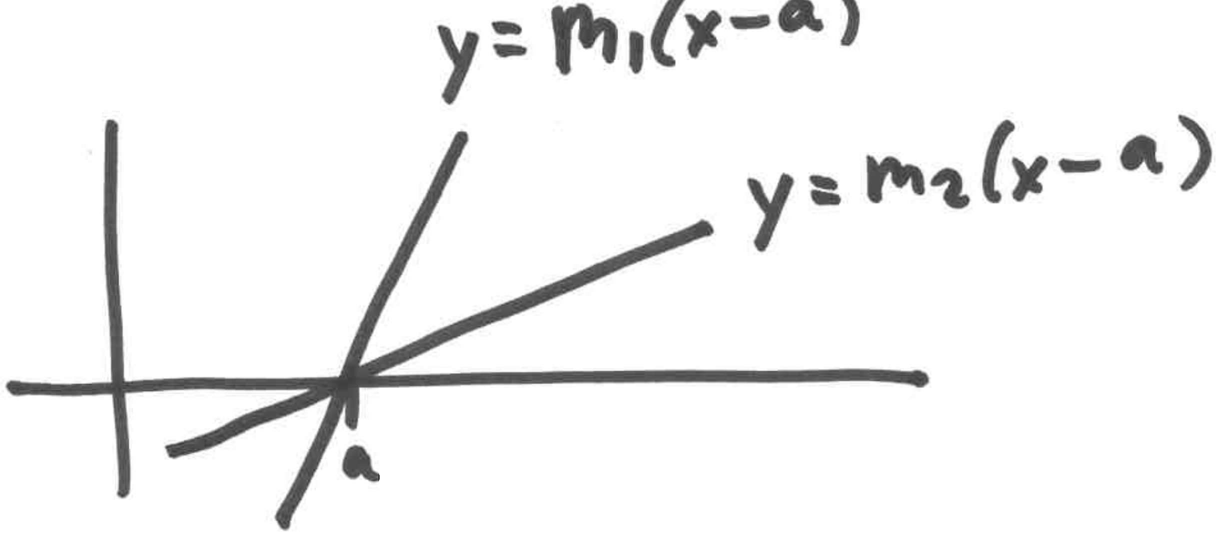
$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

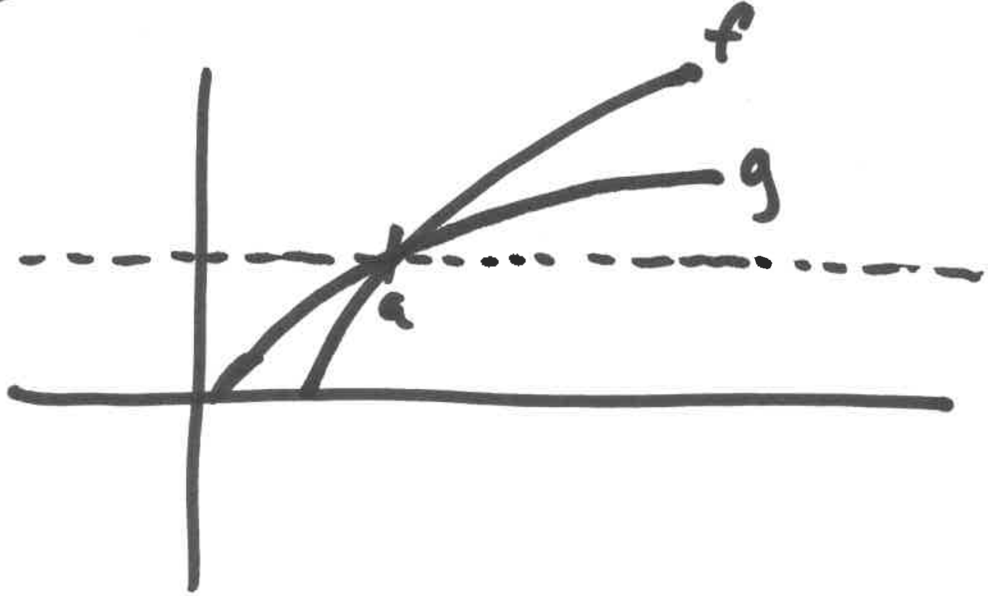
if the limit on the right exists (or is $\pm\infty$).

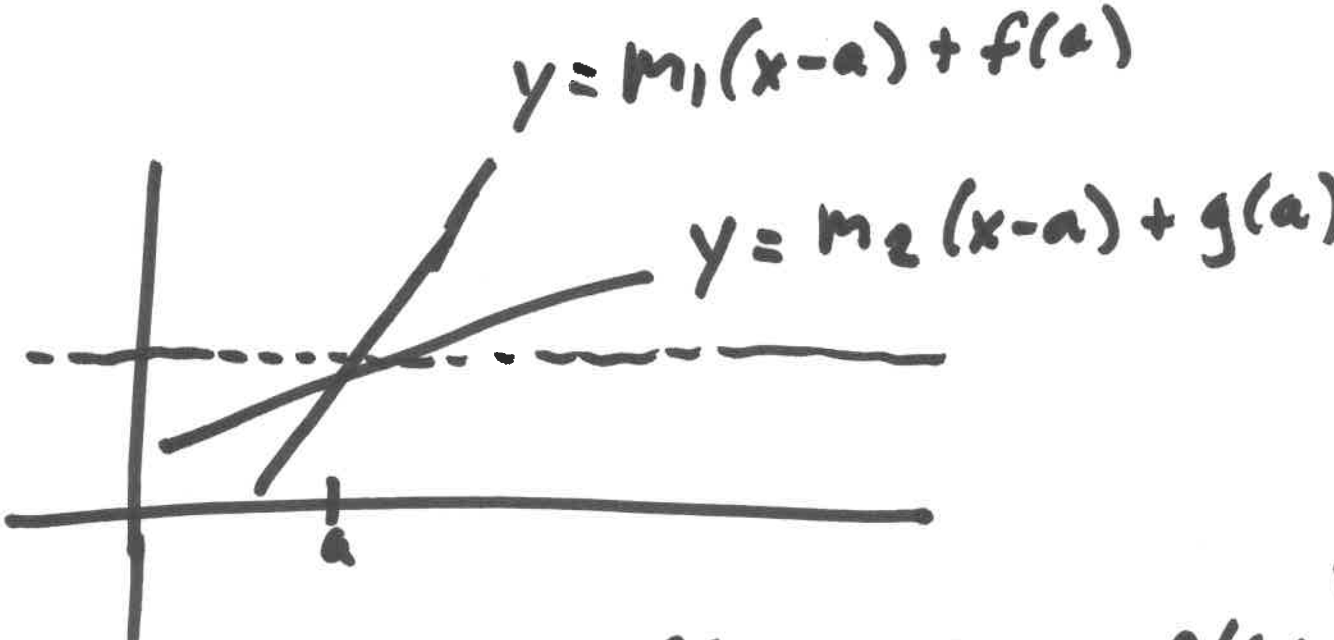




In this case, L'Hospital's Rule tells us that $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{m_1}{m_2}$.

Why is the 0-value important?





$$\frac{m_1}{m_2}$$

Note $\displaystyle\lim_{x\to a} \frac{m_1(x-a)+f(a)}{m_2(x-a)+g(a)} \neq \lim_{x\to a} \frac{f'(x)}{g'(x)}$!

Cauchy's Mean Value Theorem:

Suppose that the functions $f(x)$ and $g(x)$ are continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ $\forall x \in (a,b)$.

Then $\exists c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof of L'Hospital's Rule ($\frac{0}{0}$ case):

Suppose $f(x)$ + $g(x)$ are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except maybe at a). Suppose also that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

We want to show that $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ where $L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Define the functions

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

Note that both $F(x)$ & $G(x)$ are continuous on the open interval I.

Let $z \in I$ with $z > a$. Note $F(x)$ & $G(x)$ are continuous on $[a, z]$ and differentiable on (a, z), with $G'(x) \neq 0$ on (a, z) (since $F' = f'$ & $G' = g'$).

By Cauchy's MVT, $\exists c \in (a, z)$ such that

$$\frac{F'(c)}{G'(c)} = \frac{F(z) - F(a)}{G(z) - G(a)} \left(= \frac{F(z)}{G(z)} \right).$$

$$\Rightarrow \lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{c \to a^+} \frac{F'(c)}{G'(c)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = L.$$

Similarly, we can show also that

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = L \quad \text{as well.}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L = \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{as desired.}$$

Q: What is $\lim_{x \to 0} \frac{\sin(x)}{x}$?

A: We already know that this is 1.
But we can use L'Hospital's Rule:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1}$$

$$= 1.$$

Q: What is $\lim_{x \to 0} \frac{\tan(x)}{x}$?

A: $$\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\sec^2(x)}{1}$$

$$= \lim_{x \to 0} \frac{1}{\cos^2(x)}$$

$$= 1.$$

Q: What is $\lim_{x \to 0} \dfrac{1 - \cos(x)}{x}$?

A:

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \lim_{x \to 0} \frac{\sin(x)}{1}$$

$$= 0.$$

Q: What is $\lim_{x \to \pi} \dfrac{\sin(x)}{1-\cos(x)}$?

A: DON'T USE L'HOSPITAL'S RULE!

$\sin(x) \to 0$ as $x \to \pi$, but

$1-\cos(x) \to 2$ as $x \to \pi$.

So the limit is just 0 by substitution.

Remark: L'Hospital's Rule is also valid when $x \to a$ is replaced with $x \to a^{\pm}$ or $x \to \pm\infty$!

Q: What is $\displaystyle\lim_{x\to\infty} \frac{3x^2-2x+1}{4x^2+7x-9}$?

A: We already know that this is $\frac{3}{4}$!
But we can also apply L'Hospital's Rule:

$$\lim_{x\to\infty} \frac{3x^2-2x+1}{4x^2+7x-9} = \lim_{x\to\infty} \frac{6x-2}{8x+7}$$

$$= \lim_{x\to\infty} \frac{6}{8}$$

$$= \frac{6}{8} = \frac{3}{4}.$$

Q: Prove that $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$. $(n \in \mathbb{Z}_+)$

A: This is of the form $\frac{\infty}{\infty}$.

Applying L'Hospital's once (if $n > 1$), we get

$$\lim_{x \to \infty} \frac{e^x}{n x^{n-1}}$$

which is again of the form $\frac{\infty}{\infty}$. We can keep applying L'Hospital's Rule until we get $\lim_{x \to \infty} \frac{e^x}{(\text{constant})}$, at which point we conclude that $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$ for $n \in \mathbb{Z}_+$.

Q: What is $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$?

A: This is of the form $\frac{\infty}{\infty}$.

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}.$$

Using L'Hospital's Rule just takes us in a circle!

Two observations:

$\boxed{1}$ $\forall x,\ x^2 < x^2+1 \implies x < \sqrt{x^2+1}$

$$\implies \frac{x}{\sqrt{x^2+1}} < 1.$$

$\boxed{2}$ $f(x) = \dfrac{x}{\sqrt{x^2+1}} \implies f'(x) = \dfrac{1}{(x^2+1)^{\frac{3}{2}}}$

which is strictly positive $\forall x$.

$\implies f(x)$ is strictly increasing.

Together, $\boxed{1}$ and $\boxed{2}$ $\implies \displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1.$

What about products of the form $0 \cdot \infty$?
(Indeterminant products).

* We can always convert to the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

Since $fg = \dfrac{f}{\frac{1}{g}}$ + $fg = \dfrac{g}{\frac{1}{f}}$.

Q: What is $\lim_{x\to\infty} x^3 e^{-x^2}$?

A: This is of the form $\infty \cdot 0$.

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}}$$

$$= \lim_{x\to\infty} \frac{3x^2}{2x e^{x^2}}$$

$$= \lim_{x\to\infty} \frac{3x}{2e^{x^2}}$$

$$= \lim_{x\to\infty} \frac{3}{2\cdot 2x \cdot e^{x^2}} = 0.$$

Q: What is $\lim\limits_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$?

A: This is of the form $\infty \cdot 0$!

$$\lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{x^{-1}} = \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\cdot -\pi x^{-2}}{-x^{-2}}$$

$$= \lim_{x\to\infty} \pi\cos\left(\frac{\pi}{x}\right)$$

$$= \pi.$$

What about indeterminant differences of the form $\infty - \infty$?

★ Try to convert to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Q: What is $\lim\limits_{x\to\infty} xe^{\frac{1}{x}} - x$?

A: This is of the form $\infty - \infty$.

$$\lim_{x\to\infty} xe^{\frac{1}{x}} - x = \lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right)$$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \longleftarrow \text{form } \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}} \cdot -1x^{-2}}{-1x^{-2}} = \lim_{x\to\infty} e^{\frac{1}{x}} = 1.$$

Q: What is $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\ln(x)}\right]$?

A: $\frac{x}{x-1}-\frac{1}{\ln(x)} = \frac{x\ln(x)}{(x-1)\ln(x)} - \frac{(x-1)}{(x-1)\ln(x)}$

$= \frac{x\ln(x)-x+1}{(x-1)\ln(x)}$ ⟵ as $x\to 1$, is form $\frac{0}{0}$!

$\Rightarrow \lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\ln(x)}\right] = \lim_{x\to 1}\left[\frac{x\ln(x)-x+1}{(x-1)\ln(x)}\right]$ still $\frac{0}{0}$!

$= \lim_{x\to 1}\left[\frac{\ln(x)+x\cdot\frac{1}{x}-1}{\ln(x)+x\cdot\frac{1}{x}-\frac{1}{x}}\right] = \lim_{x\to 1}\left[\frac{\ln(x)}{\ln(x)+1-\frac{1}{x}}\right]$

$= \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+x^{-2}} = \frac{1}{1+1} = \frac{1}{2}.$

Q: What is $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$?

A: This is of the form $\infty - \infty$.

$$\frac{1}{x} - \frac{1}{e^x - 1} = \frac{(e^x - 1)}{x(e^x - 1)} - \frac{x}{x(e^x - 1)}$$

$$= \frac{e^x - x - 1}{x(e^x - 1)} \longrightarrow \text{as } x \to 0^+, \text{ is of form } \frac{0}{0}!$$

$$\Rightarrow \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \to 0^+} \frac{e^x - x - 1}{x(e^x - 1)}$$

$$= \lim_{x \to 0^+} \frac{e^x - 1}{e^x - 1 + xe^x} = \lim_{x \to 0^+} \frac{e^x}{e^x + e^x + xe^x} = \frac{1}{1+1} = \frac{1}{2}.$$

What about indeterminant forms

0^0, ∞^0, 1^∞ ?

★ Note that

$$[f(x)]^{g(x)} = e^{\ln\left([f(x)]^{g(x)}\right)} = e^{g(x)\ln(f(x))}.$$

How does this help us?

Q: What is $\lim\limits_{x\to 0^+} x^{\sqrt{x}}$?

type $0 \cdot -\infty$

A: This is of form 0^0.

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\sqrt{x}\ln(x)} = e^{\left[\lim\limits_{x\to 0^+} \sqrt{x}\ln(x)\right]}$$

$$= e^{\left[\lim\limits_{x\to 0^+} \frac{\ln(x)}{x^{-\frac{1}{2}}}\right]} = e^{\left[\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}\right]}$$

$$= e^{\left[\lim\limits_{x\to 0^+} -2x^{-1}x^{\frac{3}{2}}\right]} = e^{\left[-2\cdot \lim\limits_{x\to 0^+} x^{\frac{1}{2}}\right]}$$

$$= e^{0} = 1$$

Q: What is $\lim_{x\to\infty} x^{\frac{1}{x}}$?

A: This is of the form ∞^0.

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(x)} = e^{\left[\lim_{x\to\infty} \frac{1}{x}\ln(x)\right]}$$

$0 \cdot \infty$

$$= e^{\left[\lim_{x\to\infty} \frac{\ln(x)}{x}\right]} = e^{\left[\lim_{x\to\infty} \frac{1}{x}\right]} = e^0 = 1.$$

What about the form 0^{∞}?

⚠ This is <u>not</u> indeterminant!

Proposition: Suppose $f(x) > 0 \; \forall x$.

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$,

then $\lim_{x \to a} [f(x)]^{g(x)} = 0$.

Proof?
$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))}$$

$$= e^{\left[\lim_{x\to a} g(x)\ln(f(x))\right]}.$$

Note that $g(x)\ln(f(x))$ grows arbitrarily negative as $x \to a$, i.e., $\lim_{x\to a} g(x)\ln(f(x)) = -\infty$.

$$\Rightarrow e^{\left[\lim_{x\to a} g(x)\ln(f(x))\right]} = 0.$$

MATH 3

Lecture #20

10/25/23

Jonathan Lindbloom

MATH 3

Lecture #21

10/27/23

Jonathan Lindbloom

Optimization

General Idea:

We have already solved 1-dimensional optimization problems of the form

$$\min_{x \in [a,b]} f(x).$$

Now we will look at bivariate optimization problems with 1 constraint, i.e.,

$$\min_{C(x)=y} F(x,y).$$

For general n-dimensional optimization (with constraints), will have to wait until vector calculus.

Q: You have 100 ft. of fencing and you are told to fence off a rectangular plot of land to maximize the total area enclosed. What should the side lengths be?

$A = \ell w$

A: Want to maximize area. Constraint says

$2\ell + 2w = 100 \Rightarrow 2w = 100 - 2\ell \Rightarrow w = 50 - \ell$.

$A(\ell) = \ell(50-\ell) = 50\ell - \ell^2$, $A'(\ell) = 50 - 2\ell$.

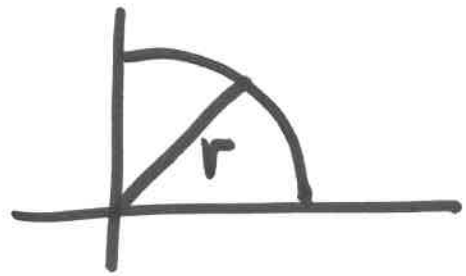
$A'(\ell) = 0 \Rightarrow 50 - 2\ell = 0 \Rightarrow \ell = 25$ + $w = 50 - \ell = 25$.

So sides should be 25 ft. × 25 ft., and max area is 25^2.

Q: What is the area of the largest rectangle that can be inscribed in a circle of radius r?

A:

☆ Just consider the 1st quadrant.



Any point (x,y) on the quarter circle defines a rectangle. We also know on this quadrant that $y = \sqrt{r^2-x^2} = (r^2-x^2)^{\frac{1}{2}}$.

So $A(x) = x(r^2-x^2)^{\frac{1}{2}}$, $A'(x) = (r^2-x^2)^{\frac{1}{2}} + \frac{x}{2}(r^2-x^2)^{-\frac{1}{2}} \cdot -2x$

$$= (r^2-x^2)^{\frac{1}{2}} - x^2(r^2-x^2)^{-\frac{1}{2}}.$$

$$A' = 0 \Rightarrow \sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}} = 0$$

$$\Rightarrow r^2 - x^2 - x^2 = 0 \Rightarrow r^2 = 2x^2$$

$$\Rightarrow x^2 = \frac{r^2}{2} \Rightarrow x = r\frac{\sqrt{2}}{2} \Rightarrow y = \sqrt{r^2 - x^2} = \sqrt{\tfrac{1}{2}r^2}$$

$$= r\frac{\sqrt{2}}{2}.$$

So the largest rectangle inscribed by the circle of radius r (in 1st quadrant) has area $A = r\frac{\sqrt{2}}{2} \cdot r\frac{\sqrt{2}}{2} = \frac{r^2}{2}$.

Considering the whole circle, the total area is $\hat{A} = 4\frac{r^2}{2} = 2r^2$.

Q: Find the point on the curve $y=\sqrt{x}$ closest to the point $(3,0)$.

A: Define $d(x) = \text{distance}\left((3,0),(x,y(x))\right)$.

$\Rightarrow d(x) = \left((3-x)^2+(0-y(x))^2\right)^{\frac{1}{2}}$

We want to find the point $(x^*, y(x^*))$ that minimizes this distance.

$$d'(x) = \frac{d}{dx}\left[(3-x)^2+(\sqrt{x})^2\right]^{\frac{1}{2}} = \frac{d}{dx}\left[(3-x)^2+x\right]^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[(3-x)^2+x\right]^{-\frac{1}{2}} \cdot \left(-2(3-x)+1\right)$$

$$= \frac{2x-6+1}{2\,d(x)} = \frac{2x-5}{2\,d(x)}.$$

$$d'(x) = 0 \Rightarrow \frac{2x-5}{2d(x)} = 0$$

$$\Rightarrow 2x - 5 = 0 \Rightarrow x^* = \frac{5}{2}.$$

$$\Rightarrow y(x^*) = \sqrt{\frac{5}{2}}.$$

So the closest point is $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$.

Q: What is the max distance between any two points on the unit circle?

A: If we are not careful, we will run into problems with $y = \pm\sqrt{1-x^2}$ (which one to choose?)

Instead, we can think of each point as determined by just an angle θ. Let the first point have angle $\theta_1 = 0$. $\Rightarrow P_1 = (1,0)$. What angle θ for the second point maximizes the distance between the two points?

First find $d(P_1, P_2)$ as a function of θ.

$$\underbrace{d(P_1, P_2(\theta))}_{\tilde{d}(\theta)} = \left((1-\cos\theta)^2 + (0-\sin\theta)^2\right)^{\frac{1}{2}}$$

$$\tilde{d}'(\theta) = \frac{1}{2}\left((1-\cos\theta)^2 + \sin^2\theta\right)^{-\frac{1}{2}} \cdot \left(2(1-\cos\theta)\sin\theta + 2\sin\theta\cos\theta\right)$$

$$= \frac{2\sin\theta - 2\sin\theta\cos\theta + 2\sin\theta\cos\theta}{2\tilde{d}(\theta)}$$

$$= \frac{\sin\theta}{\tilde{d}(\theta)}.$$

$$\tilde{d}'(\theta) = 0 \Rightarrow \frac{\sin\theta}{\tilde{d}(\theta)} = 0 \Rightarrow \sin\theta = 0$$

$$\Rightarrow \theta = \pi k, \; k \in \mathbb{Z}.$$

* But not all points $\theta = \pi k, k \in \mathbb{Z}$ give maximum distance! Only the odd k do.

Newton's Method

Suppose you want to solve $x - \ln(x) = 2$ for x. How can you do this?

☆ We cannot isolate x.

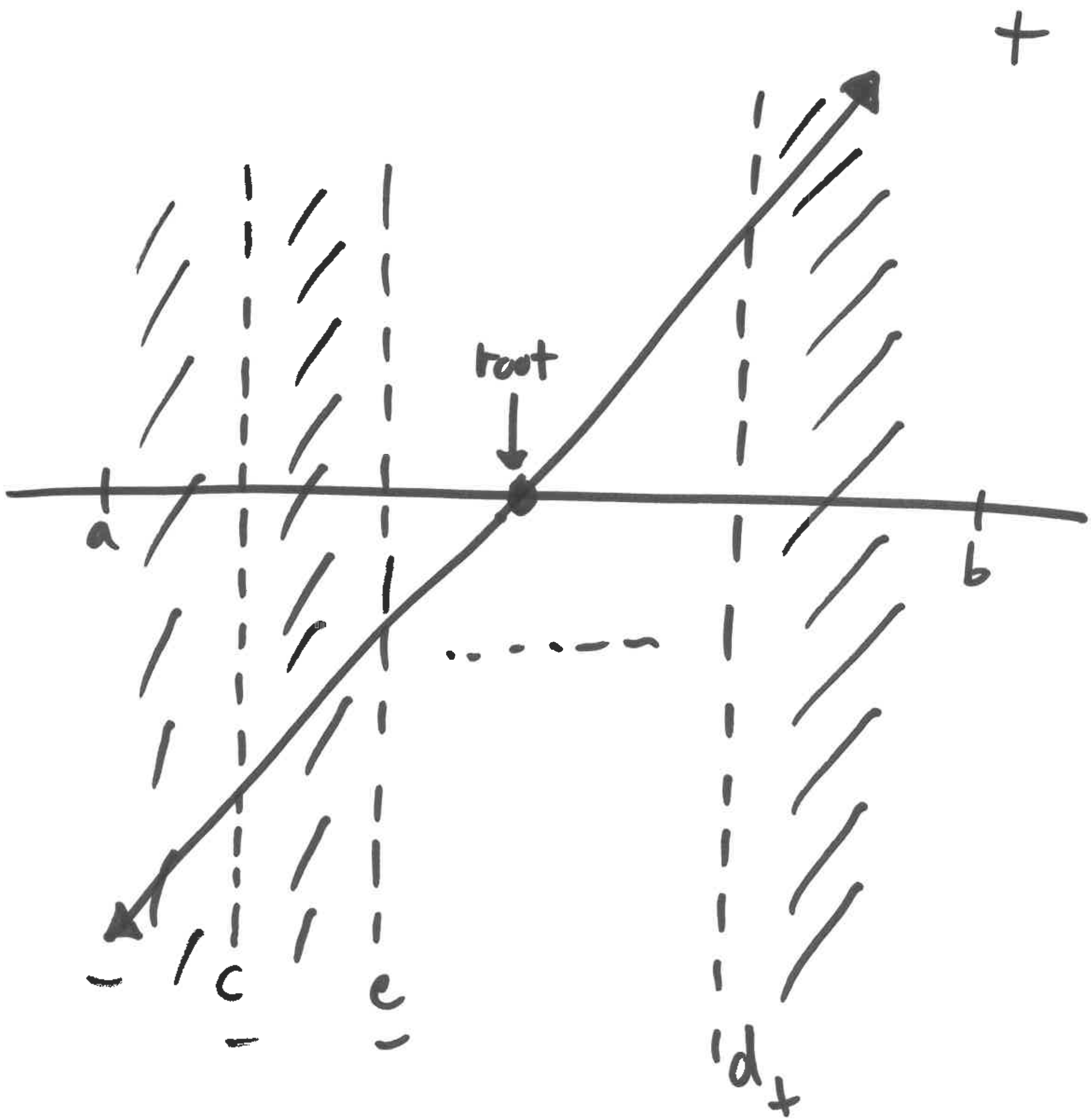
$\Downarrow$

In general, the best we can do is employ numerical methods.

What idea did we mention earlier in the class?

Bisection method, since x solves $x - \ln(x) = 2$

$\iff x$ is a root of $G(x) = x - \ln(x) - 2$.



+
root
a
b
–
c
e
d
–
–
+

<u>Game:</u> In 5 minutes, try to calculate the solution to $\cos(x) = x$ to as many correct decimal places as you can!

$$x \approx 0.7390851332151606416 6$$

Q: Sketch the graph of $f(x) = \frac{1}{3}x^3 - 9x$.

A:

→ roots? $f = x\left(\frac{1}{3}x^2 - 9\right) \Rightarrow \begin{array}{l} r_1 = -3\sqrt{3} \\ r_2 = 3\sqrt{3} \end{array}$

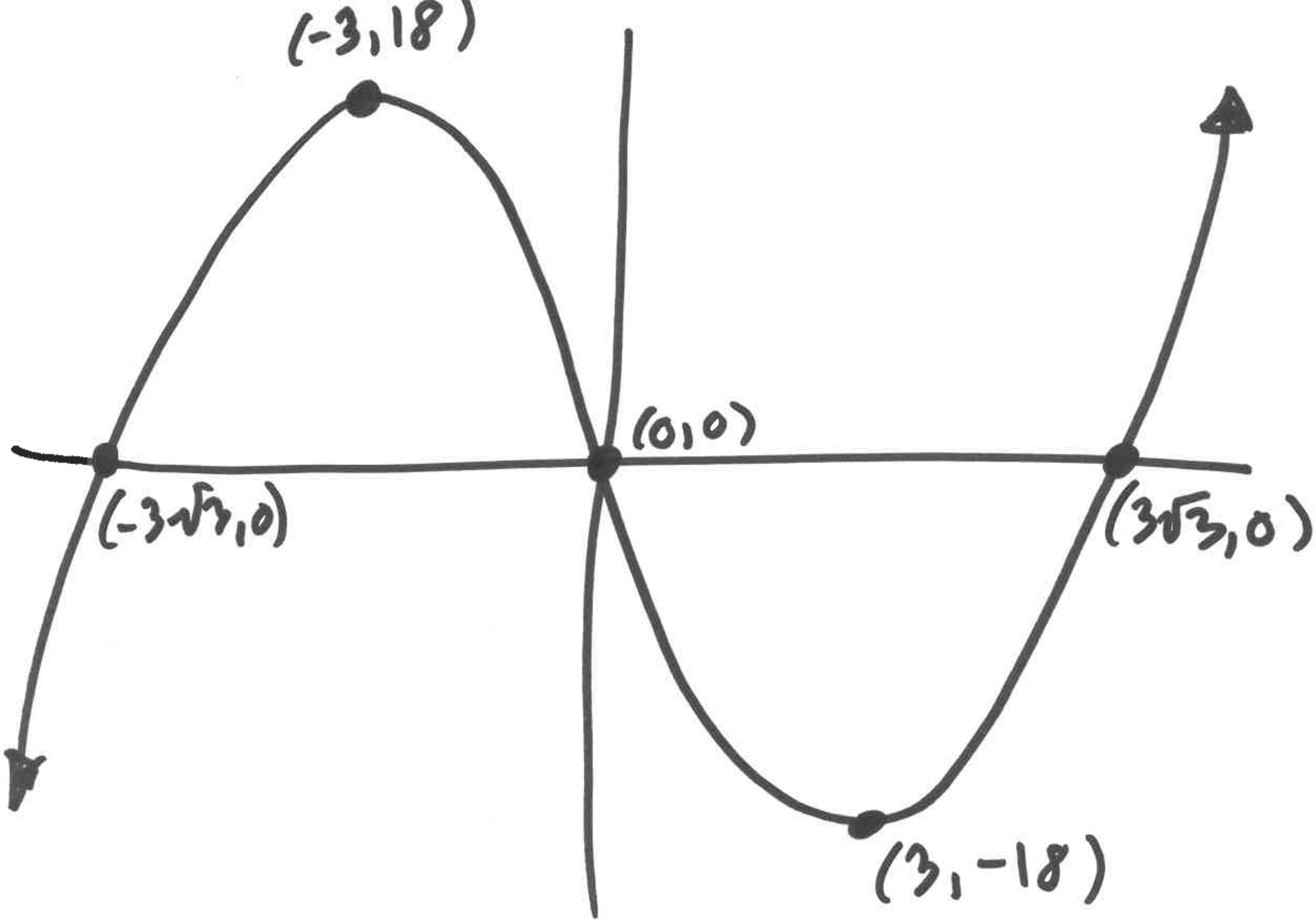
→ local max/min? $f'(x) = x^2 - 9$,

$f' = 0 \Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3$ are critical points.

$f''(x) = 2x$, $f''(-3) = -6 < 0 \Rightarrow -3$ is local max

$f''(3) = 6 > 0 \Rightarrow 3$ is local min

→ $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to +\infty} f(x) = +\infty$.


(-3, 18)
(0, 0)
(-3√3, 0)
(3√3, 0)
(3, -18)

Q: Consider $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3$

(a) On what intervals is $f(x)$ increasing/decreasing

(b) What are the critical points of $f(x)$?
Local max, local min, neither?

(c) On what intervals is $f(x)$ concave up?
Concave down?

(d) Does $f(x)$ have any inflection points?

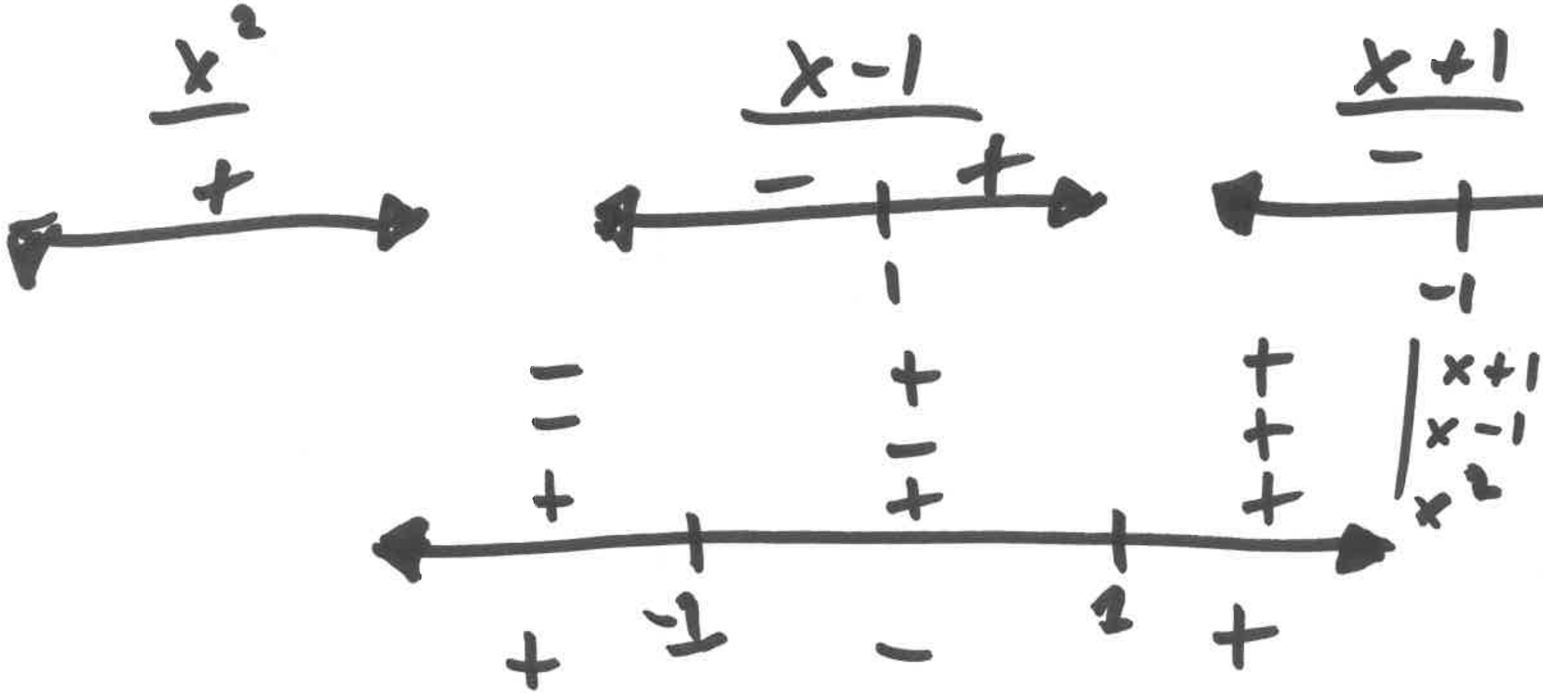
(e) Determine local min/max using 2nd derivative test?

$\boxed{(a)}$ $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3 \Rightarrow f'(x) = x^4 - x^2$

$\Rightarrow f'(x) = x^2(x^2-1) = x^2(x+1)(x-1)$.

Where is $f'(x)$ positive or negative?

Consider each term individually.



Increasing on $(-\infty,-1)$ & $(1,\infty)$; Decreasing on $(-1,1)$

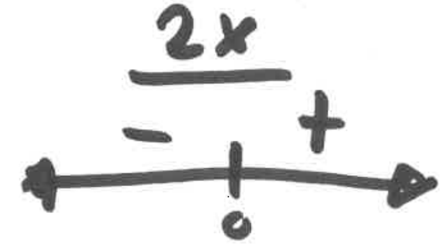
(b) Critical points are $c_1 = 0$, $c_2 = -1$, $c_3 = 1$.

By 1st derivative test,

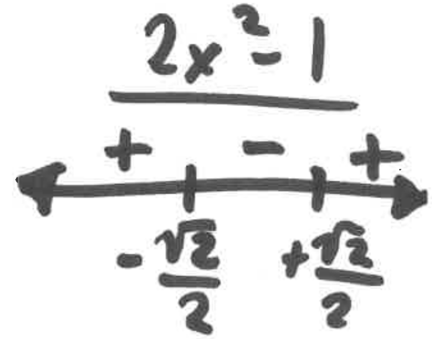
$c_1 = 0$ is neither

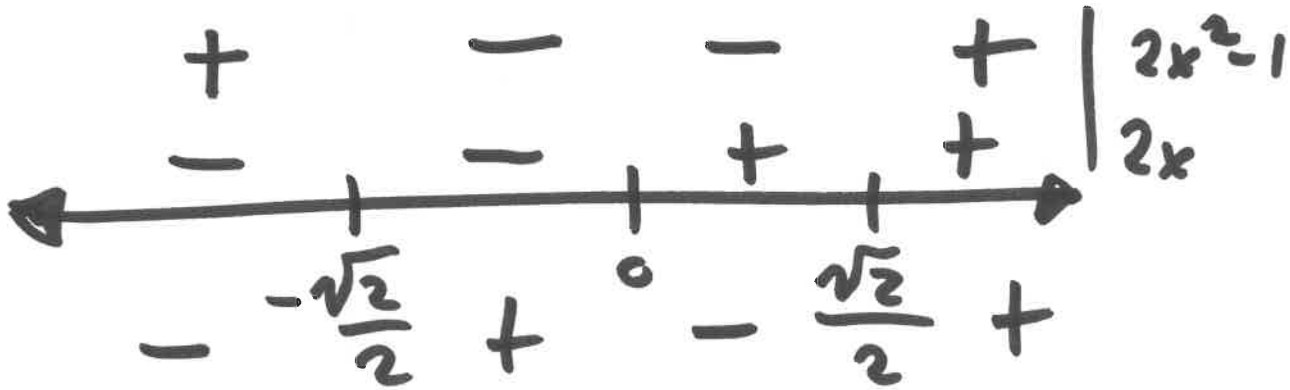
$c_2 = -1$ is local max

$c_3 = 1$ is local min



(c) $f''(x) = 4x^3 - 2x = 2x(2x^2 - 1)$





$\Rightarrow$ Concave up on $\left(-\frac{\sqrt{2}}{2}, 0\right) \,\&\, \left(\frac{\sqrt{2}}{2}, \infty\right)$; concave down on $\left(-\infty, -\frac{\sqrt{2}}{2}\right) \,\&\, \left(0, \frac{\sqrt{2}}{2}\right)$.

$\boxed{(d)}$ The only two inflection points are $\pm\frac{\sqrt{2}}{2}$ .... 0 is <u>not</u> an inflection point!

$\boxed{(e)}$ Using the 2nd derivative test,

$f''(0) = 0 \Rightarrow$ inconclusive for c_1

$f''(-1) = 4(-1)^3 - 2(-1) = -4+2 = -2 < 0$

$\Rightarrow c_2 = -1$ is a local max

$f''(1) = 4(1)^3 - 2(1) = 4-2 = 2 > 0$

$\Rightarrow c_3 = 1$ is a local min.

$\boxed{(f)}$ Can you graph the function?

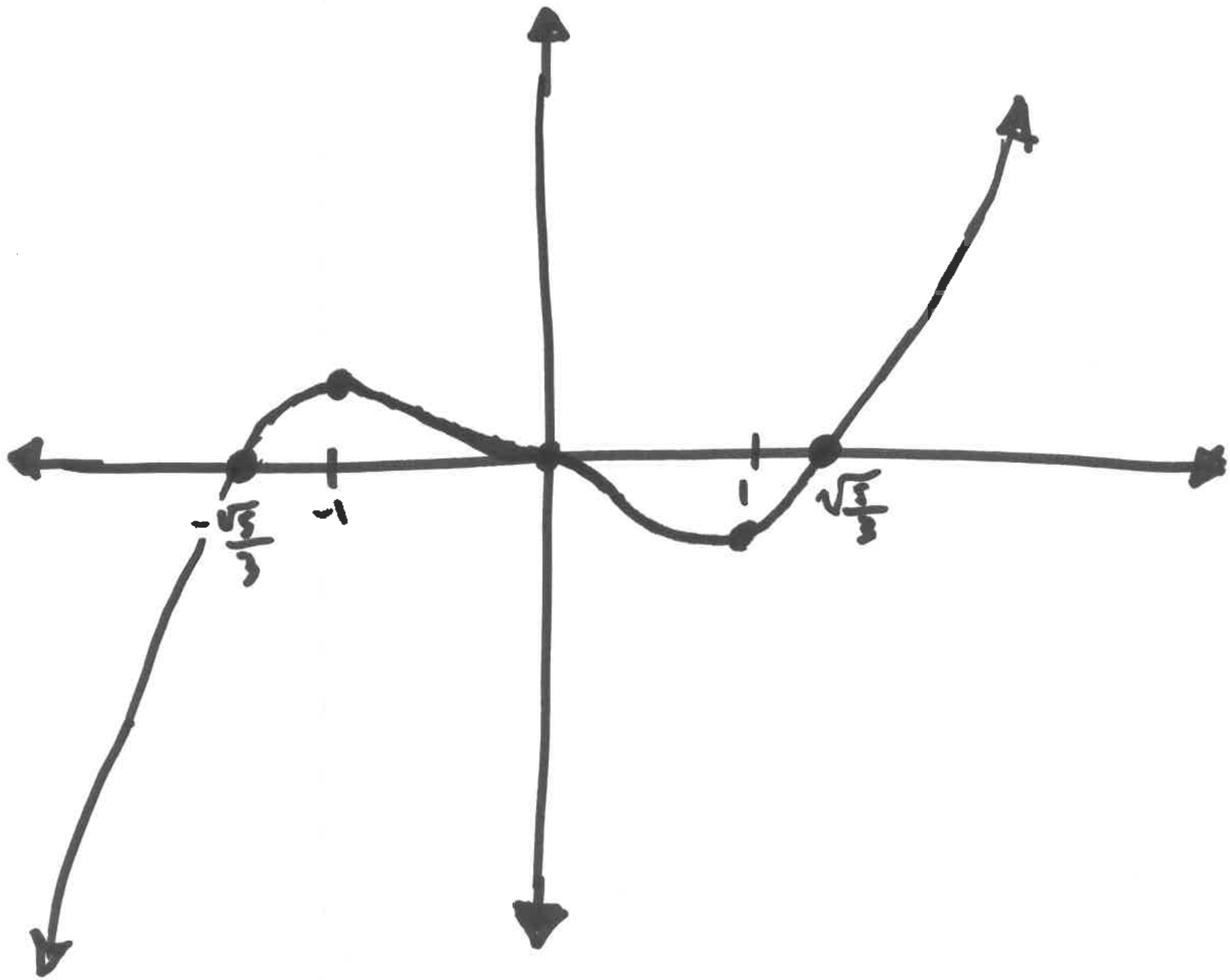
→ roots? $f(x) = x^3\left(\frac{1}{5}x^2 - \frac{1}{3}\right)$

$\Rightarrow r_1 = 0, \quad r_2 = -\sqrt{\frac{5}{3}}, \quad r_3 = \sqrt{\frac{5}{3}}.$

→ No horizontal or vertical asymptotes

→ x + y -intercepts are $(0,0)$.

→ $\lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = +\infty.$



$-\frac{\sqrt{5}}{3}$
-1
1
$\frac{\sqrt{5}}{3}$

Sketching curves?

→ x + y - intercepts

→ Asymptotes

→ Increasing/decreasing intervals

→ Local max/min

→ Concavity intervals, inflection points

Q: Sketch the curve $y = \dfrac{2x^2}{x^2-1}$.

A:

→ x-intercept? $(0,0)$

→ y-intercept? $(0,0)$

→ Critical points / increasing + decreasing?

$$f'(x) = \frac{4x(x^2-1) - 2x(2x^2)}{(x^2-1)^2}$$

$$= \frac{4x^3 - 4x - 4x^3}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

$\Rightarrow$ $c_1 = 0$ is the only critical point.

Note $f'(x)$ is positive for $x<0$ and negative for $x>0$, $\Rightarrow$ $f(x)$ increasing on $(-\infty, 0)$ and decreasing for $(0, \infty)$.

$\Rightarrow$ $c_1 = 0$ is local max!

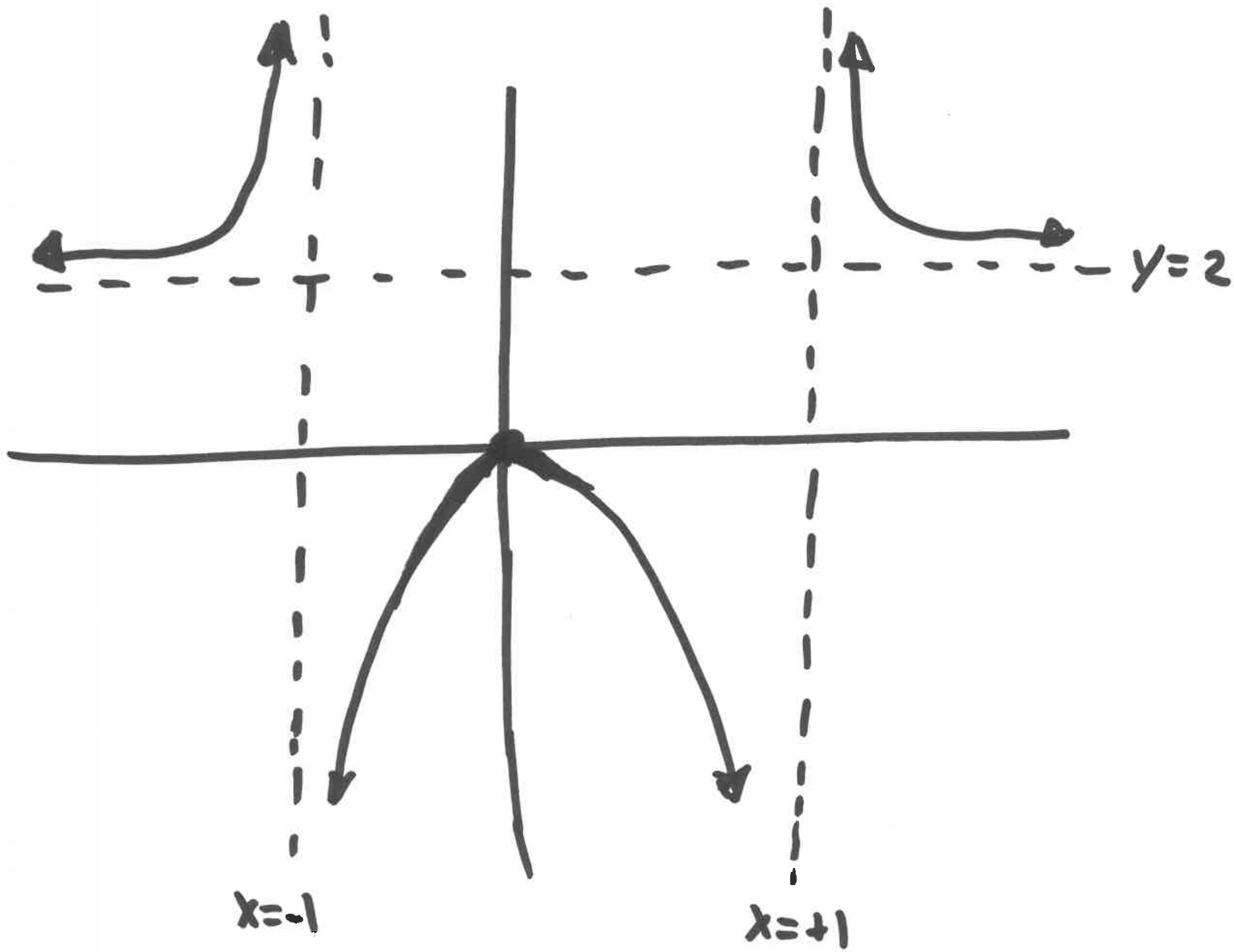
$\rightarrow$ Asymptotes? $\lim\limits_{x\to+\infty} f(x) = 2$, $\lim\limits_{x\to-\infty} f(x) = 2$.

So $y=2$ is only horizontal asymptotes.

Vertical ones?

$$\lim_{x\to 1^+} f(x) = \infty \qquad \lim_{x\to -1^+} f(x) = -\infty$$

$$\lim_{x\to 1^-} f(x) = -\infty \qquad \lim_{x\to -1^-} f(x) = \infty$$


y = 2
x = -1
x = +1

Q: Find + classify the critical points of $f(x) = x(6-x)^{\frac{2}{3}}$. Then find any inflection points. Finally, sketch the curve.

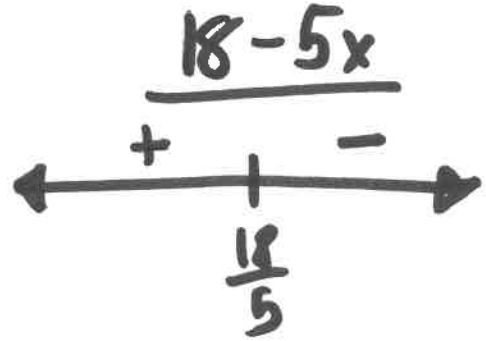
A: $f'(x) = \dfrac{18-5x}{3(6-x)^{\frac{1}{3}}}$, $f''(x) = \dfrac{10x-72}{9(6-x)^{\frac{4}{3}}}$.

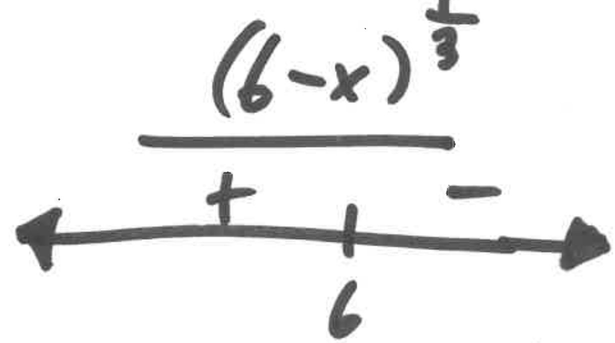
What are the critical points? When $f'(x) = 0$ OR when $f'(x)$ DNE. So $c_1 = \frac{18}{5} = 3.6$ and $c_2 = 6$!

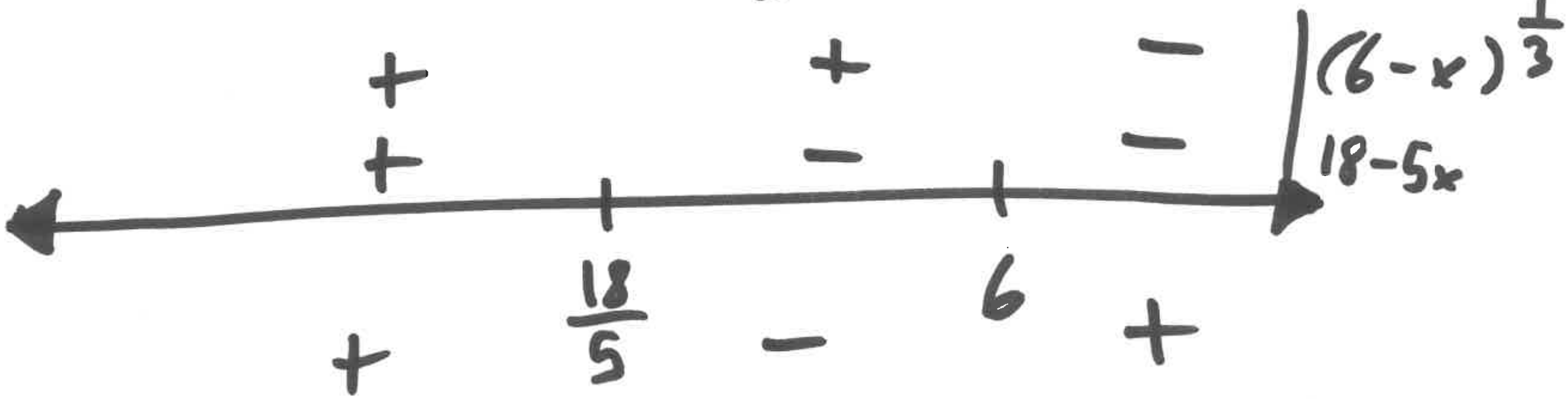
$f''(\frac{18}{5}) = -1.245 < 0 \Rightarrow c_1 = \frac{18}{5}$ is a local max.

But we can't use the 2nd derivative test for $c_2 = 6$!

$$f'(x) = \frac{18-5x}{3(6-x)^{\frac{1}{3}}}.$$







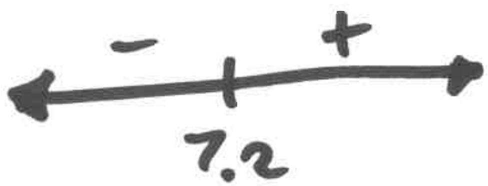
So $c_1 = \frac{18}{5}$ is local max

and $c_2 = 6$ is a local min!

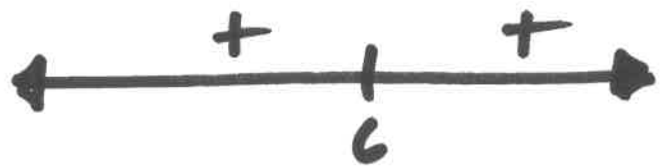
Concavity and inflection points?

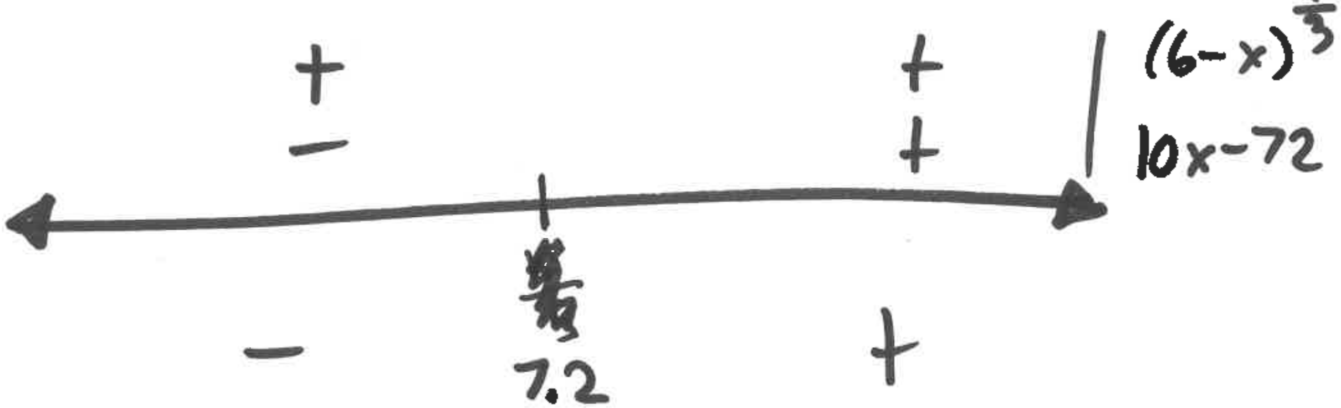
$$f''(x) = \frac{10x-72}{9(6-x)^{\frac{4}{3}}}.$$

$\underline{10x-72}$



$(6-x)^{\frac{4}{3}}$





$\Rightarrow$ 7.2 is an inflection point. $f(x)$ is concave down on $(-\infty, 7.2)$, concave up on $(7.2, \infty)$.

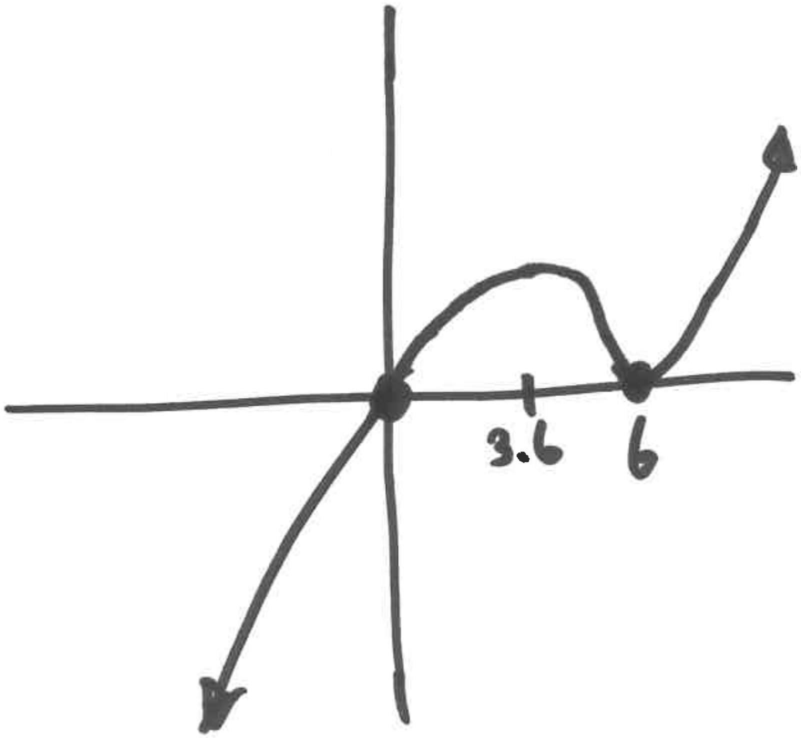
Graph?

→ x-intercepts at $x=0$ & $x=6$

→ y-intercept at $y=0$

→ No horizontal/vertical asymptotes

→ $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = +\infty$



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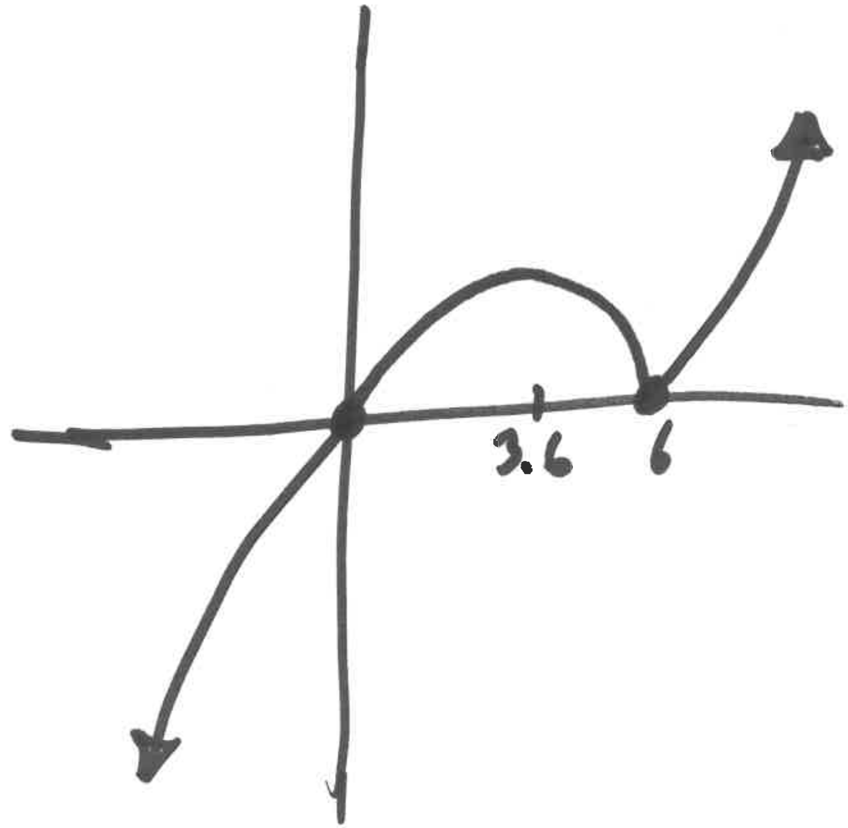


correct

Q: Consider $f(x) = x^2 - x - \ln(x)$.

(a) Find where $f(x)$ is increasing/decreasing.

(b) Find the local max/min values of $f(x)$.

(c) Find where $f(x)$ is CU/CD + inflection points.

(d) Sketch the curve.

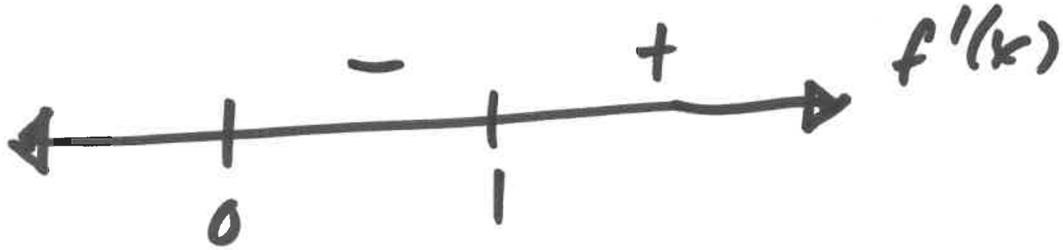
A: $f'(x) = 2x - 1 - \frac{1}{x}$

$f' = 0 \Rightarrow 2x - 1 - \frac{1}{x} = 0 \Rightarrow 2x^2 - x - 1 = 0$

$\Rightarrow (2x+1)(x-1) = 0 \Rightarrow \left.\begin{array}{l} x = -\frac{1}{2} \\ x = 1 \end{array}\right\}$ are critical points

$\boxed{\text{(a)}}$

irrelevant, since $f(x)$ not defined here!



$\boxed{(b)}$ local max/min? Just at $x=1$.

$f''(x) = 2 + \frac{1}{x^2}$, $f''(1) = 2 + \frac{1}{1} = 3 > 0$

$\Rightarrow x=1$ is a local min.

$\boxed{(c)}$ Since $f''(x) = 2 + \frac{1}{x^2} > 0$, $f(x)$ is concave up on $(0, \infty)$.

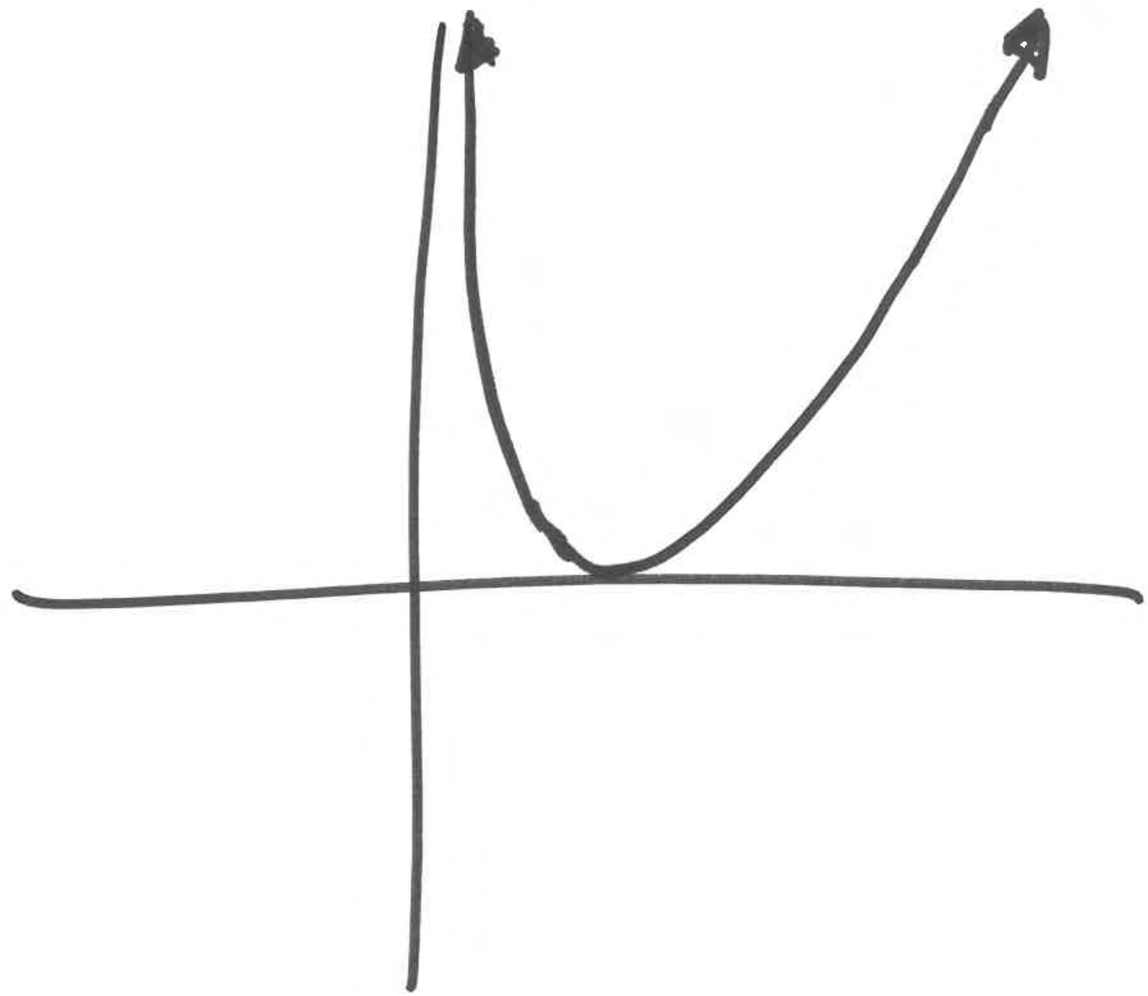
<u>No</u> inflection points.

Graph? $\boxed{(d)}$

→ x-intercept? Hard to find.

→ Defined only for $x > 0$.

→ $\lim_{x \to 0^+} f(x) = +\infty$, $\lim_{x \to \infty} f(x) = +\infty$.



Q: Find [redacted] all critical points of the function $f(x) = \cos(x + \cos(3x))$ on the interval $[0, 1.3]$.

A:

$$f'(x) = -\sin(x + \cos(3x))(1 - 3\sin(3x))$$

$$f' = 0 \Rightarrow \begin{cases} 1 - 3\sin(3x) = 0 \rightarrow c_1 \approx 0.1133, \quad c_2 \approx 0.9339 \\ \sin(x + \cos(3x)) = 0 \rightarrow c_3 \approx 0.8877, \quad c_4 \approx 0.9794 \end{cases}$$

Q: What does the graph of $f(x) = kx^4 - 4x^2 + 1$ look like for varying k? Draw all qualitatively different curves.

A: Critical points? $f'(x) = 4kx^3 - 8x$

$= 4x(kx^2 - 2)$

$\Rightarrow c_1 = 0$

$c_2 = -\sqrt{\frac{2}{k}}$

$c_3 = +\sqrt{\frac{2}{k}}$.

☆ $c_1 = 0$ is always a critical point, but c_2 + c_3 are only critical points when $k > 0$!

So there are 2 cases to think about:

① $k > 0$ and ② $k \le 0$. $f''(x) = 12kx^2 - 8$.

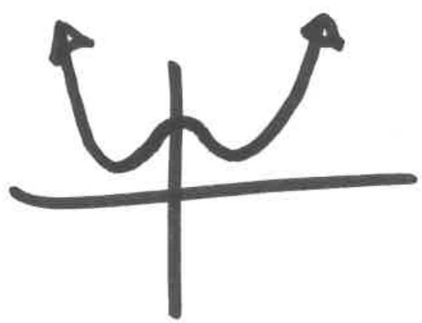
In ①, $f''(0) = -8 < 0 \Rightarrow c_1 = 0$ is local max

$f''(-\sqrt{\frac{2}{k}}) = 24 - 8 > 0 \Rightarrow c_2$ is local min

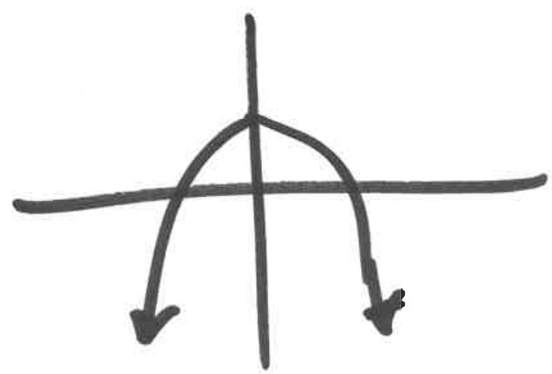
$f''(\sqrt{\frac{2}{k}}) = 24 - 8 > 0 \Rightarrow c_3$ is local min

In ②, $f''(0) = -8 < 0$ again, so local max.

$k > 0$



$k \le 0$



MATH 3

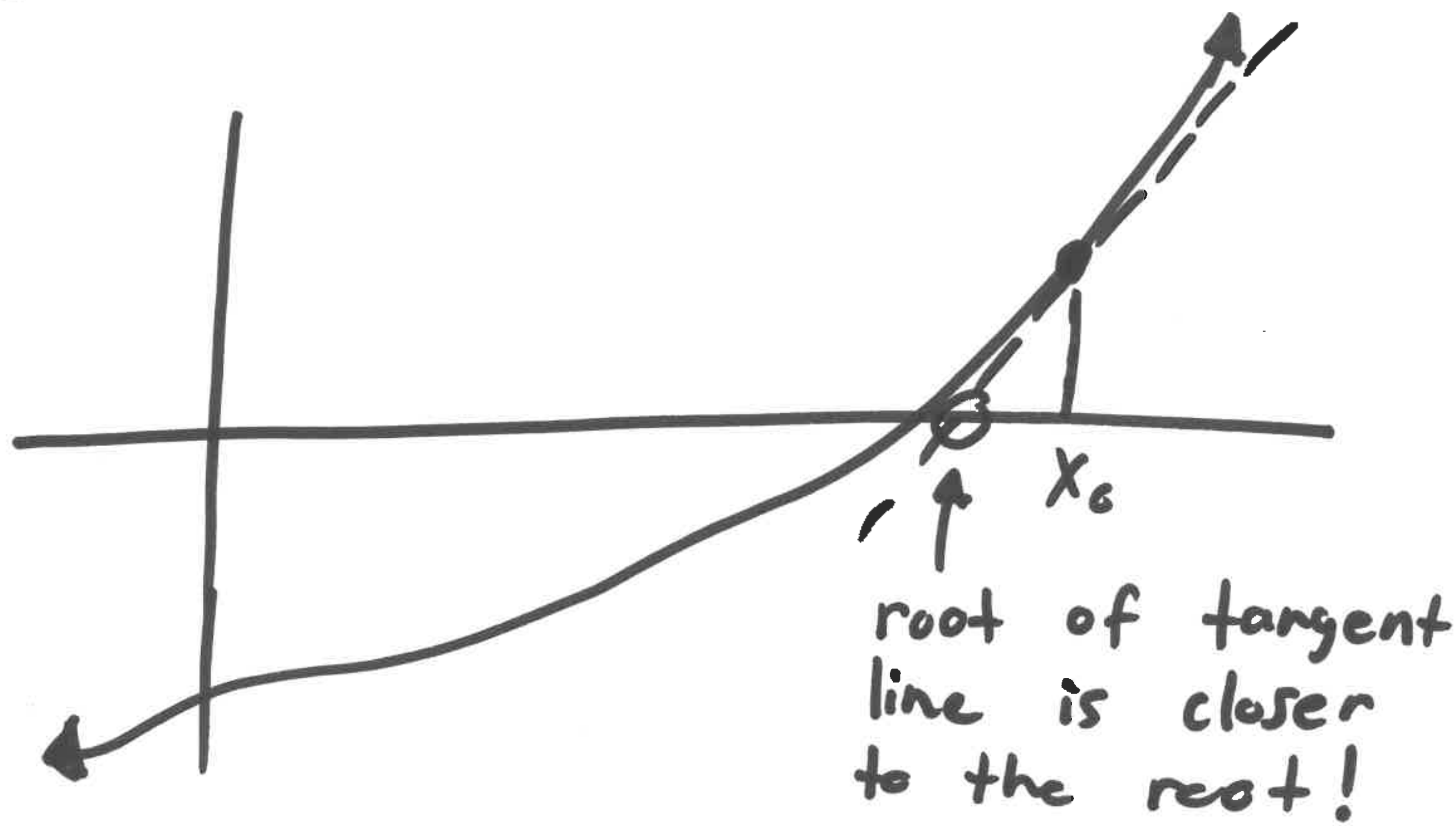
Lecture #22

10/30/23

Jonathan Lindbloom

Newton's method is a root-finding method that uses gradient information to help find the roots of a function.

Idea:



Start at x_0. Build linearization of $f(x)$ around x_0, $L_0(x) = f(x_0) + f'(x_0)(x - x_0)$.

What is the root of the linearization?

$$L_0(x) = 0 \Rightarrow f(x_0) + f'(x_0)(x - x_0) = 0$$

$$\Rightarrow x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

So take our approximate root to be $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Rinse and repeat! $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.

Hopefully it gets better.

Game: In 5 minutes, try to calculate the solution to $\cos(2x) = x$ to as many correct decimal places as possible! (using Newton's method)

$$x \approx 0.514933264661129413 80$$

We just discussed Newton's method for root-finding of a function $f(x)$,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Can we also use it for optimization?

i.e., to solve

$$x^* = \underset{x}{\operatorname{argmax}}\ g(x)\ ?$$

Yes! Want to find roots of $g'(x)$; applying Newton's method to $g'(x)$ we get the iteration

$$x_{k+1} = x_k - \frac{g'(x_k)}{g''(x_k)}.$$

Antiderivatives

Def: A function $F(x)$ is called an antiderivative of $f(x)$ on an interval I if $F'(x) = f(x)\ \forall x \in I$.

Prop: If $F_1(x)$ and $F_2(x)$ are both antiderivatives of $f(x)$ on I, then $F_1(x) = F_2(x) + C$ on I for some $C \in \mathbb{R}$.

Q: What are the antiderivatives of the following functions?

$f(x) = \cos(x) \Rightarrow F(x) = \sin(x) + C$

$f(x) = 2x \Rightarrow F(x) = x^2 + C$

$f(x) = e^x \Rightarrow F(x) = e^x + C$

$f(x) = \sec^2(x) \Rightarrow F(x) = \tan(x) + C$

$f(x) = \dfrac{1}{1+x^2} \Rightarrow F(x) = \arctan(x) + C$

Q: What is the antiderivative of

$f(x) = x^n$?

If $n \neq -1$,

$$f(x) = x^n \Rightarrow F(x) = \frac{1}{n+1} x^{n+1} + C$$

If $n = -1$,

$$f(x) = \frac{1}{x} \Rightarrow F(x) = \ln|x| + C.$$

If $f(x) = g(x) + h(x)$ and $G(x)$, $H(x)$ are the antiderivatives of g + h, then the antiderivative of $f(x)$ is

$$F(x) = G(x) + H(x) + C.$$

If $f(x) = c\,g(x)$ and $G(x)$ is the antiderivative of g, then the antiderivative of $f(x)$ is

$$F(x) = c\,G(x) + C.$$

"Inverse" Chain Rule:

If $f(x) = g'(h(x)) \cdot h'(x)$ for some functions $g(x)$ and $h(x)$, then the antiderivative of $f(x)$ is

$$F(x) = g(h(x)).$$

Q: What is the antiderivative of the following functions?

$$f(x) = 2x\cos(x^2) \Rightarrow F(x) = \sin(x^2) + C$$

$$f(x) = 4e^{4x} \Rightarrow F(x) = e^{4x} + C$$

$$f(x) = \frac{5}{2}x^{\frac{3}{2}} \Rightarrow F(x) = x^{\frac{5}{2}} + C$$

$$f(x) = \sin(x)\cos(\cos(x)) \cdot \sin(\sin(\cos(x))) \Rightarrow F(x) = \cos(\sin(\cos(x))) + C$$

$$f(x) = \frac{3}{9x^2+1} \Rightarrow F(x) = \arctan(3x) + C$$

"Inverse" Product Rule:

If $f(x) = g'(x)h(x) + g(x)h'(x)$

for some functions $g(x)$ and $h(x)$,

then the antiderivative of $f(x)$ is

$$F(x) = g(x)h(x) + C.$$

Q: What are the antiderivatives of the following functions?

$$f(x) = \sin(x) + x\cos(x) \Rightarrow F(x) = x\sin(x) + C$$

$$f(x) = xe^x + e^x \Rightarrow F(x) = xe^x + C$$

$$f(x) = \cos^2(x) - \sin^2(x) \Rightarrow F(x) = \sin(x)\cos(x) + C$$

"Inverse" Quotient Rule:

If $f(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$

for some functions $g(x)$ & $h(x)$, then the antiderivative of $f(x)$ is

$$F(x) = \frac{g(x)}{h(x)} + C.$$

Q: What is the antiderivative of

$$f(x) = \frac{e^x(\cos(x)+\sin(x))}{\cos^2(x)} ?$$

A: $F(x) = \dfrac{e^x}{\cos(x)}$. Check by differentiating!

Multiple antidifferentiation:

Let $f(x)$ be a function and $F(x)+C_1$ denote its antiderivatives.

What is the antiderivative of $F(x)+C_1$? If $G'(x)=F(x)$, then the antiderivative of $F(x)+C_1$ is

$$H(x) = G(x) + C_1 x + C_2 .$$

Relation to linear motion?

Velocity is derivative of position
$\Rightarrow$ position is antiderivative of velocity.

Also, position is second antiderivative of acceleration.

$$v(t) = k$$
$$\Rightarrow p(t) = kt + C$$

$$a(t) = k$$
$$\Rightarrow v(t) = kt + C_1$$
$$\Rightarrow p(t) = \frac{k}{2}t^2 + C_1 t + C_2.$$

Q: If a particle has constant acceleration $a(t) = k$, initial position $p(0) = 1$, and initial velocity $v(0) = 1$, what is the position function?

A: $a(t) = k \Rightarrow v(t) = kt + C_1$

$\Rightarrow p(t) = \frac{k}{2}t^2 + C_1 t + C_2$.

$p(0) = 1 \Rightarrow 0 + 0 + C_2 = 1 \Rightarrow C_2 = 1$.

$v(0) = p'(0) = 1 \Rightarrow k(0) + C_1 = 1 \Rightarrow C_1 = 1$.

So $p(t) = \frac{k}{2}t^2 + t + 1$.

MATH 3

Lecture #23

11/1/23

Jonathan Lindbloom

Computing

Areas

Summation notation:

If we have a sum of a sequence of numbers

$$a_1 + a_2 + \cdots + a_{n-1} + a_n,$$

we can abreviate this by

$$\sum_{i=1}^{n} a_i.$$

Ex: If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, then $f(x) = \sum_{i=0}^{n} a_i x^i$.

Q: What is the sum

$$1 + 2 + 3 + \cdots + N-1 + N \,?$$

A: *Idea: play a matching game.

$$\left.\begin{array}{lcl}
(0) + N & = & N \\
1 + N-1 & = & N \\
2 + N-2 & = & N \\
\vdots & & \\
N/2 - 1 + N/2 + 1 & = & N
\end{array}\right\}$$

$$N/2 \quad = N/2$$

there are $\frac{N}{2}$ N's here, sum to $\frac{N^2}{2}$

add

$$\frac{N^2}{2} + \frac{N}{2} = \boxed{\frac{N(N+1)}{2}}$$

* Assuming N even!

Some Sums:

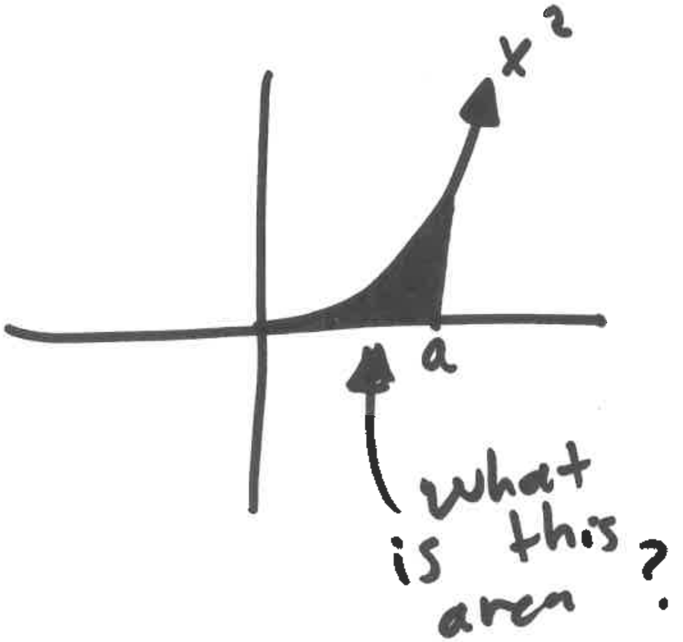
$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

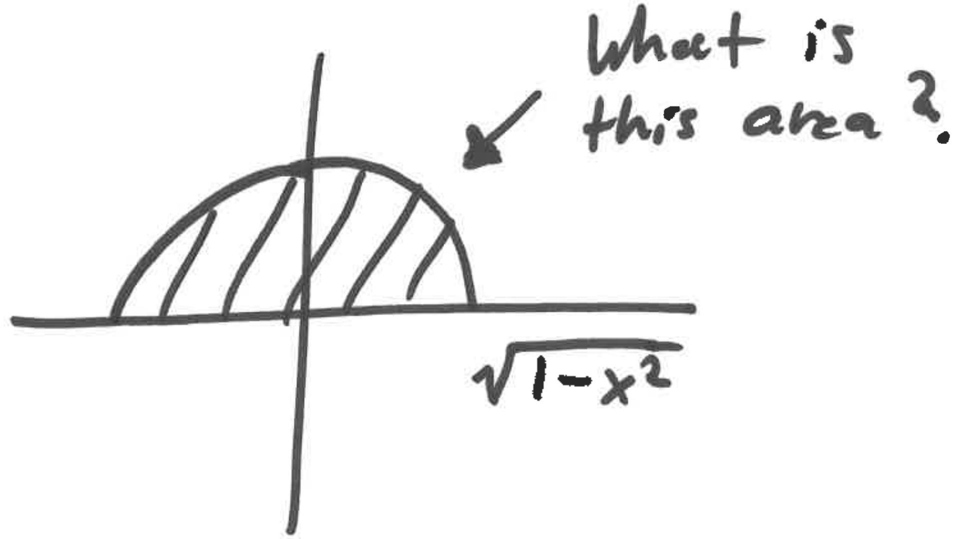
$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{i=1}^{N} i^3 = \left[\frac{N(N+1)}{2}\right]^2$$

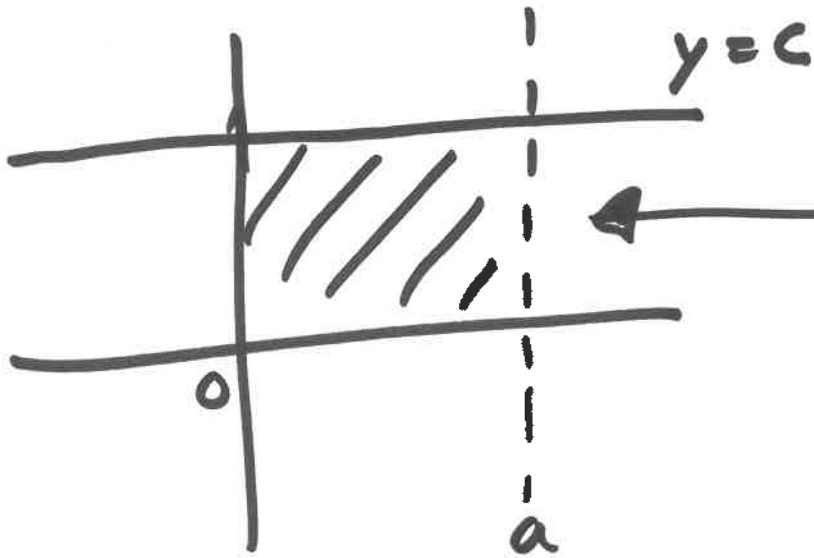
$\displaystyle\sum_{i=0}^{N-1} r^i =$ [redacted] $\displaystyle = \frac{1-r^N}{1-r}.$

Question: How can we estimate areas using limits?



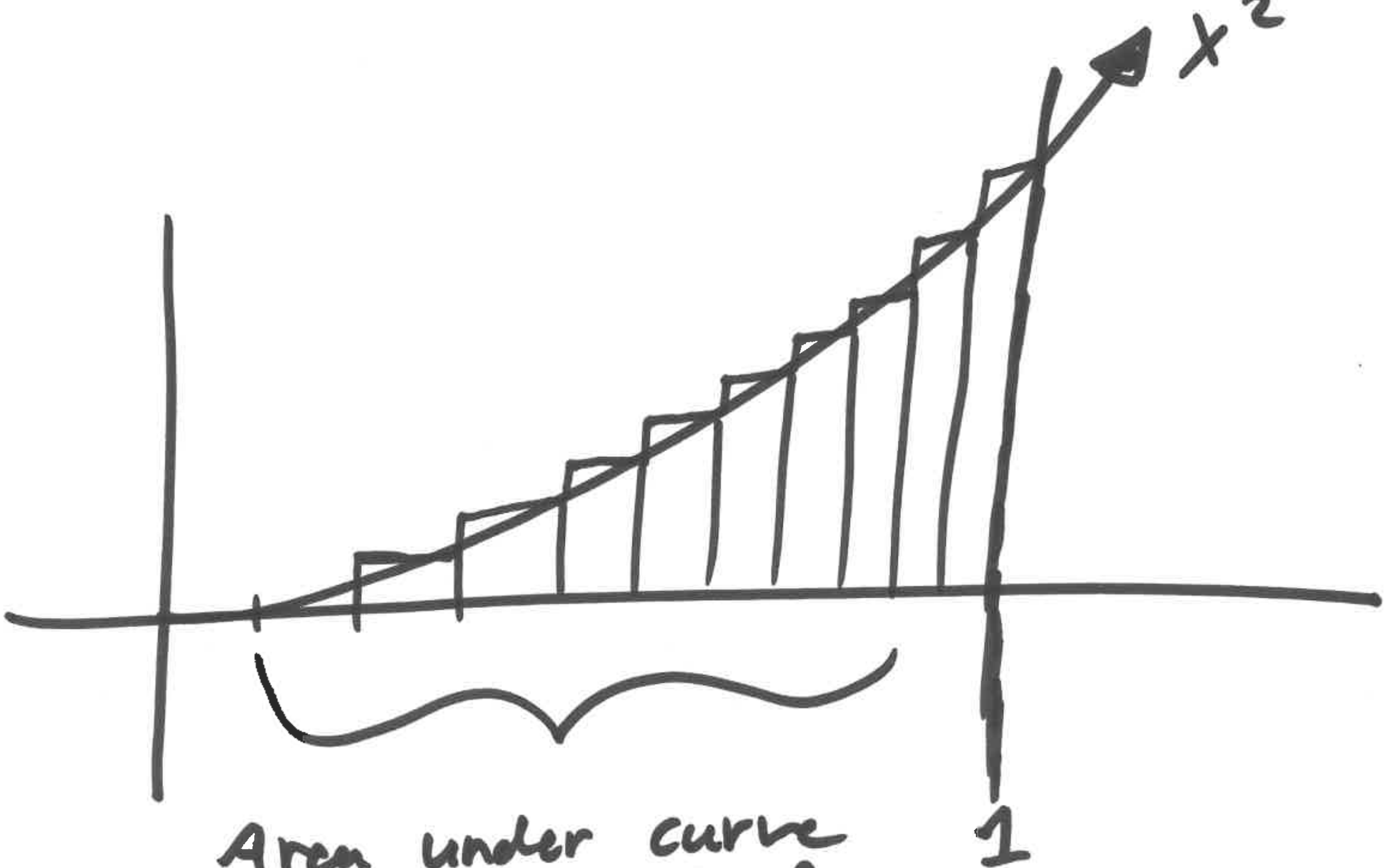


Simpler question: What is the area bounded by a constant function?



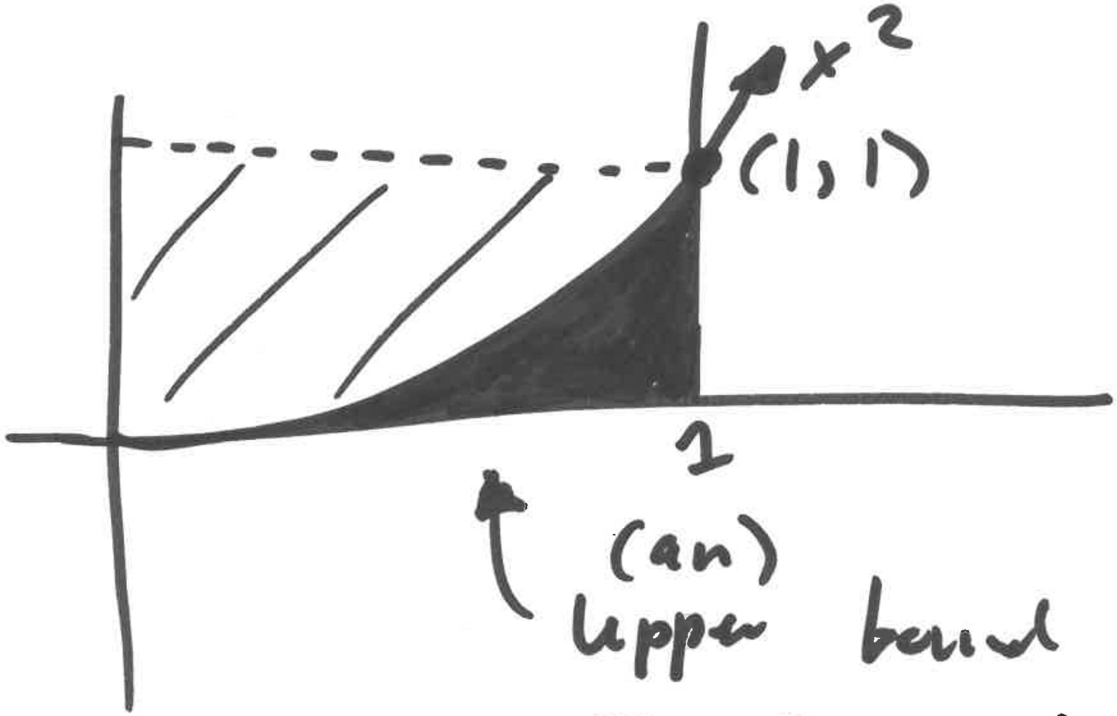
area is $= c \cdot a$.

Idea: approximate area under a curve with a suitable set of rectangles.



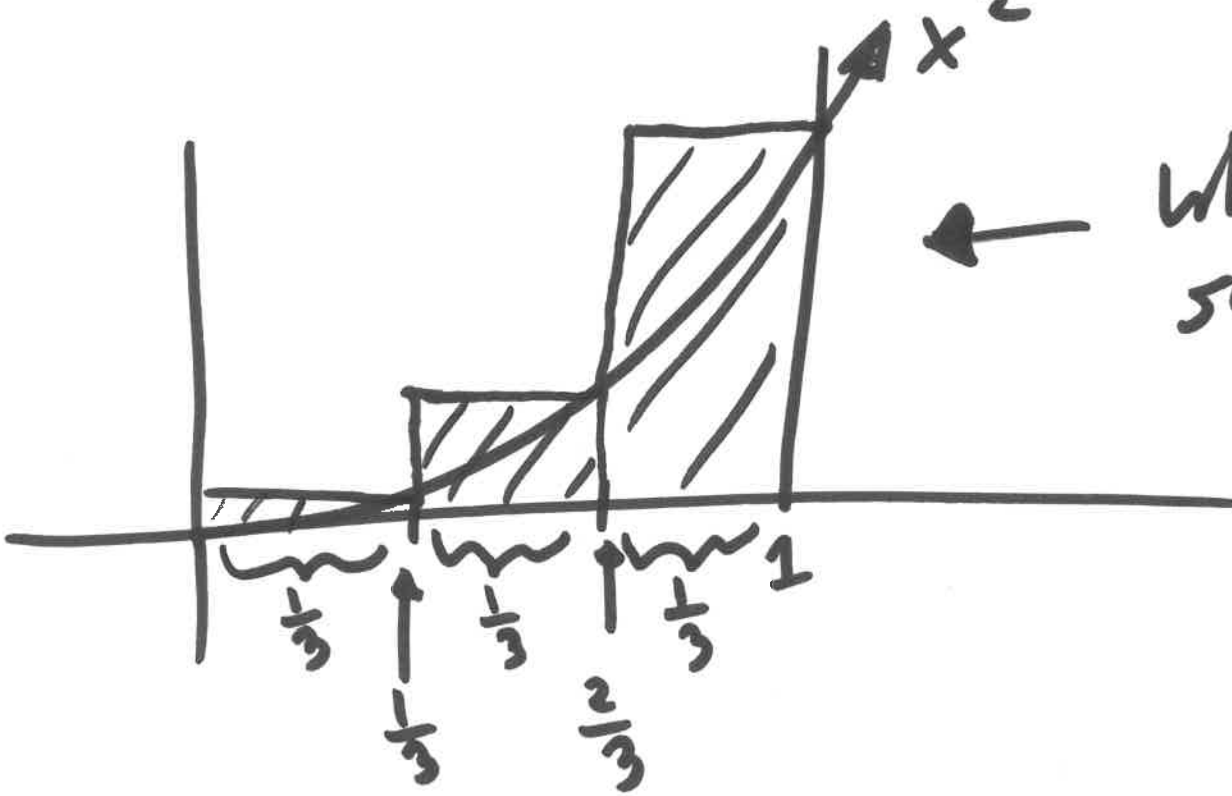
Area under curve is $\approx \sum$ area of rectangles

Using just 1 rectangle? Upper bound on area under curve?



(an) upper bound on area is 1!

Using just 3 rectangles?



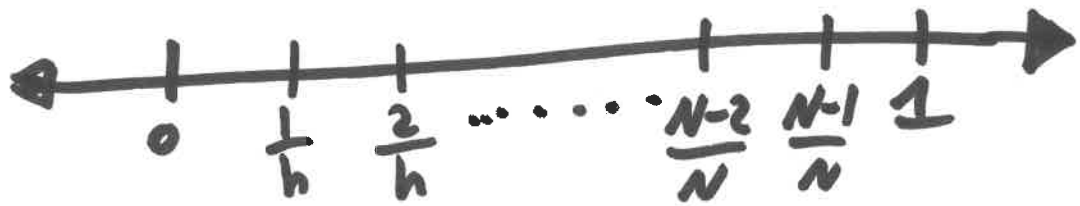
What is the sum of the areas?

$$\sum \text{areas} = \frac{1}{3} \times \left(\frac{1}{3}\right)^2 + \frac{1}{3} \times \left(\frac{2}{3}\right)^2 + \frac{1}{3} \times (1)^2$$

$$= \frac{14}{27} \approx 0.5185$$

Take the interval $[0,1]$ and partition it into N equal-sized subintervals.

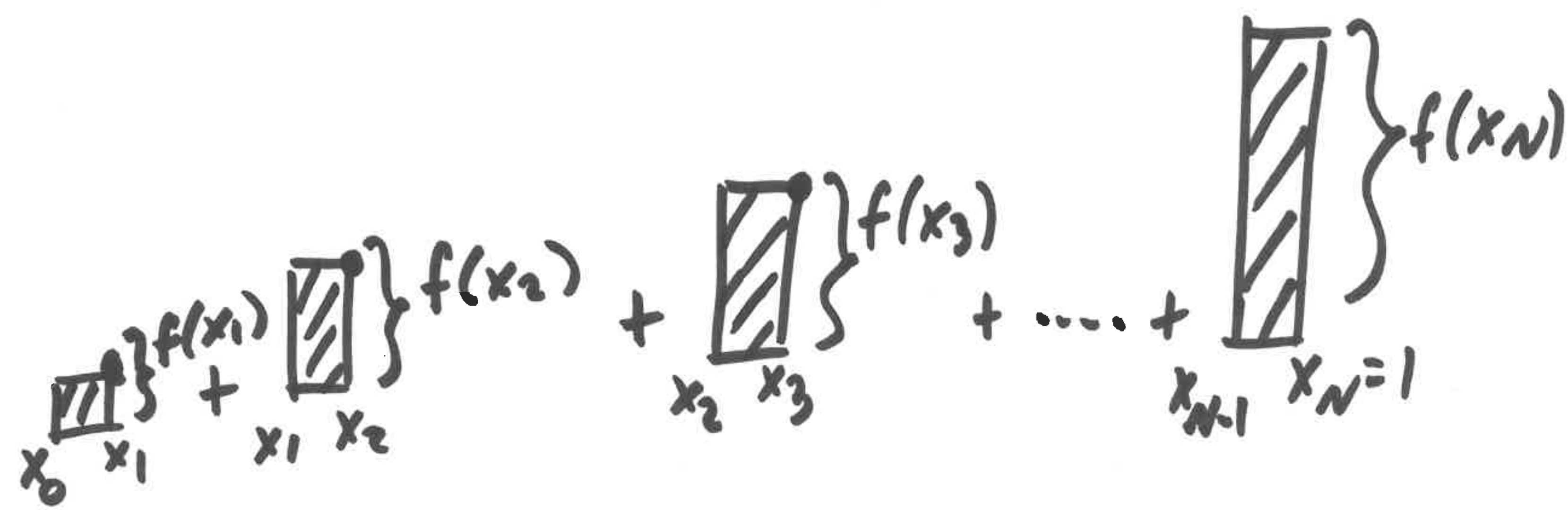
Define $h = \frac{1}{N}$ & $x_i = \frac{i}{N}$ for $i = 0, \dots, N$.



- All rectangles will have the same base width of $\frac{1}{h}$.

- What is height of rectangle? Must make a decision, e.g., left or right endpoint.

Right endpoint? Height of the rectangle on interval $[x_i, x_{i+1}]$ is $f(x_{i+1})$.



How to express the sum?

Sigma notation!

$$A = \sum_{i=1}^{N} \frac{1}{N} \cdot f(x_i) = \frac{1}{N} \sum_{i=1}^{N} x_i^2$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left(\frac{i}{N}\right)^2 = \frac{1}{N^3} \sum_{i=1}^{N} i^2 .$$

$$A = \frac{1}{N^3} \sum_{i=1}^{N} i^2 = \frac{1}{N^3} \cdot \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{(N+1)(2N+1)}{6N^2} = \frac{2N^2+3N+1}{6N^2}.$$

<u>Question</u>: What happens as we send # of rectangles N to $+\infty$?

$$\lim_{N \to +\infty} \left[\frac{2N^2+3N+1}{6N^2} \right] = \frac{2}{6} = \frac{1}{3}$$

In this example, does the choice of where we evaluate $f(x)$ on each $[x_i, x_{i+1}]$ matter?

Let's re-do this example, but now use the midpoint rule, which is to pick the height of each rectangle to be $\dfrac{f(x_i)+f(x_{i+1})}{2}$.

Will this change the sum as $N \to \infty$?

Why is midpoint rule valid? Since the IVT!

The total area is

$$A = \sum_{i=0}^{N-1} \left[\frac{1}{N} \cdot \frac{f(x_i) + f(x_{i+1})}{2} \right]$$

$$= \frac{1}{2N} \sum_{i=0}^{N-1} \left[\left(\frac{i}{N}\right)^2 + \left(\frac{(i+1)}{N}\right)^2 \right]$$

$$= \frac{1}{2N} \left(\frac{1}{N^2} \sum_{i=0}^{N-1} i^2 + \frac{1}{N^2} \sum_{i=0}^{N-1} (i+1)^2 \right)$$

$$= \frac{1}{2N} \left(\frac{1}{N^2} \sum_{i=1}^{N-1} i^2 + \frac{1}{N^2} \sum_{i=1}^{N} i^2 \right)$$

$$= \frac{1}{2N^3} \cdot \left(\frac{(N-1)N(2(N-1)+1)}{6} + \frac{N(N+1)(2N+1)}{6} \right)$$

$$= \frac{1}{2N^3}\left(\frac{(N-1)N(2N-1) + N(N+1)(2N+1)}{6}\right)$$

$$= \frac{1}{12N^2}\cdot\left((N-1)(\cancel{N}2N-1) + (N+1)(2N+1)\right)$$

$$= \frac{1}{12N^2}\left(2N^2-3N+1 + 2N^2+3N+1\right)$$

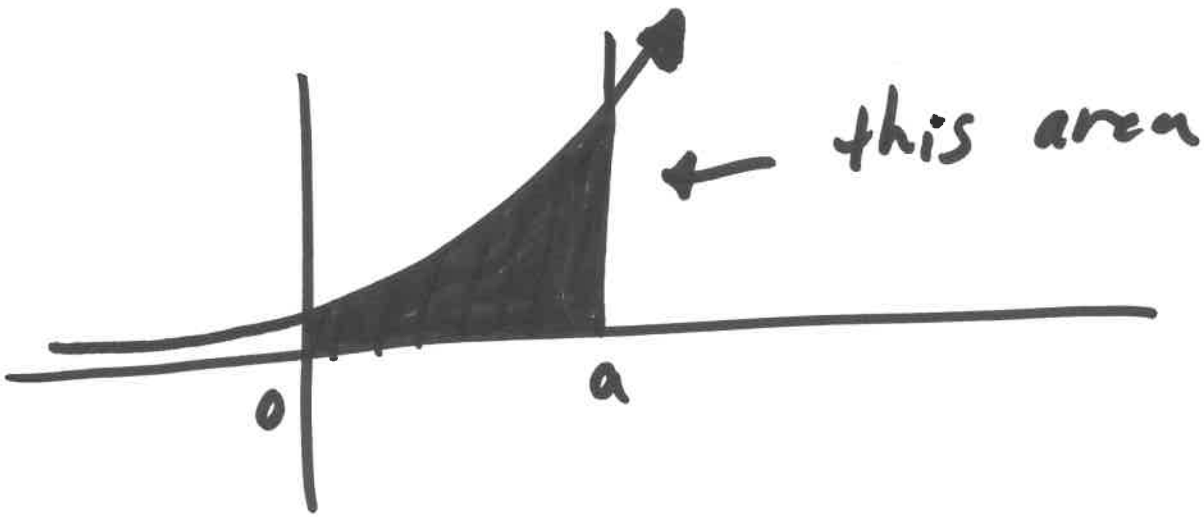
$$= \frac{4N^2 + 2}{12N^2}$$

$$= \frac{2N^2+1}{6N^2}. \qquad \cancel{A} \lim_{N\to\infty} \frac{2N^2+1}{6N^2} = \frac{2}{6}$$

$$= \frac{1}{3}$$ The same!

Let's do a trickier example.

Let $f(x) = e^x$, and suppose we want the area under the curve from $x = 0$ to $x = a$.



Same idea. Split the interval $[0, a]$ into N equal-sized parts. What are the endpoints?

$$x_i = \frac{ia}{N}, \quad i = 0, \ldots, N.$$

$x_0 = 0$, $x_N = a$.

Let's use the left endpoints to determine the height of each rectangle, i.e., the rectangle on interval $[x_i, x_{i+1}]$ to have height

$$f(x_i) = e^{x_i}.$$

What is the sum of all the rectangles?

$$A = \sum_{i=0}^{N-1} \underbrace{\left(\frac{a}{N}\right)}_{\text{base width}} \underbrace{f(x_i)}_{\text{height}}$$

$$= \frac{a}{N} \sum_{i=0}^{N-1} e^{x_i} = \frac{a}{N} \sum_{i=0}^{N-1} e^{\frac{ia}{N}}$$

$$= \frac{a}{N} \sum_{i=0}^{N-1} \left(e^{\frac{a}{N}}\right)^i$$

Can we evaluate this sum?

Yes! Using geometric series formula.

$$= \frac{a}{N} \frac{1-(e^{\frac{a}{N}})^N}{1-e^{\frac{a}{N}}} = \frac{a}{N} \frac{1-e^a}{1-e^{\frac{a}{N}}}.$$

Now, what is the limit as $N \to \infty$?

$$\lim_{N\to\infty} \left[\frac{a}{N} \cdot \frac{1-e^a}{1-e^{\frac{a}{N}}} \right] = a(1-e^a) \cdot \left[\lim_{N\to\infty} N(1-e^{\frac{a}{N}}) \right]^{-1}.$$

What is $\lim_{N\to\infty} N(1-e^{\frac{a}{N}})$?

L'Hopital's Rule! Is of form $\infty \cdot 0$.

$$\lim_{N\to\infty} N\left(1-e^{\frac{a}{N}}\right) = \lim_{N\to\infty} \frac{1-e^{\frac{a}{N}}}{\frac{1}{N}}$$

$$= \lim_{N\to\infty} \frac{1-e^{aN^{-1}}}{N^{-1}} = \lim_{N\to\infty} \frac{-e^{aN^{-1}} \cdot a \cdot \cancel{-1 \cdot N^{-2}}}{\cancel{-N^{-2}}}$$

$$= \lim_{N\to\infty} -ae^{\frac{a}{N}} = -a.$$

So the limit of the area as $N\to\infty$ is

$$a(1-e^a)\cdot[-a]^{-1} = -(1-e^a) = e^a - 1.$$

$$(= e^a - e^0)$$

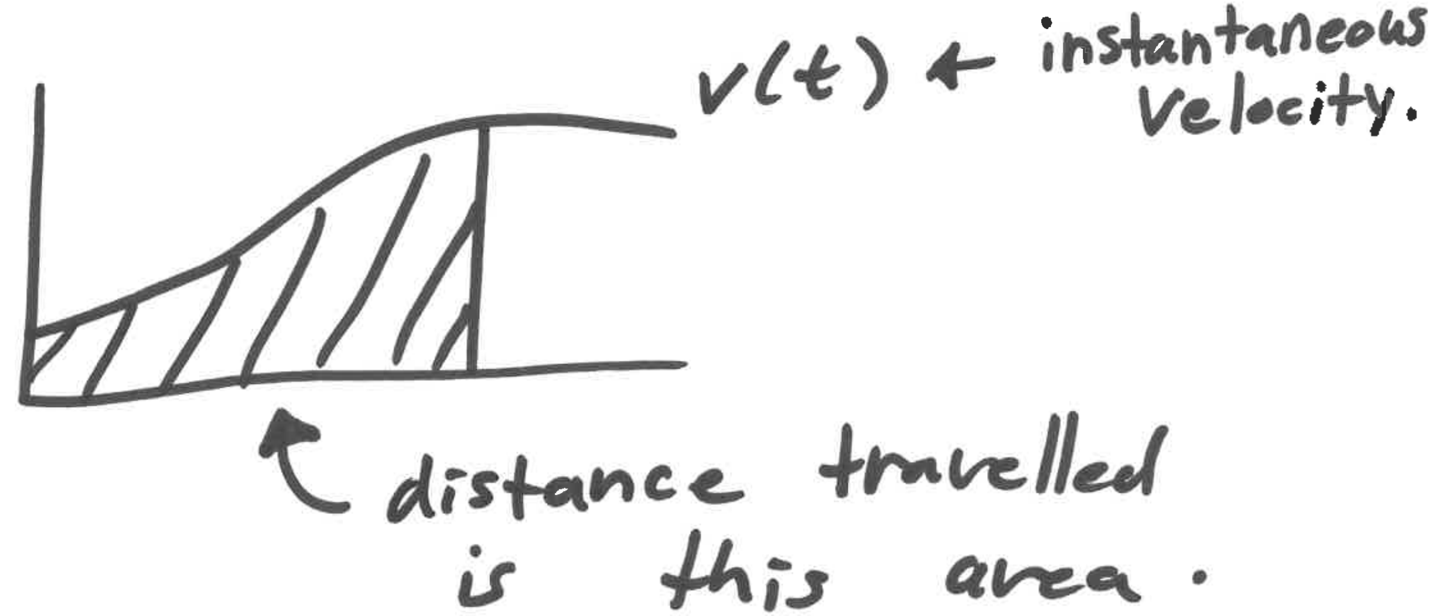
Velocity

&

Distance

Question: Given initial position and velocity, can you determine distance travelled?

★ **Fact:**



How to approximate distance travelled?
Estimate the area under this curve.

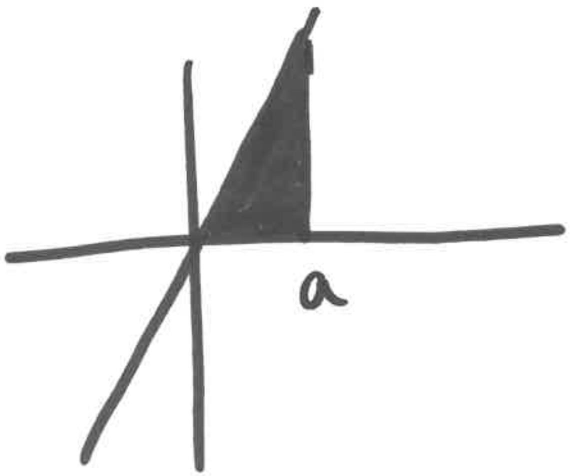
MATH 3

Lecture #24

11/3/23

Jonathan Lindbloom

Ex: What is the area under the curve $f(x) = 3x$ from $x=0$ to $x=a$?
(above x-axis)



Partition the interval $[0,a]$ into N equal subintervals.

The ith interval $[x_i, x_{i+1}]$ looks like $\left[\frac{i}{N}a, \frac{i+1}{N}a\right]$.

Build our Riemann sum approximation: (right endpoint)

$$A_N = \sum_{i=1}^{N} \underbrace{\left(\frac{a}{N}\right)}_{\text{base}} \cdot \underbrace{f(x_i)}_{\text{height}} = \sum_{i=1}^{N} \frac{a}{N} \cdot 3x_i$$

$$= \sum_{i=1}^{N} \frac{a}{N} \cdot 3 \cdot \frac{i}{N} a = \sum_{i=1}^{N} \frac{3a^2}{N^2} \cdot i$$

$$= \frac{3a^2}{N^2} \sum_{i=1}^{N} i = \frac{3a^2}{N^2} \cdot \frac{N(N+1)}{2}$$

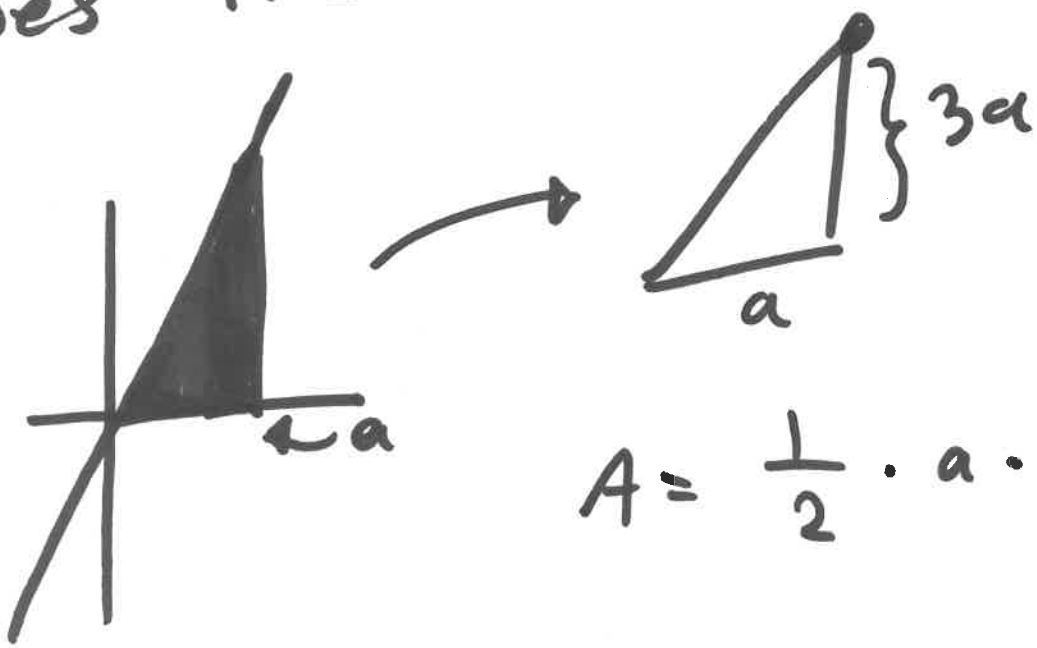
$$= \frac{3a^2 N^2 + 3a^2 N}{2N^2}$$

What happens as $N \to \infty$?

$$\lim_{N\to\infty} A_N = \lim_{N\to\infty} \frac{3a^2N^2 + 3a^2N}{2N^2}$$

$$= \frac{3a^2}{2}$$

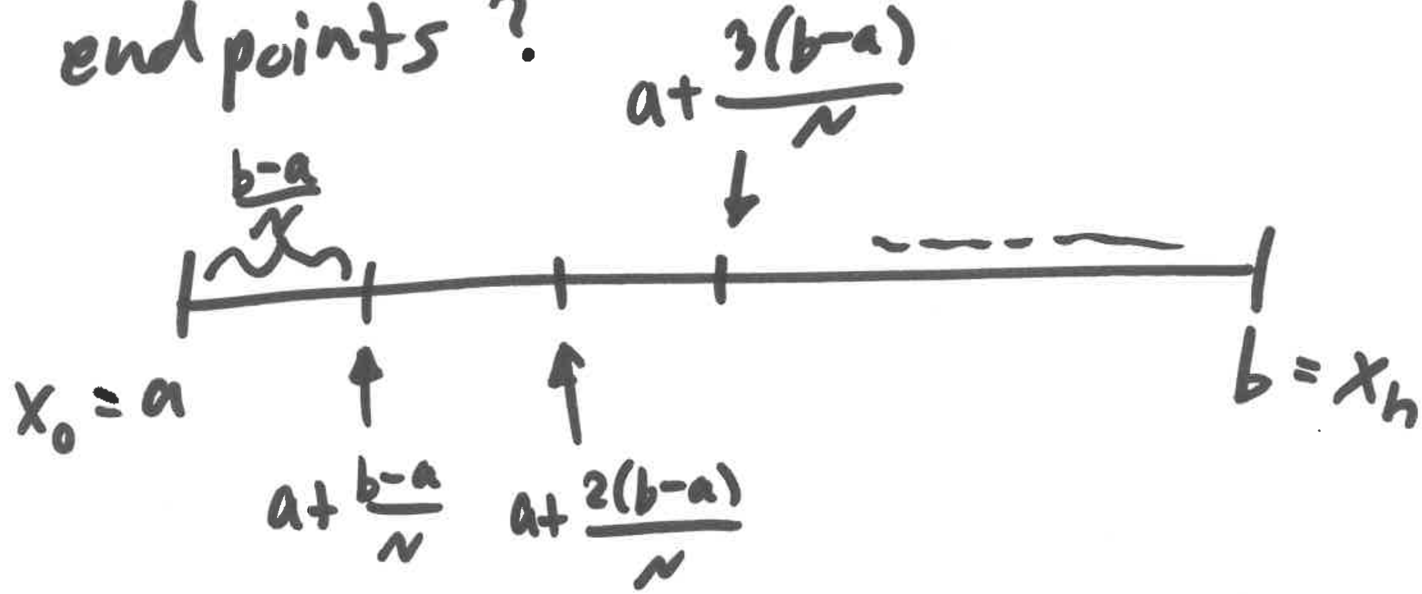
Does this make sense?



$$A = \frac{1}{2} \cdot a \cdot 3a = \frac{3a^2}{2}$$ ✓

The Definite Integral

If we divide the interval $[a,b]$ into n equal (-lengthed) subintervals, what are the endpoints?



$$x_i = a + \frac{i}{n}(b-a), \quad i = 0, \ldots, n.$$

Definite integral: Let $f(x) : [a,b] \to \mathbb{R}$.

→ Divide $[a,b]$ into n equal subintervals with endpoints $x_i = a + \frac{i}{n}(b-a)$, $i = 0, \ldots, n$.

→ For each subinterval $[x_{i-1}, x_i]$, pick arbitrarily a "sample point" $x_i^* \in [x_{i-1}, x_i]$.

→ Then we write that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \underbrace{\frac{(b-a)}{n}}_{(\Delta x)}.$$

if the limit exists and is identical under any choice of the x_i^*. We say that $\int_a^b f(x)dx$ is the definite integral of f from a to b, and that $f(x)$ is integrable on $[a,b]$.

$*$ Limit definition:

$\forall \varepsilon > 0,\ \exists N$ s.t. $n > N$ AND any x_i^*,

$$\Rightarrow \left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*) \overbrace{\frac{b-a}{n}}^{(\Delta x)} \right| < \varepsilon .$$

We can make the sum as close to the integral as we like by using more + thinner rectangles.

upper limit

$$\int_a^b$$

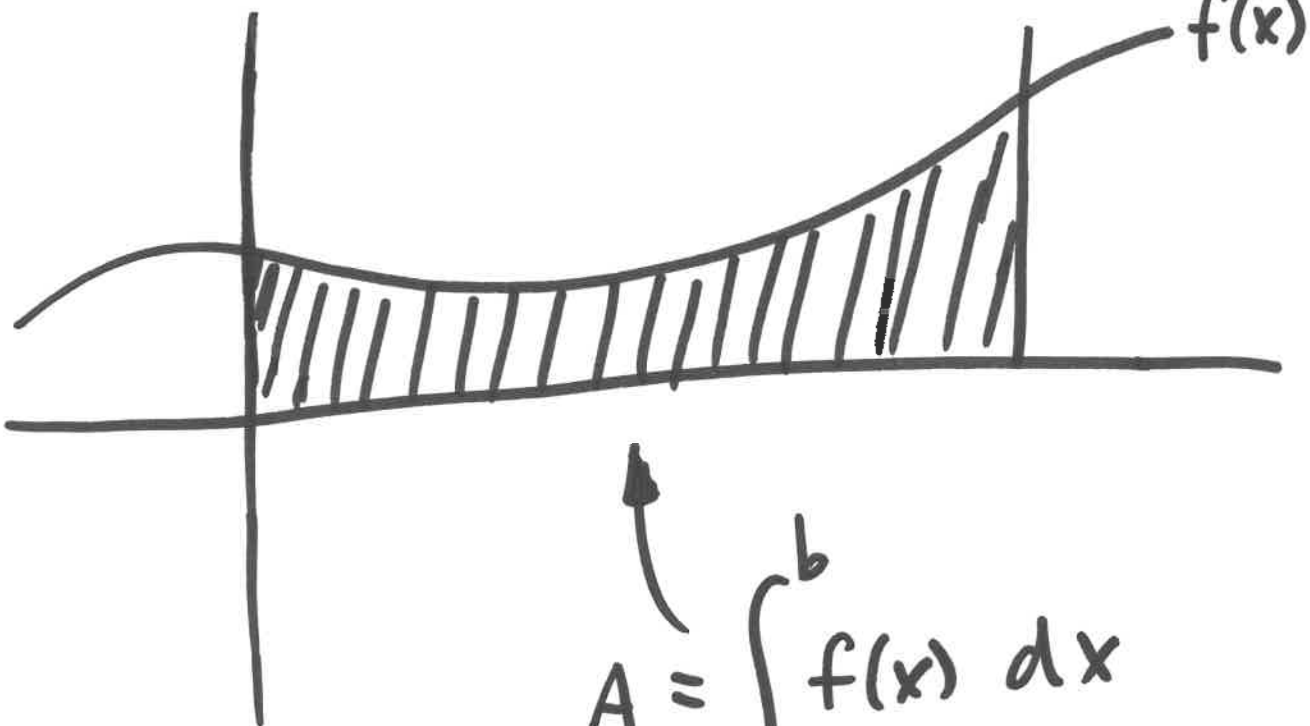
integral sign

limits of integration

lower limit

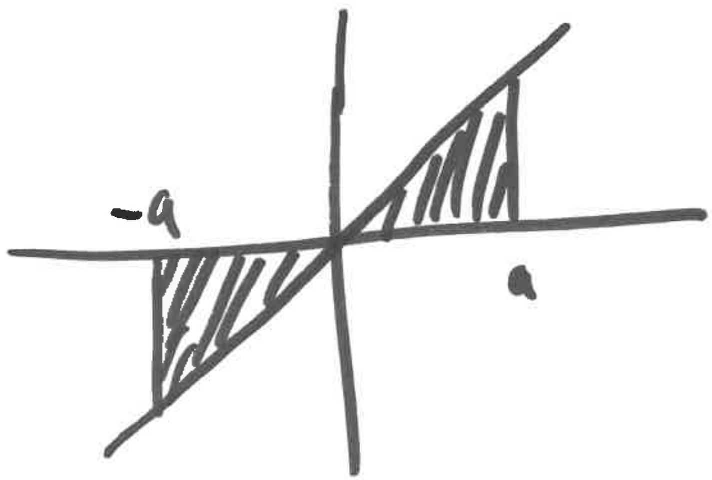
f(x) is the integrand

dx tells us the variable of integration



$$A = \int_a^b f(x)\, dx$$

What is the integral of $f(x) = x$ from $x = -a$ to $x = a$?



Let's approximate with a Riemann sum.

① Split the interval $[-a, a]$ into n equal-length subintervals. Endpoints are $x_i = -a + \frac{i}{n} \cdot 2a$.

② Right endpoint method: $A_n = \sum_{i=1}^{n} \left(\frac{2a}{n}\right) \cdot f(x_i)$

$$= \sum_{i=1}^{n} \left(\frac{2a}{n}\right)\left(-a + \frac{2a}{n} i\right)$$

$$= \sum_{i=1}^{n} \left[-\frac{2a^2}{n} + \frac{4a^2}{n^2} i\right]$$

$$= \sum_{i=1}^{n} -\frac{2a^2}{n} + \sum_{i=1}^{n} \frac{4a^2}{n^2} i$$

$$= -\frac{2a^2}{n} \sum_{i=1}^{n} 1 + \frac{4a^2}{n^2} \sum_{i=1}^{n} i$$

$$= -\frac{2a^2}{n} \cdot n + \frac{4a^2}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= -2a^2 + \frac{2a^2(n+1)}{n}$$

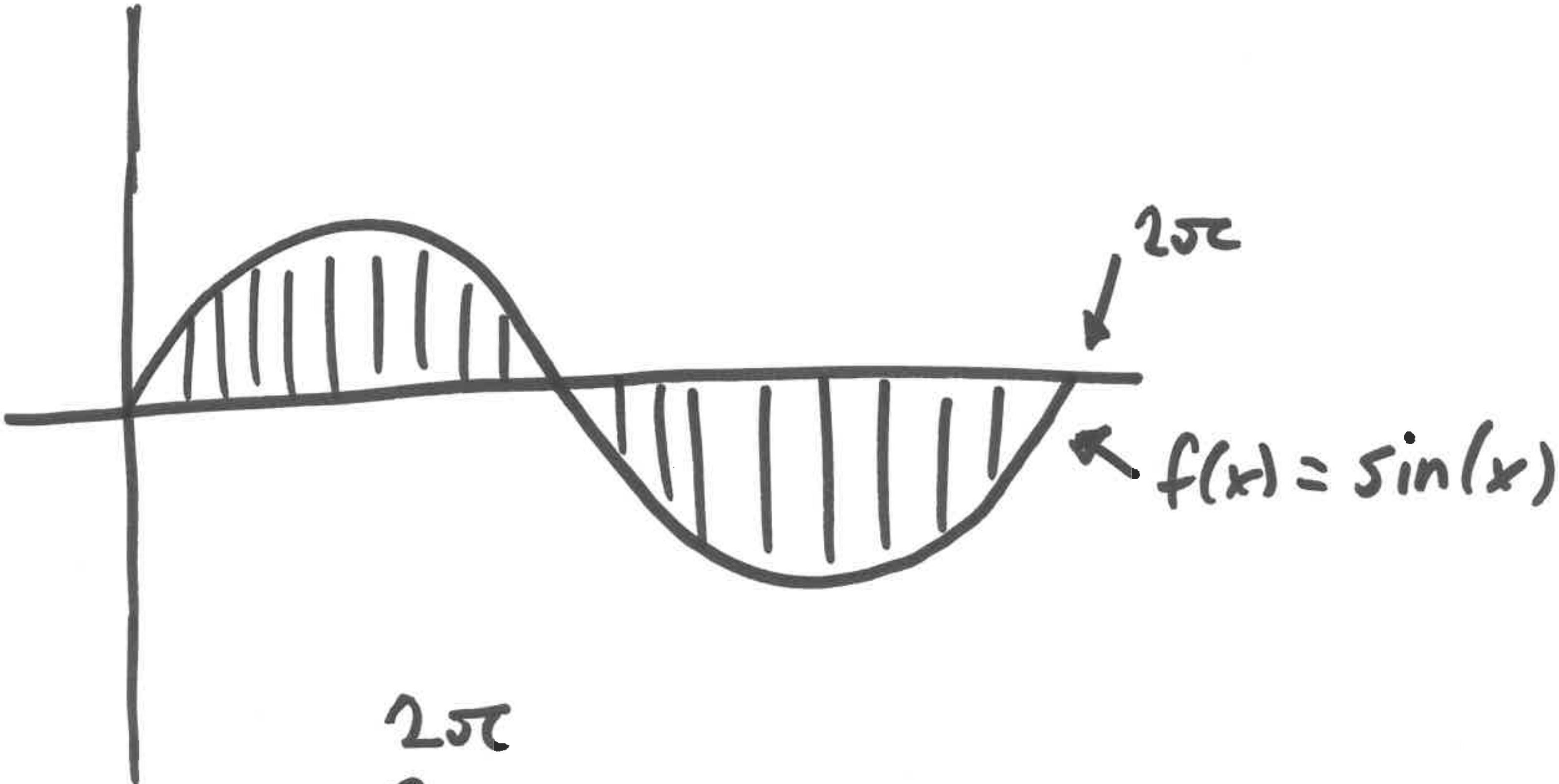
What is the limit as $n \to \infty$?

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[-2a^2 + \frac{2a^2(n+1)}{n}\right]$$

$$= -2a^2 + 2a^2$$

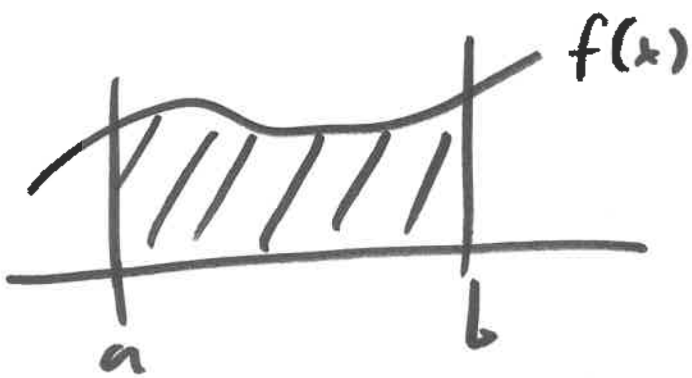
$$= 0.$$

So $\int_{-a}^{a} x \, dx = 0$ for all choices of a.



$$\int_0^{2\pi} \sin(x)\,dx = 0$$

Thm: If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$, i.e., $\int_a^b f(x)\,dx$ exists.



Some properties:

$\boxed{1}$ $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$\boxed{2}$ $$\int_a^a f(x)\,dx = 0$$

$\boxed{3}$ $$\int_a^b c\,dx = c\int_a^b 1\,dx = c(b-a)$$

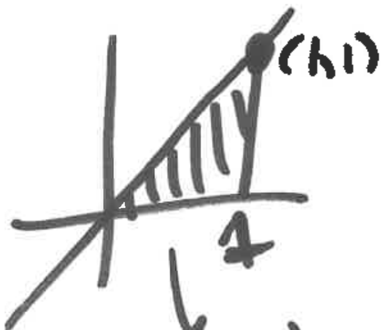
$\boxed{4}$ $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$\boxed{5}$ $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

<u>Q:</u> What is $\int_0^1 x + x^2 \, dx$?

<u>A:</u> $= \int_0^1 x \, dx + \int_0^1 x^2 \, dx$.

last lecture, we saw this was $\frac{1}{3}$



A= $\frac{1}{2}$

$= \frac{1}{2} + \frac{1}{3}$

$= \frac{5}{6}$

$\boxed{6}$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

for $a \le b \le c$.

$\boxed{7}$ $\forall x \in [a,b]\ f(x) \geq 0 \implies \int_a^b f(x)\,dx \geq 0$

$\boxed{8}$ $\forall x \in [a,b]\ f(x) \geq g(x) \implies \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$\boxed{9}$ $\forall x \in [a,b]\ m \le f(x) \le M \implies$

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Diagram for $\boxed{\#6}$

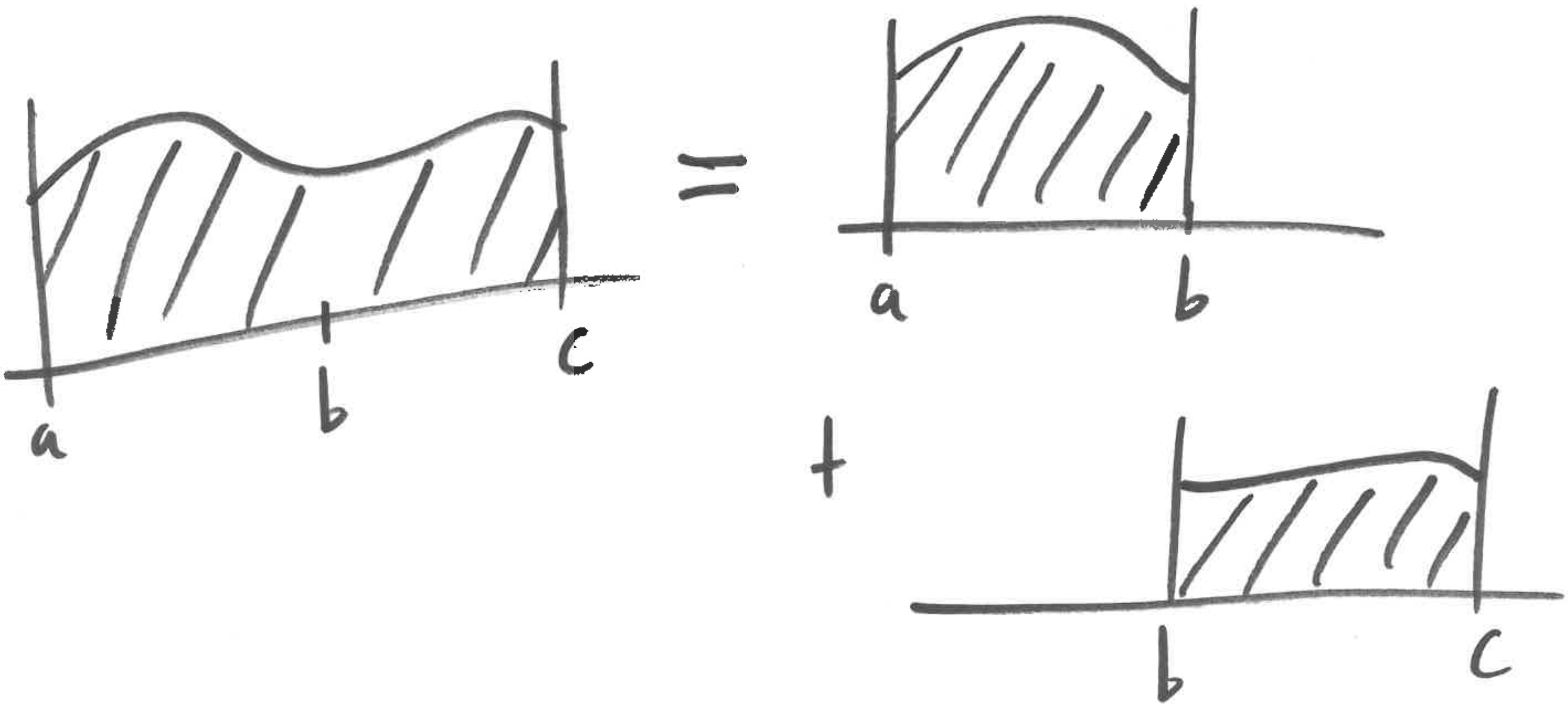
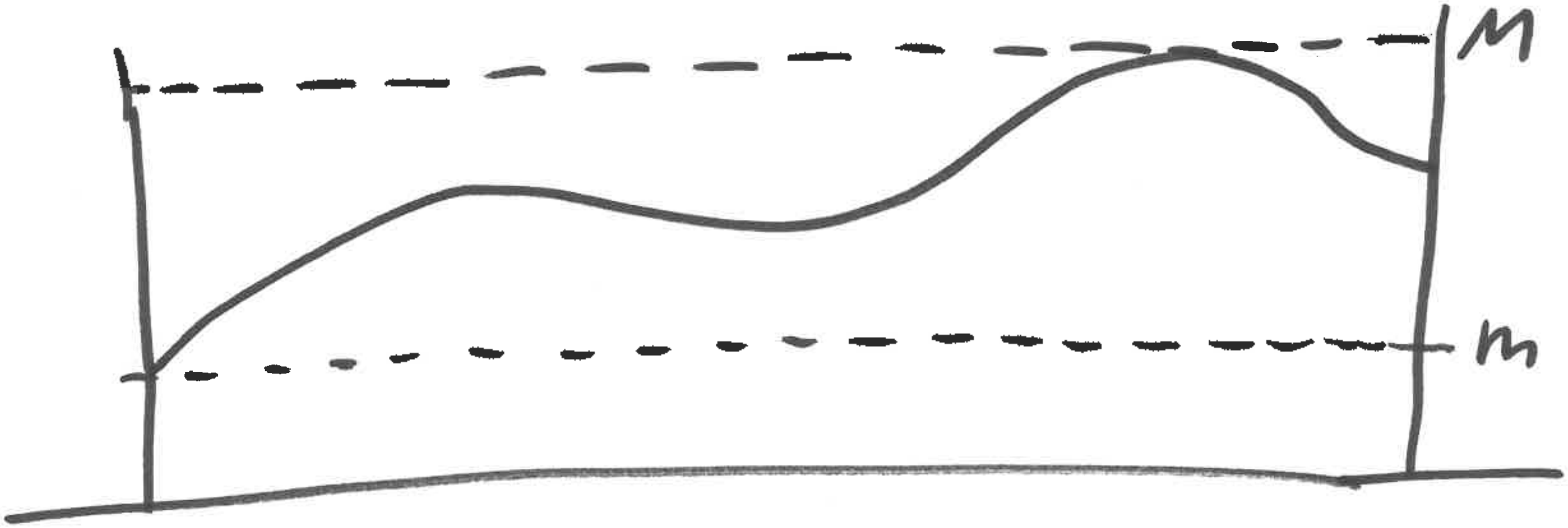
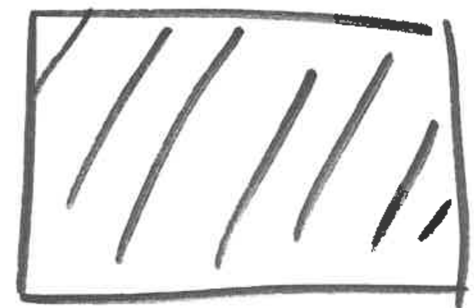


Diagram for $\boxed{\#9}$:



Is less than (bounded within)



Is greater than

Can you think of a bounded function $f(x): [a,b] \to \mathbb{R}$ that is <u>not</u> Riemann integrable over $[a,b]$?

Consider $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

Recall that $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n}$

if the limit exists, and recall that the limit needs to be the same for all choices of the x_i^* for the limit to exist.

Consider always choosing the sample points $x_i^* \in [x_{i-1}, x_i]$ to be rational numbers.

Then

$$\sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} = \frac{b-a}{n} \sum_{i=1}^{n} 1 = \frac{b-a}{n} \cdot n = b-a,$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} = b-a.$$

But now consider always choosing the sample points $x_i^* \in [x_{i-1}, x_i]$ to be irrational numbers.

Then $$\sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} = \frac{b-a}{n} \sum_{i=1}^{n} 0 = 0,$$

$\Rightarrow \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \frac{b-a}{n} = 0$! Since this is different, $f(x)$ is <u>not</u> Riemann integrable.

MATH 3

Lecture #25

11/6/23

Jonathan Lindbloom

Fundamental Theorem of Calculus

FTC, Part 2: If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of $f(x)$, i.e., any function $F(x)$ s.t. $F'(x) = f(x)$.

Q: What is $\int_2^6 \frac{1}{x^2}\,dx$?

A: This was from a written assignment! Let $f(x) = x^{-2}$.

Then $F(x) = -x^{-1}$ is an antiderivative of $f(x)$, and

$$\int_2^6 \frac{1}{x^2}\,dx = F(6) - F(2)$$
$$= -\frac{1}{6} - \left(-\frac{1}{2}\right)$$
$$= \frac{1}{3}$$

Q: What is $\int_0^1 x^2\,dx$?

A: We did this before using Riemann sums! Let $f(x) = x^2$. Then $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x)$, and by FTC (part 2) we have

$$\int_0^1 x^2\,dx = F(1) - F(0) \left(= \left[\tfrac{1}{3}x^3\right]_0^1 \right)$$
$$= \frac{1}{3} - 0$$
$$= \frac{1}{3}.$$

Q: What is $\int_0^a e^x \, dx$?

A: We computed this before using Riemann sums! Let $f(x) = e^x$. Then $F(x) = e^x$ is an antiderivative of $f(x)$, and by FTC (part 2) we have

$$\int_0^a e^x \, dx = F(a) - F(0)$$
$$= e^a - e^0$$
$$= e^a - 1.$$

Q: Let
$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ x^2, & 1 < x \le 2. \end{cases}$$

What is $\int_0^2 f(x)\,dx$?

A: strategy: break up integral onto subintervals.

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 x^2\,dx$$

$$= \left[\tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{3}x^3\right]_1^2 = \tfrac{1}{2}(1)^2 - \tfrac{1}{2}(0)^2 + \tfrac{1}{3}(2)^3 - \tfrac{1}{3}(1)^3$$

$$= \frac{1}{2} + \frac{8}{3} - \frac{1}{3} = \frac{17}{6}.$$

FTC, Part 1: If f is continuous on $[a,b]$, then the function $g(x)$ defined by

$$g(x) = \int_a^x f(t)\,dt \quad , \quad a \leq x \leq b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Freshel function: Consider $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$.

What is $S'(x)$?

$$S'(x) = \frac{d}{dx} S(x) = \frac{d}{dx} \int_0^x \sin(\pi t^2/2)\,dt$$

$$= \sin(\pi t^2/2).$$

Proof of FTC 1?

We want to show that $\dfrac{d}{dx}\underbrace{\int_a^x f(t)\,dt}_{g(x)} = f(x)$ for continuous $f(x)$.

Note that $g(x+h)-g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

$$= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt,$$

so $$\boxed{\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.} \qquad \boxed{1}$$

shows up in limit def. of derivative!

ϕ Assume $h>0$. By the EVT, $\exists\, u_h, v_h \in [x, x+h]$ s.t. $f(u_h) \leq f(x) \leq f(v_h)$ $\forall x \in [x, x+h]$.

From the integral properties, we can "integrate this inequality" to get

$$\int_x^{x+h} f(u_h)\,dt \leq \int_x^{x+h} f(t)\,dt \leq \int_x^{x+h} f(v_h)\,dt$$

$$\| \qquad\qquad\qquad\qquad\qquad \|$$

$$f(u)\int_x^{x+h} 1\,dt \leq \int_x^{x+h} f(t)\,dt \leq f(v)\int_x^{x+h} 1\,dt$$

$$\| \qquad\qquad\qquad\qquad\qquad \|$$

$$\boxed{2} \quad \boxed{f(u_h)h \leq \int_x^{x+h} f(t)\,dt \leq f(v_h)h.}$$

Since $h > 0$, dividing inequality by h gives

$$f(u_h) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v_h)$$

Using $\boxed{1}$, we insert to get

$$\boxed{3} \quad \boxed{f(u_h) \le \frac{g(x+h) - g(x)}{h} \le f(v_h).}$$

What happens as we send $h \to 0$?

$$\underset{\substack{\min\limits_{[x,x+h]} f(x)}}{\underset{\parallel}{f(u_h)}} \to f(x) \quad \text{and} \quad \underset{\substack{\max\limits_{[x,x+h]} f(x)}}{\underset{\parallel}{f(v_h)}} \to f(x) \; !$$

Applying squeeze theorem, we conclude that

$$\lim_{h\to 0} f(u_h) = \lim_{h\to 0} f(v_h) = f(x) \Rightarrow \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x).$$

But this just says that

$$\frac{d}{dx} g(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad \blacksquare$$

FTC, Part 1 (shorter):

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

Differentiation and integration are "inverse" operations.

FTC, Part 2 (another statement):

$$\boxed{\int_a^x F'(t)\,dt = F(x) - F(a)}$$

Proof of FTC 2? We want to show that

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F'(x) = f$.

if $F(x)$ is an antiderivative of $f(x)$.

Let $g(x) = \int_a^x f(t)\,dt$. FTC1 tells us that

$\frac{d}{dx} g(x) = f(x)$. From earlier in course,

$\Rightarrow F(x) = g(x) + C$ for some $C \in \mathbb{R}$,

$\forall x \in [a,b]$.

What is $F(b)-F(a)$? By the formula,

$$F(b)-F(a) = (g(b)+C) - (g(a)+C)$$

$$= g(b) - g(a)$$

$$= \int_a^b f(t)\,dt - \cancelto{0}{\int_a^a f(t)\,dt}$$

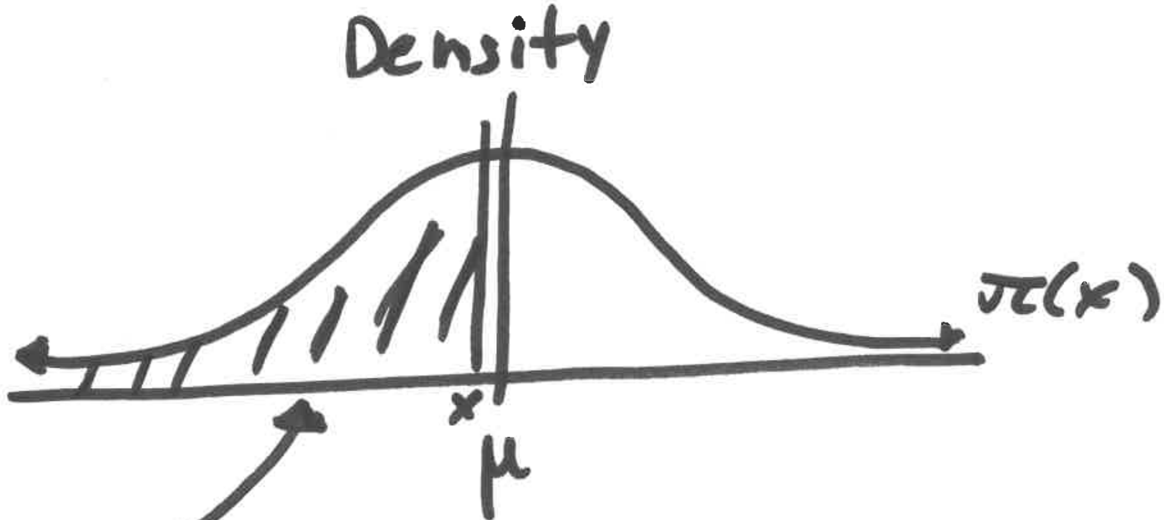
$$= \int_a^b f(t)\,dt.$$

This shows that

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

The Gaussian: Let $X \sim \mathcal{N}(\mu, \sigma^2)$

be a Gaussian (Normal) random variable.



This area gives

$\Pr(X \leq x)$.

It turns out that

$$\Pr(X \leq x) = \int_{-\infty}^{x} \pi(t)\,dt = \lim_{\ell \to -\infty} \int_{\ell}^{x} \pi(t)\,dt, \quad \text{and}$$

$$\lim_{t \to -\infty} \int_t^x \pi(t)\,dt = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]. \quad (= F(x)).$$

Here $\operatorname{erf}(\cdot)$ is the error function, given by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt.$$

left area under curve

<u>Q:</u> We have a formula for $\int_{-\infty}^{x} \pi(t)\,dt$,

can we figure out what $\pi(t)$ is?

Yes! From FTC 1, we know that

$$\frac{d}{dx}\int_{-\infty}^{x} \pi(t)\,dt = \frac{d}{dx}\lim_{t \to -\infty}\int_{t}^{x}\pi(t)\,dt$$

need extra justification here!

$$= \lim_{t\to-\infty}\frac{d}{dx}\int_{t}^{x}\pi(t)\,dt = \pi(x).$$

But what is the derivative?

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left[\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

chain rule!

$$= \frac{1}{2}\left[\operatorname{erf}(z)'\right]_{z=\frac{x-\mu}{\sigma\sqrt{2}}} \cdot \underbrace{\frac{d}{dx}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)}_{=\frac{1}{\sigma\sqrt{2}}}.$$

The last piece we need is the derivative of the error function.

$$\frac{d}{dz}\operatorname{erf}(z) = \frac{d}{dz}\left[\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt\right]$$

★ used FTC 1 again!

$$= \frac{2}{\sqrt{\pi}}\frac{d}{dz}\int_0^z e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}e^{-z^2}.$$

So

$$\frac{d}{dx}F(x) = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{\pi}}\,e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2}\cdot\frac{1}{\sigma\sqrt{2}}$$

$$= \underbrace{\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}_{\text{Gaussian density function.}} = \pi(x).$$

Indefinite Integrals

__Notation:__ If $F(x)$ is an antiderivative of $f(x)$, then we write

$$\int f(x)\,dx = F(x).$$

We say that $\int f(x)\,dx$ is the indefinite integral of $f(x)$.

☆ $\int_a^b f(x)\,dx$ represents a number, while $\int f(x)\,dx$ is a function.

Ex:

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int k\,dx = k\int 1\,dx = k(x+c) = kx + \tilde{c}.$$

MATH 3

Lecture #26

11/8/23

Jonathan Lindbloom

Some Examples

Q: What is $\int 3x^2+4x+1 \, dx$?

A: $\int 3x^2+4x+1 \quad dx$

$= \int 3x^2 \, dx + \int 4x \, dx + \int 1 \, dx$

$= 3\int x^2 \, dx + 4\int x \, dx + \int 1 \, dx$

$= 3 \cdot \frac{1}{3}x^3 + C_1 + 4 \cdot \frac{1}{2}x^2 + C_2 + x + C_3$

$= x^3 + 2x^2 + x + C.$

Substitution for Integration (indefinite)

Note that

$$\frac{d}{dx} f(g(x)) = f'(g(x))\,g'(x).$$

Suppose that we have an antiderivative $F(x)$ of $f(x)$, i.e., $F'(x) = f(x)$.

Then

$$\frac{d}{dx} F(g(x)) = F'(g(x))\,g'(x)$$
$$= f(g(x))\,g'(x).$$

$\Rightarrow$ $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$!

This means that

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If we can write an integrand this way (identify $f(x)$, $g(x)$, $g'(x)$, $F(x)$), then we can use this fact to evaluate an integral.

Consider $\int e^{7t}\,dt$.

$$\int e^{7t}\,dt = \frac{1}{7}\int 7e^{7t}\,dt$$

☆ We see the derivative of the "inner" function outside!

Take $g(x) = 7t$, $f(x) = e^{t}$,

$\Rightarrow F(x) = e^{t}$.

$$\Rightarrow \frac{1}{7}\int 7e^{7t}\,dt = \frac{1}{7}e^{7t} + C.$$

Q: What is $\int \sin^2(x)\cos(x)\,dx$?

A: Take $g(x) = \sin(x)$, $f(x) = x^2$;

$\Rightarrow F(x) = \frac{1}{3}x^3$ & $g'(x) = \cos(x)$.

So $\int \sin^2(x)\cos(x)\,dx = F(g(x)) + C$

$= \frac{1}{3}\sin^3(x) + C$

Q: What is $\int x e^{-x^2} dx$?

$$\to -\frac{1}{2} \cdot \int -2x e^{-x^2} dx$$

A: Take $g(x) = -x^2$,

$f(x) = e^x$, $\Rightarrow g'(x) = -2x$, $F(x) = e^x$.

So

$$-\frac{1}{2} \int -2x e^{-x^2} dx = -\frac{1}{2} F(g(x)) + C$$

$$= -\frac{1}{2} e^{-x^2} + C.$$

<u>Q:</u> $\int x^3 \cos(x^4+2)\,dx$?

$$\parallel$$

$$\frac{1}{4}\int 4x^3\cos(x^4+2)\,dx$$

Take $g(x) = x^4+2$, $f(x) = \cos(x)$,

$\Rightarrow F(x) = \sin(x)$, $g'(x) = 4x^3$.

So $\frac{1}{4}\int 4x^3\cos(x^4+2)\,dx$

$$= \frac{1}{4}F(g(x)) = \frac{1}{4}\sin(x^4+2) + C.$$

$+C$

There is another way to think about this:

$$\int x^3 \cos(x^4+2)\,dx \quad ? \rightarrow = \frac{1}{4}\int 4x^3 \cos(x^4+2)\,dx$$

Let $u = x^4+2$. $\Rightarrow \dfrac{du}{dx} = 4x^3$,

(formally) $\Rightarrow$ $du = 4x^3\,dx$, $\Rightarrow dx = \dfrac{1}{4x^3}\,du$.

Substituting, we get

$$\frac{1}{4}\int 4x^3 \cos(x^4+2)\,dx = \frac{1}{4}\int \cancel{4x^3}\cos(u)\cdot\frac{1}{\cancel{4x^3}}\,du$$

$$= \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(u) + C$$

$$= \frac{1}{4}\sin(x^4+2) + C.$$

Q: What is $\int \tan(x)\,dx$?

A: $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$

$$= \int \sin(x)[\cos(x)]^{-1}\,dx$$

$$= -\int \underbrace{-\sin(x)}_{\text{derivative of}} [\underbrace{\cos(x)}]^{-1}\,dx$$

Let $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$

$$\Rightarrow dx = -\frac{1}{\sin(x)}\,du$$

Inserting,

$$-\int -\sin(x)[\cos(x)]^{-1}\,dx = -\int \cancel{-}\cancel{\sin}(x)\cdot u^{-1}\cdot \cancel{-}\frac{1}{\cancel{\sin}(x)}\,du$$

$$= -\int u^{-1}\,du = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C$$

$$\left(= \ln|\sec(x)| + C\right).$$

Substitution for Integration (definite)

When we apply substitution to definite integrals, there is one extra step we need to take.

Consider $\int_0^1 3x\,dx = 3\left[\frac{1}{2}x^2\right]_0^1 = \frac{3}{2}$.

Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow dx = \frac{1}{3}du$

Substituting,

$$\int_0^1 3x\,dx = \int_0^1 u \cdot \frac{1}{3}du = \frac{1}{3}\int_0^1 u\,du = \frac{1}{3}\left[\frac{1}{2}u^2\right]_0^1$$

wrong $\rightarrow$ $= \frac{1}{6}$! Where did we go wrong?

We have to also change the bounds!

$$\int_{x=0}^{x=1} 3x\,dx = \int_{x=0}^{x=1} u \cdot \frac{1}{3}\,du = \frac{1}{3}\int_{x=0}^{x=1} u\,du$$

$$= \frac{1}{3}\int_{u=0}^{u=3} u\,du = \frac{1}{3}\left[\frac{1}{2}u^2\right]_0^3 = \frac{1}{3}\,\frac{3^2}{2} = \frac{3}{2}$$

This gives us the correct integral.

☆ If you forget to change bounds, you will generally get the wrong answer!

Q: What is $\int_0^1 \cos(\frac{\pi}{2}t)\,dt$?

A: Let $u = \frac{\pi}{2}t$. $\Rightarrow du = \frac{\pi}{2}dt$,

$\Rightarrow dt = \frac{2}{\pi}du$.

Inserting:

$$\int_0^1 \cos(\tfrac{\pi}{2}t)\,dt = \int_0^{\frac{\pi}{2}} \cos(u)\cdot\frac{2}{\pi}\,du$$

$$= \frac{2}{\pi}\Big[\sin(u)\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(\sin(\tfrac{\pi}{2}) - \sin(0)\right)$$

$$= \frac{2}{\pi}$$

Q: What is $\int \frac{x}{1+x^4}\,dx$?

Not obvious at first. Let $u = x^2$

$\Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x}\,du$.

Inserting, $\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+u^2} \cdot \frac{1}{2x}\,du$

$= \frac{1}{2} \int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C$

$= \frac{1}{2}\arctan(x^2) + C$.

Q: What is $\int \frac{3x^{27}}{8+x^{28}}\,dx$?

A: $= 3\int \frac{x^{27}}{8+x^{28}}\,dx$. Let $u = x^{28}$

$\Rightarrow du = 28x^{27}\,dx$

$\Rightarrow dx = \frac{1}{28x^{27}}\,du$

$= 3\int \frac{\cancel{x^{27}}}{8+u} \cdot \frac{1}{28\cancel{x^{27}}}\,du$

$= \frac{3}{28}\int \frac{1}{8+u}\,du$

$= \frac{3}{28}\ln|u+8| + C = \frac{3}{28}\ln|x^{28}+8| + C.$

Q: What is $\int_1^2 x\sqrt{x-1}\,dx$?

Let $u = x-1$, $\Rightarrow du = dx$.

So $$\int_1^2 x\sqrt{x-1}\,dx = \int_{x=1}^{x=2} x\,u^{\frac{1}{2}}\,du$$

$$= \int_{x=1}^{x=2} (1+u)\,u^{\frac{1}{2}}\,du = \int_{x=1}^{x=2} u^{\frac{1}{2}}\,du + \int_{x=1}^{x=2} u^{\frac{3}{2}}\,du$$

$$= \int_{u=0}^{u=1} u^{\frac{1}{2}}\,du + \int_{u=0}^{u=1} u^{\frac{3}{2}}\,du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{5}u^{\frac{5}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{5} = \frac{10}{15} + \frac{6}{15} = \frac{16}{15}.$$

Q: What is $\int_0^a x\sqrt{x^2+a^2}\,dx$? $(a>0)$

A: Let $u = x^2+a^2 \Rightarrow du = 2x\,dx$

$$\Rightarrow dx = \frac{1}{2x}\,du.$$

Inserting:

$$\int_0^a x(x^2+a^2)^{\frac{1}{2}}\,dx = \int_{x=0}^{x=a} x\,u^{\frac{1}{2}} \cdot \frac{1}{2x}\,du$$

$$= \frac{1}{2}\int_{x=0}^{x=a} u^{\frac{1}{2}}\,du = \frac{1}{2}\int_{a^2}^{2a^2} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{a^2}^{2a^2} = \frac{1}{2}\left(\frac{2}{3}(2a^2)^{\frac{3}{2}} - \frac{2}{3}(a^2)^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(2\sqrt{2}-1\right)a^3.$$

MATH 3

Lecture #27

11/10/23

Jonathan Lindbloom

Integration
by
Parts
(indefinite)

Product rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

$$\Rightarrow \int \left[f'(x)g(x) + f(x)g'(x)\right] dx = f(x)g(x)$$

$$\Rightarrow \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)g(x)$$

$$\Rightarrow f(x)g(x) - \int g f'\,dx = \int f(x)g'(x)\,dx$$

Q: What is $\int x \sin(x)\, dx$?

A:

$$\int \underbrace{x}_{f(x)} \underbrace{\sin(x)}_{g'(x)} = f(x)g(x) - \int g(x) f'(x)\, dx$$

$f'(x) = 1$ $\qquad$ $g(x) = -\cos(x)$

$$= -x\cos(x) - \int 1 \cdot -\cos(x)\, dx$$

$$= -x\cos(x) + \int \cos(x)\, dx$$

$$= -x\cos(x) + \sin(x) + C$$

Q: What is $\int x e^{2x}\,dx$?

A: $\int x e^{2x}\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

$f(x)$ ⟹ $f' = 1$

$g'(x)$ ⟹ $g = \frac{1}{2}e^{2x}$

$$= \frac{x}{2}e^{2x} - \int 1 \cdot \frac{1}{2}e^{2x}\,dx$$

$$= \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{x}{2}e^{2x} - \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C$$

$$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$$

Q: $\int \sqrt{x}\,\ln(x)\,dx$?

A: $$\int \underset{f(x)}{x^{\frac{1}{2}}}\ \underset{g'(x)}{\ln(x)}\,dx$$

$\Rightarrow f' = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad\Rightarrow g = \ldots$?

This substitution may not work.

Try another!

$$\int x^{\frac{1}{2}} \ln(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln(x) - \int \frac{1}{x}\cdot\frac{2}{3}x^{\frac{3}{2}}\, dx$$

$g'(x)$

$\Rightarrow g = \frac{2}{3}x^{\frac{3}{2}}$

$f(x)$

$\Rightarrow f' = \frac{1}{x}$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln(x) - \frac{2}{3}\int x^{\frac{1}{2}}\, dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln(x) - \frac{2}{3}\cdot\frac{2}{3}x^{\frac{3}{2}} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln(x) - \frac{4}{9}x^{\frac{3}{2}} + C.$$

Q: $\int \ln(x)\, dx$?

A:
$$= \int \underbrace{1}_{g'(x)} \underbrace{\ln(x)}_{f(x)}\, dx$$

$$\Rightarrow f' = \frac{1}{x}$$

$$\Rightarrow g = x$$

$$= f(x)g(x) - \int f'(x)g(x)\, dx$$

$$= x\ln(x) - \int \frac{1}{x}\, x\, dx$$

$$= x\ln(x) - \int 1\, dx$$

$$= x\ln(x) - x + C.$$

Q: $\int \arcsin(x)\,dx$?

A: $= \int \underbrace{\arcsin(x)}_{f(x)} \cdot \underbrace{1}_{g'(x)}\,dx$

$\Rightarrow f' = \frac{1}{\sqrt{1-x^2}}$ $\quad \Rightarrow g = x$

$= f(x)g(x) - \int f'(x)g(x)\,dx$

$= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

? substitution!

$$\int \frac{x}{\sqrt{1-x^2}}\,dx \,?. \quad \text{Let } u = 1-x^2.$$

$$\Rightarrow du = -2x\,dx,$$

$$\Rightarrow dx = -\frac{1}{2x}\,du$$

$$\downarrow$$

$$= \int x u^{-\frac{1}{2}}\,dx = \int x u^{-\frac{1}{2}} \cdot -\frac{1}{2x}\,du$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = -\frac{1}{2}\cdot 2u^{\frac{1}{2}} + C$$

$$= -u^{\frac{1}{2}} + C = -\sqrt{1-x^2} + C.$$

$$= x\arcsin(x) + \sqrt{1-x^2} + C.$$

Box trick?

$$\int x \cos(4x)\, dx \, ?$$

differentiate $\longrightarrow$

x $\rightarrow$	1
$\frac{1}{4}\sin(4x)$	$\leftarrow$ $\cos(4x)$

$\longleftarrow$ integrate

$$= \bigcirc - \int \bigcirc$$

$$= \frac{x}{4}\sin(4x) - \int \frac{1}{4}\sin(4x)\, dx$$

$$= \frac{x}{4}\sin(4x) - \frac{1}{4}\int \sin(4x)\, dx$$

$$= \frac{x}{4}\sin(4x) + \frac{1}{16}\cos(4x) + C$$

Q: $\int (\ln(x))^2 \, dx$

$[\ln(x)]^2$	$2\ln(x)\cdot\frac{1}{x}$
x	1

$$= x[\ln(x)]^2 - \int 2\ln(x)\, dx$$

$$= x[\ln(x)]^2 - 2(x\ln(x) - x) + C \, .$$

from earlier

Integration by Parts (definite)

$$\int_a^b f(x)g'(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

Q: What is $\int_0^1 \arctan^{-1}(x)\,dx$?

$\arctan(x) \rightarrow$	$\frac{1}{1+x^2}$
$x \leftarrow$	1

A: $= \int_0^1 1\cdot \arctan^{-1}(x)\,dx$

$$= \left[x\arctan(x)\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

← need to use substitution!

$$= \overset{\frac{\pi}{4}}{\cancel{\arctan(1)}} - \int_0^1 \frac{x}{1+x^2}\,dx$$

Let $u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x}\,du$

$$= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{2\cancel{x}}\cdot\frac{\cancel{x}}{1+x^2}\,du = \frac{\pi}{4} - \frac{1}{2}\int_{u=1}^{u=2} u^{-1}\,du$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln(u)\right]_1^2 = \frac{\pi}{4} - \frac{\ln(2)}{2}$$

Q: $\displaystyle\int_0^{\frac{\pi}{3}} \sin(x)\ln(\cos(x))\,dx$?

$\ln(\cos(x))$	$\frac{1}{\cos(x)}\cdot -\sin(x)$
$-\cos(x)$	$\sin(x)$

A:

$$= \Big[-\cos(x)\ln(\cos(x))\Big]_0^{\frac{\pi}{3}} - \int_0^{\pi/3} \cos(x)\cdot\frac{1}{\cos(x)}\cdot\sin(x)\,dx$$

$$= \left(-\cos\left(\tfrac{\pi}{3}\right)\ln\left(\cos\left(\tfrac{\pi}{3}\right)\right)\right) - \left(-\cos(0)\ln(\cos(0))\right) - \int_0^{\pi/3}\sin(x)\,dx$$

$$= -\frac{1}{2}\ln\left(\frac{1}{2}\right) - \Big[-\cos(x)\Big]_0^{\frac{\pi}{3}} = -\frac{1}{2}\ln\left(\frac{1}{2}\right) - \left(-\cos\left(\tfrac{\pi}{3}\right) + \cos(0)\right)$$

$$= -\frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{2} - 1$$

Q: $\int e^x \sin(x)\,dx$?

e^x	e^x
$-\cos(x)$	$\sin(x)$

A:
$$= -e^x\cos(x) + \int e^x\cos(x)\,dx$$

e^x	e^x
$\sin(x)$	$-\cos(x)$

$$= -e^x\cos(x) + e^x\sin(x) - \int e^x \sin(x)\,dx$$

Just went in a circle?

$$\int e^x\sin(x)\,dx = e^x\big(\sin(x)-\cos(x)\big) - \int e^x\sin(x)\,dx$$

$$\Rightarrow \int e^x\sin(x)\,dx = \frac{1}{2}e^x\big(\sin(x)-\cos(x)\big) + C.$$

$$\int t^3 e^t \, dt \quad ?$$

t^3	$3t^2$
e^t	e^t

$$= t^3 e^t - 3\int t^2 e^t \, dt$$

t^2	$2t$
e^t	e^t

$$= t^3 e^t - 3\left(t^2 e^t - 2\int t e^t \, dt\right)$$

t	1
e^t	e^t

$$= t^3 e^t - 3t^2 e^t + 6\int t e^t \, dt$$

$$= t^3 e^t - 3t^2 e^t + 6\left(t e^t - \int e^t \, dt\right)$$

$$= t^3 e^t - 3t^2 e^t + 6 t e^t - 6 e^t + C.$$

Q: What is $\dfrac{d}{dx}\displaystyle\int_a^{f(x)} g(t)\,dt$?

A: Define $F(x) := \displaystyle\int_a^{x} g(t)\,dt$,

and we know by FTC1 that

$$\frac{d}{dx}F(x) = g(x).$$

Note that $\displaystyle\int_a^{f(x)} g(t)\,dt = F(f(x))$,

$$\Rightarrow \frac{d}{dx}\int_a^{f(x)} g(t)\,dt = \frac{d}{dx}F(f(x)) = F'(f(x))\cdot f'(x)$$
$$= g(f(x))\cdot f'(x).$$

$\underline{Q!}$ $\int e^{\sqrt{x}}\, dx$?

$\underline{A!}$ Let $u = x^{\frac{1}{2}} \Rightarrow du = \frac{1}{2} x^{-\frac{1}{2}}\, dx$

$\Rightarrow dx = 2x^{\frac{1}{2}}\, du$

$$= \int e^{u} \cdot 2x^{\frac{1}{2}}\, du = 2\int u e^{u}\, du$$

u	1
e^{u}	e^{u}

$$= 2\left(ue^{u} - \int e^{u}\, du \right)$$

$$= 2ue^{u} - 2e^{u} + C$$

$$= 2\sqrt{x}\, e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

Q: $\int x \ln(1+x)\,dx$?

$\ln(u)$	u^{-1}
$\frac{1}{2}u^2$	u

Let $u = 1+x \Rightarrow du = dx.$

$$= \int x \ln(u)\,du = \int (u-1)\ln(u)\,du$$

$\ln(u)$	u^{-1}
u	1

$$= \int u\ln(u)\,du - \int \ln(u)\,du$$

$$= \left[\frac{1}{2}u^2\ln(u) - \int \frac{1}{2}u\,du\right] - \left[u\ln(u) - \int 1\,du\right]$$

$$= \frac{1}{2}u^2\ln(u) - \frac{1}{4}u^2 - u\ln(u) + u + C$$

$$= \frac{1}{2}(1+x)^2\ln(1+x) - \frac{1}{4}(1+x)^2 - (1+x)\ln(1+x) + (1+x) + C.$$

MATH 3

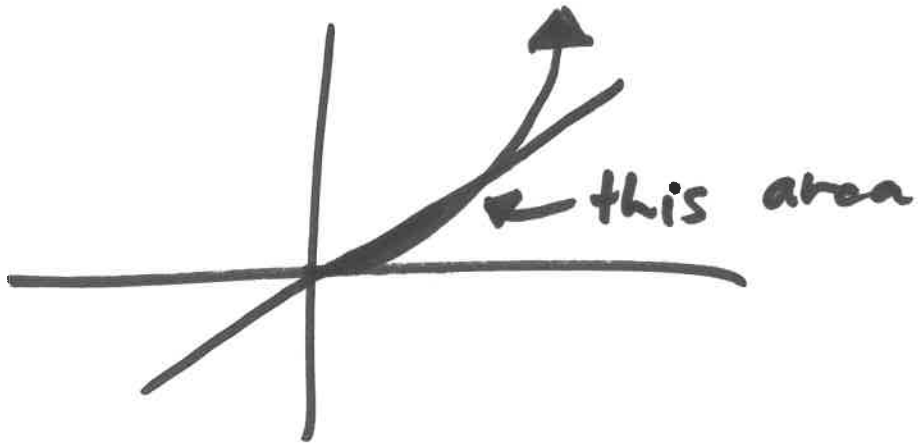
Lecture #28

11/13/23

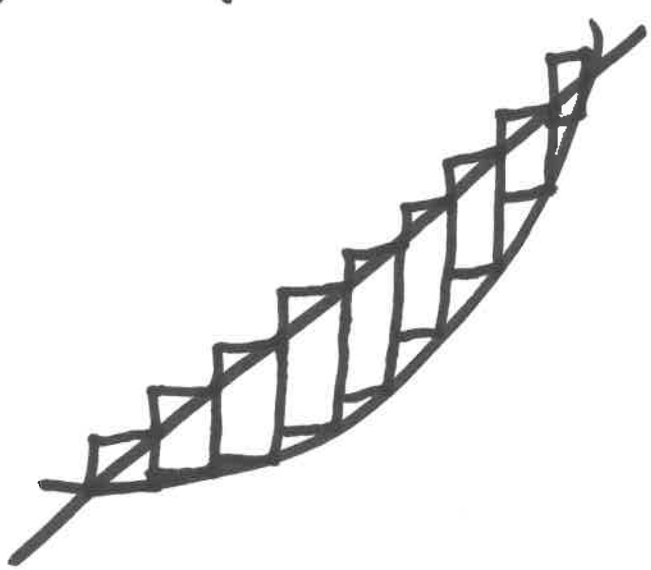
Jonathan Lindbloom

Area Between Curves

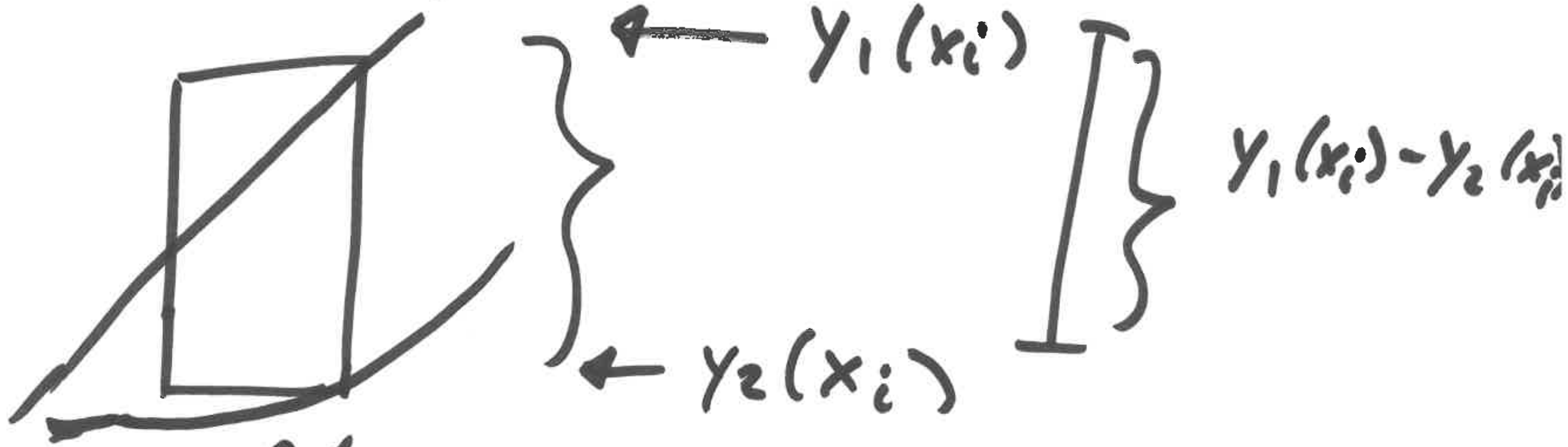
Q: What is the area between the curves $y_1 = x$ and $y_2 = x^2$?



A: Let's approximate the area by rectangles!



Split the interval $[0,1]$ into N equal parts, $x_i = \frac{i}{N}$, $i = 0, \ldots, N$. With right-endpoint rule, ith rectangle looks like



$$\frac{b-a}{N} = \frac{1-0}{N} = \frac{1}{N}$$

So our Riemann sum is

$$A_N = \sum_{i=1}^{N} \frac{1}{N} \cdot \left(y_1(x_i) - y_2(x_i) \right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} x_i - x_i^2 = \frac{1}{N} \sum_{i=1}^{N} \frac{i}{N} - \frac{i^2}{N^2}$$

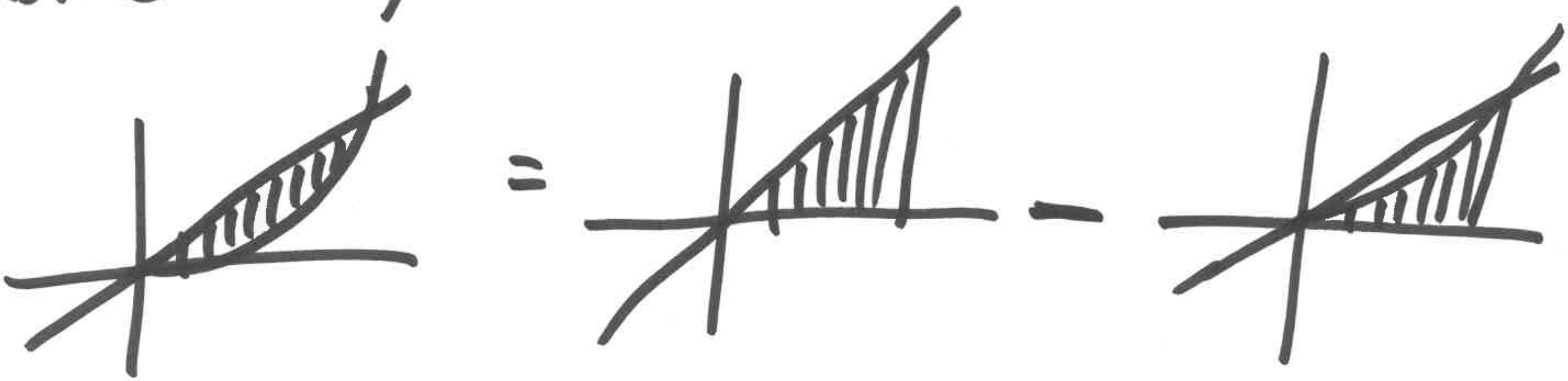
$$= \frac{1}{N^2} \sum_{i=1}^{N} i - \frac{1}{N^3} \sum_{i=1}^{N} i^2$$

$$= \frac{1}{N^2} \frac{N(N+1)}{2} - \frac{1}{N^3} \frac{N(N+1)(2N+1)}{6} .$$

As $N \to \infty$?

$$\lim_{N \to \infty} A_N = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

Another way?



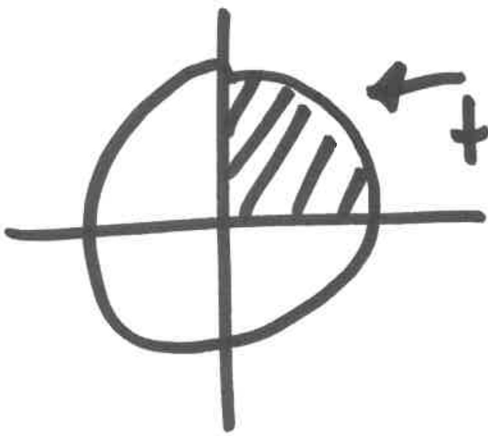
$$= \int_0^1 x \, dx - \int_0^1 x^2 \, dx$$

$$= \left[\frac{1}{2}x^2\right]_0^1 - \left[\frac{1}{3}x^3\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$ ✓

Q: Compute the area of the unit circle using an integral.

A:



find this, then ×4.

Want:

$$\int_0^1 \sqrt{1-x^2}\, dx.$$

Let $x = \sin\theta$. $\Rightarrow$ $dx = \cos\theta\, d\theta$.

$$\int_{x=0}^{x=1} \sqrt{1-x^2}\, dx = \int_{x=0}^{x=1} \sqrt{1-\sin^2\theta} \cdot \cos\theta\, d\theta = \int_{x=0}^{x=1} \cos^2\theta\, d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos^2\theta \, d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2}\left[1+\cos(2\theta)\right] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 \, d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\, d\theta$$

$$= \frac{\pi}{4} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2\theta)\, d\theta$$

Let $u = 2\theta$. $\Rightarrow du = 2\,d\theta$

$\Rightarrow d\theta = \frac{1}{2}du$

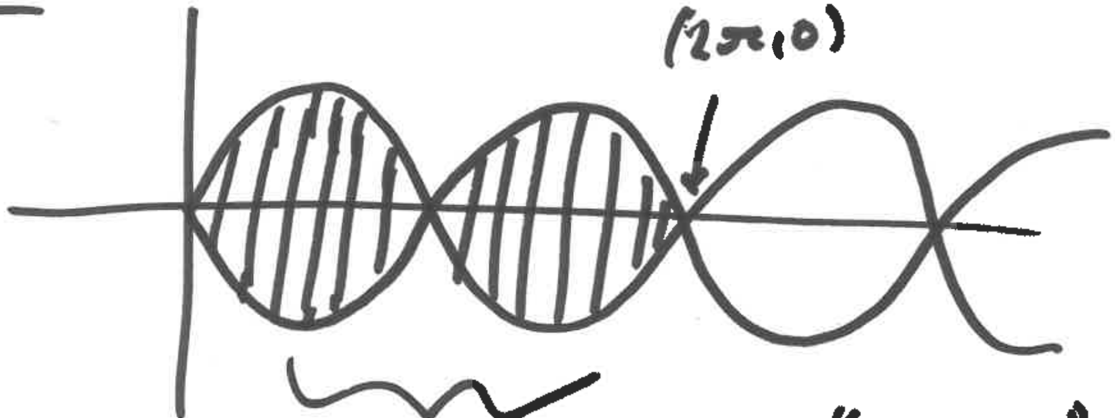
$$= \frac{\pi}{4} + \frac{1}{2}\int_{u=0}^{u=\pi} \cos(u) \cdot \frac{1}{2}\, du$$

πr^2

with $r = 1$.

$$= \frac{\pi}{4} + \frac{1}{4}\left[\sin(u)\right]_0^{\pi} = \frac{\pi}{4} \xrightarrow{\times 4} \pi$$

Q:



What is this area?

"upper" curve is $y_2 = \sin(x)$

"lower" curve is $y_1 = -\sin(x)$

☆ We have to be careful, since if we just do

$$A = \int_0^{2\pi} y_2(x) - y_1(x)\,dx = \int_0^{2\pi} \sin(x)\,dx - \int_0^{2\pi} -\sin(x)\,dx$$

$$= 2\int_0^{2\pi} \sin(x)\,dx = 2\left[-\cos(x)\right]_0^{2\pi} = 0 \;!$$

In general, the area between two cures $y_1(x)$ and $y_2(x)$ from $x=a$ to $x=b$ is given by

$$A = \int_a^b |y_1(x) - y_2(x)| \, dx.$$

$$\int_0^{2\pi} |y_2(x) - y_1(x)|\, dx = \int_0^{2\pi} |\sin(x) - (-\sin(x))|\, dx$$

$$= 2\int_0^{2\pi} |\sin(x)|\, dx = 2\left(\int_0^{\pi} \sin(x)\, dx + \int_{\pi}^{2\pi} -\sin(x)\, dx\right)$$

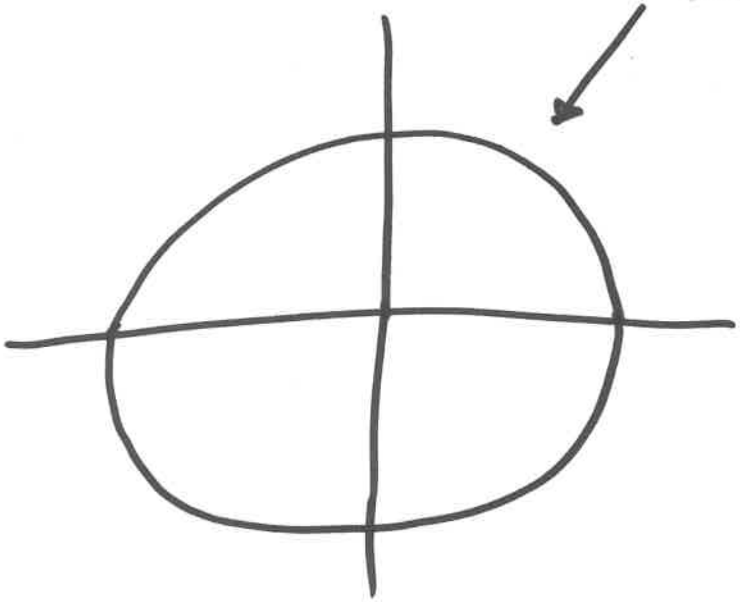
$$= 2\left[-\cos(x)\right]_0^{\pi} + 2\left[\cos(x)\right]_{\pi}^{2\pi}$$

$$= 2(1+1) + 2(1+1)$$

$$= 4 + 4$$

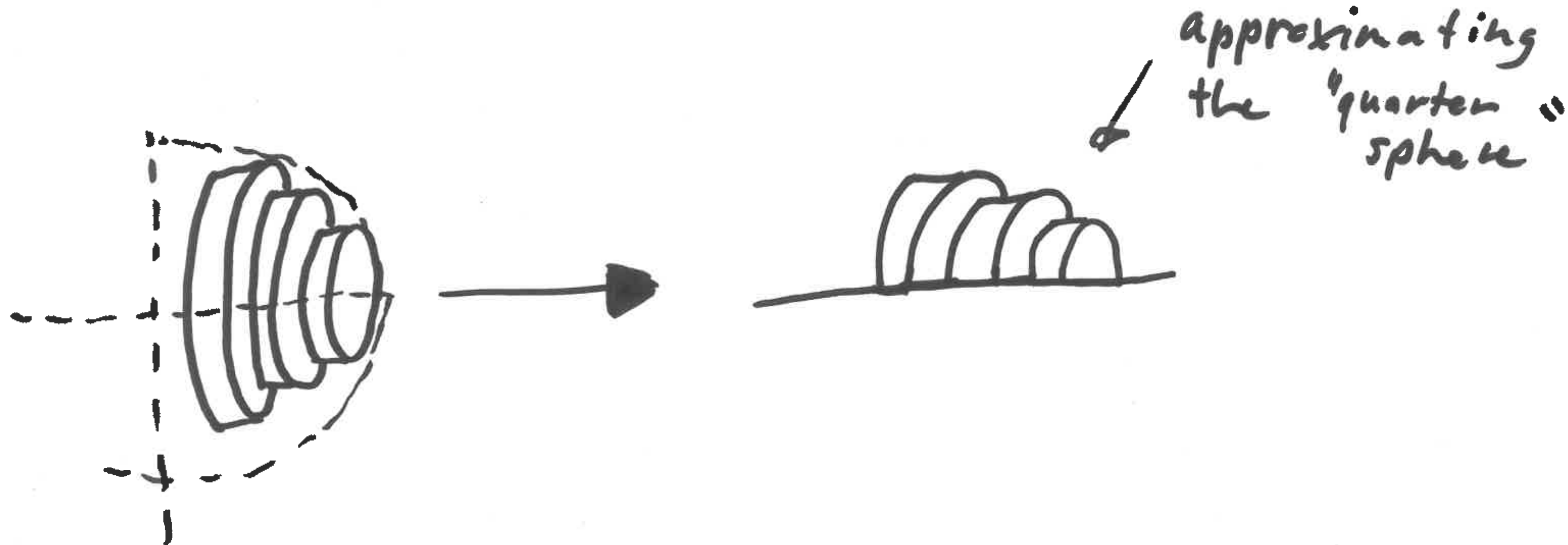
$$= 8$$

How can we compute the volume of a sphere?



We already know how to compute the area of a 2D circle

☆ Instead of assigning a rectangle to each subinterval, assign a cylindrical disk!



Split the interval $[0,1]$ into N equal parts with endpoints $x_i = \frac{i}{N}$, $i = 0, \ldots, N$.

In the ith disk, using right-endpoints the ith disk is a half cylinder with base a circle of radius $\sqrt{1-x_i^2}$ and height $\frac{1}{N}$.

So the volume of the ith half-disk is

$$A_i = \frac{1}{2} \cdot \pi \left(\sqrt{1-x_i^2}\right)^2 \cdot \frac{1}{N} = \frac{\pi}{2N}\left(1 - x_i^2\right)$$

$$= \frac{\pi}{2N}\left(1 - \frac{i^2}{N^2}\right).$$

The volume of all the half-disks is

$$A_N = \sum_{i=1}^{N} \frac{\pi}{2N}\left(1 - \frac{i^2}{N^2}\right) = \frac{\pi}{2N} \sum_{i=1}^{N} 1 - \frac{\pi}{2N^3} \sum_{i=1}^{N} i^2$$

$$= \frac{\pi}{2N} \cdot N - \frac{\pi}{2N^3} \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{\pi}{2} - \frac{\pi}{12} \frac{(N+1)(2N+1)}{N^2}$$

As $N \to \infty$?

$$\lim_{N\to\infty} A_N = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

Multiplying by 4 (since 4 quarter spheres),

we get $\underbrace{V = \frac{4}{3}\pi}_{\text{volume of sphere with unit radius!}}$

Review of Topics:

① Limits

→ Definition

→ Properties

→ L'Hospital's Rule

② Continuity

→ Definition

→ Properties

→ Intermediate Value Theorem (IVT)

(3) Derivatives

→ Definition
→ Properties
→ Implicit differentiation
→ Derivatives of trig/inverse trig functions
→ Product/Quotient rules
→ Chain rule
→ Linear approximations
→ tangent lines

(4) Applications of Derivatives

→ Fermat's Theorem
→ Finding absolute max/min of functions
→ Extreme Value Theorem
→ Rolle's/Mean Value Theorem
→ Intervals of increasing/~~de~~decreasing
→ Intervals of concavity
→ Exponential growth/decay
→ Related Rates
→ Curve sketching
→ Newton's Method

⑤ Antiderivatives & Integration

- → Antiderivatives → Areas under/between curves
- → Riemann sums & def. of definite integrals
- → Indefinite integrals → Fundamental Theorem of Calculus (Part 1 & 2)
- → Integration by Substitution
- → Integration by parts
- → General strategies/techniques for computing definite/indefinite integrals