Extremal Functions for Graph Linkages

Paul Wollan

School of Mathematics
Georgia Institute of Technology
wollan@math.gatech.edu

Abstract. A graph G is *k-linked* if for every set of vertices $\{s_1, ..., s_k, t_1, ..., t_k\}$ there exist k disjoint paths P_i with ends s_i and t_i. Robertson and Seymour showed that if G is $2k$-connected and G contains a K_{3k} minor, then G is k-linked. This, combined with results of Kostochka and Thomason, implies that there exists a function $f(k) = O(k\sqrt{\log k})$ such that every $f(k)$-connected graph is k-linked. Bollobás and Thomason improved this result to show that $22k$-connectivity suffices to imply G is k-linked. We give a simple induction argument that improves this constant to $16k$. With more focused analysis, we are able to further reduce the constant. We use the same induction method to obtain the optimal edge bound in the $k = 3$ case. We show that every 6-connected graph on n vertices with $5n - 14$ edges is 3-linked. This is the best bound possible, in that the result does not hold for 5-connected graphs, and there exist arbitrarily large 6-connected graphs with n vertices and $5n - 15$ edges that are not 3-linked. This is joint work with Robin Thomas.