

**Quasi permutation matrices  
and  
product of idempotents**

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## Plan

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## 1 Introduction

J.M. Howie (1966)  $X$  a finite set. The mappings from  $X$  to  $X$  that are not onto can be presented as a product of idempotents.

J.A. Erdos (1967): Every singular square matrix over a field can be expressed as a product of idempotent matrices.

True for matrices over division rings or commutative euclidean domains Laffey (1983).

Hannah-O'Meara studied decomposition of elements of a regular ring (1989).

Bhaskara Rao characterized singular matrices over a commutative PID that can be decomposed as a product of idempotent matrices (2009).

Different notions extending the classical commutative Euclidean rings have been introduced (Cooke), Leutbecher (even non commutative rings but with commutative goals) and many more (recently: Lebowski).

There are connections between decompositions into products of idempotents and factorizations of invertible matrices into product of elementary matrices (Ruitenburg (1993), Salce-Zanardo (2015), L.-Facchini (2016) )

## 2 Decomposition of singular matrices

Some examples of decompositions:

**Examples 2.1.** (a)  $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-1 & 0 \end{pmatrix}.$

(a')  $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$

(b)  $\begin{pmatrix} a & ac \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1+c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-ca+c & c-cac+c^2 \\ a-1 & ac-c \end{pmatrix}.$

(c)  $\begin{pmatrix} ac & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix},$

(d) with  $b \in U(R)$ ,  $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b(b^{-1}a) & b \\ 0 & 0 \end{pmatrix}$  is factorized as in (c).

(e) If  $u \in U(R)$  is a unit, the matrix

$$\begin{pmatrix} ab+u & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^{-1}ab+1 & u^{-1}a \\ -b(u^{-1}ab+1) & -bu^{-1}a \end{pmatrix},$$

is a product of idempotent matrices.

(f) If there exists  $x \in R$  such that  $a+bx \in U(R)$  then the matrix

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is a product of idempotent matrices.

If  $R$  is a ring. Can we say that all singular matrices over  $R$  are product of idempotent matrices ?

Yes for the following rings:

- $R$  is a field.
- $R$  is a division ring.
- $R$  is a euclidean domain.
- $R$  is a commutative principal ideal domain with  $IP_2$ .

- $R$  is unit regular ring.

The decomposition is not always possible:

Consider  $k$  a field and  $R = k[x, y]$ . The matrix  $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$  is not decomposable into product of idempotent matrices.

**Definition 2.2.** In any ring a pair  $(a, b)$  is right Euclidean if there exists a finite sequence of divisions of the form  $a = bq_1 + r_1; b = r_0 = r_1q_1 + r_2; \dots; r_n = r_{n+1}q_{n+1}$  (reaching an exact division).

A ring  $R$  is *right quasi-euclidean* if any pair in  $R$  is a right Euclidean pair.

**Theorem 2.3.** Let  $R$  be a right and left quasi-Euclidean domain. Then every matrix  $A \in M_n(R)$  with  $l.\text{ann}(A) \neq 0$  (equivalently,  $r.\text{ann}(A) \neq 0$ ) is a product of idempotent matrices.

Examples of right quasi-Euclidean rings:

- Usual right euclidean rings.
- Unit regular rings.

Some Properties:

A ring  $R$  is right quasi-Euclidean if and only it is right  $GE_2$  and right  $K$ -Hermite.

If  $R$  is right quasi-Euclidean then for any  $n$   $M_n(R)$  is right-quasi euclidean.

Recall that a ring  $R$  is *Dedekind finite* if for  $a, b \in R$ ,  $ab = 1$  implies  $ba = 1$ . A right and left quasi-Euclidean ring is not necessarily Dedekind finite (Bergman).

### 3 Decomposition of elements in a ring.

Some relevant properties:

- A ring  $R$  is *idempotent complete rings* if, for every  $x \in R$ ,  $\ell(x) \neq 0$  and  $r(x) \neq 0$  imply that  $x$  is a product of finitely many idempotents of  $R$ .
- Remark that if an element  $a$  is a product of idempotents then  $\ell(a) \neq 0, r(a) \neq 0$ .
- A ring  $R$  satisfies the property The property (\*) if

(\*) “for every  $a \in R$ ,  $\ell(a) \neq 0$  if and only if  $r(a) \neq 0$ ”,

Note that if a ring  $R$  satisfies the property (\*), then  $R$  must be Dedekind finite.

Boolean rings, integral domains, matrix rings over a field are examples of idempotent complete rings.

A unit regular ring  $R$  satisfies property (\*).

Recall that a ring is *abelian* if all its idempotents are central. Local rings, reduced rings (i.e., rings with no non-zero nilpotent element) and commutative rings are abelian. The next two results are from the introduction of a paper with A. Facchini.

**Proposition 3.1.** *Let  $R$  be an abelian ring. Then  $R$  is idempotent complete if and only if either  $R$  is an integral domain or  $R$  is a boolean ring.*

We say that a ring  $R$  is *indecomposable* if its central idempotents are only 0 and 1.

**Proposition 3.2.** *If  $R$  is a ring that is not indecomposable and  $n \geq 2$  is an integer, then the ring  $M_n(R)$  is not idempotent complete.*

Turning to special elements in a ring let us first mention that over any ring  $R$ , a strictly upper triangular matrix  $A \in M_n(R)$  is always a product of idempotent matrices.

For a unit regular ring  $R$ , Hannah and O’Meara proved that an element  $a \in R$  is a product of idempotent elements of  $R$  if and only if  $Rr(a) =$

$R(1 - a)R$ . In particular, for a simple unit regular ring, any zerodivisor is a product of idempotent elements.

Here are results extracted from a paper by Hannah and O'Meara:

**Theorem 3.3.** *Let  $R$  be a unit regular ring and  $k$  a positive integer then  $a \in R$  is a product of  $k$  idempotents if and only if  $(1 - a)R$  can be embedded in  $k \cdot r(a)$  where  $r(a) := \{x \in R \mid ax = 0\}$ .*

**Corollaire 3.4.** *An element  $R$  of a unit regular ring is a product of idempotents if and only if  $Rr(a) = r(1 - a)R$ .*

Hannah and O'Meara also studied the case of self injective von Neumann regular rings:

**Theorem 3.5.** *Let  $R$  be a right self-injective von Neumann regular ring  $R$ , an element  $a \in R$  is a product of idempotents if and only if  $Rr(a) = l(a)R = R(1 - a)R$ .*

## 4 Nonnegative decompositions

Question (Jain, Manipal, December 2014) Can singular nonnegative real matrices be decomposed into a product of nonnegative idempotent matrices?

In some cases the answer is YES

- a) Stricly upper nonnegative triangular matrices.
- b) Rank 1 nonnegative matrices.
- c) Nilpotent nonnegative matrices.

Some ingredients for the proofs.

- a) This is easy and based on the following useful decomposition

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

- b) This is based on the form of a nonnegative matrix of rank 1:

$M \in M_n(\mathbb{R}^+)$  be a nonnegative matrix of rank one. Then there exist nonnegative column vectors  $a, b \in \mathbb{R}^n$  such that  $M = ab^T$ . Let  $\beta$  be the nonnegative real number given by  $\beta = b^T a = a \cdot b$ . Then:

- If  $\beta = 0$ ,  $M$  is a product of two nonnegative idempotents.
- If  $\beta > 0$  then  $M = \beta xy^T$  where  $x, y$  are positive vectors such that  $y^T x = 1$  (i.e.  $xy^T$  is an idempotent matrix).

In fact, a rank 1 nonnegative singular matrix, can be written as a product of *three* idempotent matrices.

- c) For any nonnegative nilpotent matrix  $N$  there exists a permutation matrix  $P$  such that  $PNP^{-1}$  is upper triangular.

A matrix (with coefficients in any ring) is a *quasi-permutation matrix* (QP) if it has at most one non zero element in each row and each

column. Such a matrix can be singular. A special role is played by QP matrices obtained by erasing a "1" in a permutation matrix:

$$P_{\sigma,l} = \sum_{\substack{i=1 \\ i \neq l}}^n e_{i,\sigma(i)}.$$

If the permutation associated with such a matrix is without fixed point, then this matrix is in fact nilpotent (hence product of nonnegative idempotents). This is one of the key ingredients for the proof of the part a) of the following theorem.

**Theorem 4.1.** (a) *Let  $R$  be any domain and  $A \in M_n(R)$  a singular quasi-permutation matrix. Then  $A$  is a product of idempotent matrices.*

(b) *Any singular nonnegative quasi-permutation matrix  $A \in M_n(\mathbb{R}^+)$  is a product of nonnegative idempotent matrices.*

The proof of b) is based on

- the shape of nonnegative matrices having nonnegative inverses.
- the properties of quasi-permutations.

**Theorem 4.2.** *Let  $A \in M_n(\mathbb{R}^+)$  be singular and such that there exists  $X \in M_n(\mathbb{R}^+)$  with  $A = AXA$ , then  $A$  is a product of nonnegative idempotents.*

The proof is based on the description of such a matrix  $A$  (Jain and Goel) and on the above factorization result for QP matrices.

## 5 Final remarks and perspectives

Let  $A \in M_n(\mathbb{R}^+)$  be a singular matrix. If there exists  $X \in M_n(\mathbb{R}^+)$  such that  $A = AXA$ , then  $A$  is a product of nonnegative idempotents. The converse is untrue: the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

but it can be checked that all the von Neuman inverses of  $A$  (matrices  $X$  such that  $A = AXA$ ) are not Nonnegative.

We have seen that rank one nonnegative matrices  $A \in M_n(\mathbb{R}^+)$  can be presented as a product of nonnegative idempotents. This is untrue for rank two matrices: It can be shown that the nonnegative (even totally nonnegative) singular matrix of rank 2

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

is not a product of nonnegative idempotent matrices. The way of proving this assertion is very direct: we first look for the shape of nonnegative idempotent matrices  $E$  that satisfy  $AE = A$  and then show that these  $E$  cannot be really a factor of  $A$ .

It would be worth to analyze the stochastic matrices with respect to decomposition as products of nonnegative idempotents.

**Thank you !**