

Diagonalizing algebras of operators on infinite-dimensional vector spaces

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Thank you, George!

- 1 Classical diagonalization
- 2 Topology in $\text{End}(V)$
- 3 An infinite-dimensional Wedderburn-Artin theorem
- 4 Infinite-dimensional diagonalization

Diagonalization: the basics

Notation fixed throughout:

- $K =$ field (no further assumptions, unless explicitly stated)
- $V = K$ -vector space
- $\text{End}(V) =$ ring of K -linear endomorphisms

Def: A linear transformation $T \in \text{End}(V)$ is **diagonalizable** if there is a basis of V consisting of eigenvectors for T .

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In case $V = K^d$ for $d = \dim(V) < \infty$:

- $\text{End}(V) = \mathbb{M}_d(K)$
- $T \in \mathbb{M}_d(K)$ diagonalizable \iff some UTU^{-1} diagonal, $U \in \mathbb{M}_d(K)$

Classical diagonalization

A blast from the past: some “basic” linear algebra

Classical diagonalizability (operators): For $T \in \mathbb{M}_d$, the following are equivalent:

- (1) T is diagonalizable;
- (2) The minimal polynomial of T splits into distinct linear factors;
- (3) $K[T] \cong K^n$ as algebras for some $n \leq d$.

In (2), the linear factors are $x - \lambda_i$ for the distinct eigenvalues λ_i of T .

In (3), the n orthogonal idempotents in $K[T] \cong K^n$ are projections onto the various eigenspaces.

Diagonalizable subalgebras

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Question: What happens in case $\dim(V)$ is infinite?

The trouble with infinite-dimensional operators

Example: Assume $\mathbb{Q} \subseteq K$ and V has basis $\{v_1, v_2, v_3, \dots\}$. Define:

$$S(v_n) = v_{n+1}$$

Right shift operator, has no eigenvalues, not diagonalizable.

$$T(v_n) = n \cdot v_n$$

Infinitely many distinct eigenvalues, is diagonalizable.

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Each operator generates a subalgebra of $\text{End}(V)$:

$$A := K[S] \cong K[x] \cong K[T] =: B$$

Here B is **diagonalizable**, A is **not**, and they are **isomorphic!**

The trouble with infinite-dimensional operators

There are isomorphic subalgebras $A, B \subseteq \text{End}(V)$ with one diagonalizable and the other not.

Conclusion: Diagonalizability of $A \subseteq \text{End}(V)$ **cannot** be determined purely by its internal algebraic structure!

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Taking a cue from operator algebras, we will consider **topology** on A .

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Topologies on spaces of functions

For sets X and Y , we have a topological space of functions:

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Open neighborhood of $S \in \text{End}(V)$: for given $x_1, \dots, x_n \in V$,

$$U = \{T \in \text{End}(V) \mid S(x_1) = T(x_1), \dots, S(x_n) = T(x_n)\}$$

(Note: This is the topology used to interpret the “density” in the Jacobson Density Theorem.)

Topological algebra of $\text{End}(V)$

The finite topology makes $\text{End}(V)$ a “nice” topological ring. It is:

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- Left linear (neighborhood basis of 0 consisting of open left ideals)
- Complete (Cauchy nets converge)

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Conclusion: $C \subseteq \text{End}(V)$ is diagonalizable $\iff \overline{C}$ is diagonalizable.

\therefore We restrict our classification to **closed** commutative subalgebras.

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Def: A topological module ${}_R M$ is **pseudocompact** if it satisfies these equivalent conditions:

- M is an inverse limit in ${}_R \mathbf{TMod}$ of discrete, finite length modules;
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Fact [Gabriel]: The full subcategory ${}_R \mathbf{PC}$ of p.c. modules in ${}_R \mathbf{TMod}$ is “nice” (abelian, ${}_R \mathbf{PC}^{op}$ is Grothendieck and “locally finite”).

“Topologically left artinian” rings

A topological replacement for the left artinian property:

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Example 3: A noetherian complete local ring (R, \mathfrak{m}) with its \mathfrak{m} -adic topology is pseudocompact:

$$R \cong \varprojlim R/\mathfrak{m}^n$$

is an inverse limit of discrete, finite length modules

Infinite-Dimensional Wedderburn-Artin Theorem

For a topological ring R , the following are equivalent:

- 1 R is a product of discrete simple left modules in ${}_R\mathbf{TMod}$
- 2 R is left pseudocompact and every closed left ideal has a closed complement
- 3 $R \cong \prod \text{End}_{D_i}(V_i)$ for right vector spaces V_i over division rings D_i
- 4 R is left pseudocompact and Jacobson semisimple
- 5 Every short exact sequence in ${}_R\mathbf{PC}$ splits
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Proof utilizes P. Gabriel's duality theory (especially for the conditions on ${}_R\mathbf{PC}$), as it's often easier to reason about objects in the locally finite category ${}_R\mathbf{PC}^{op}$.

Commutative pseudocompact semisimple rings

When R is a commutative topological ring, what form does the WA theorem take?

Def: $\text{Spec}_0(R)$ is the set of open prime ideals of R

Fact: When R is pseudocompact, each $\mathfrak{m} \in \text{Spec}_0(R)$ is maximal

Corollary For a commutative topological ring R , the following are equivalent:

- R is pseudocompact semisimple
- R is a (topological) direct product of discrete fields
- $R \cong \prod_{\mathfrak{m} \in \text{Spec}_0(R)} R/\mathfrak{m}$ as topological rings

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Pseudocompactness for diagonalizable subalgebras

Further restrict the WA theorem to characterize algebras like $K^{\mathcal{B}}$.

Corollary: For a commutative topological K -algebra, TFAE:

- A is p.c. semisimple, with all open maximal ideals of codimension 1
- $A \cong K^X$ for some set X
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Call such A a **function algebra**. These form a category **Func**(K).

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Cor: Any closed subalgebra $A \subseteq K^X$ is isomorphic to K^Y for some set Y with $|Y| \leq |X|$. (**Proof:** $K^Y \hookrightarrow K^X \iff X \twoheadrightarrow Y$)

Function algebras over algebraically closed fields

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Def: A topological K -algebra is **K -pseudocompact** if it is an inverse limit of discrete algebras that have finite K -dimension.

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Proposition: For a commutative topological algebra over an algebraically closed field K , the following are equivalent:

- $A \cong K^X$ is a function algebra
- A is K -p.c. and Jacobson semisimple
- A is K -p.c. and topologically reduced (i.e., $x^n \rightarrow 0 \implies x = 0$)

Infinite-Dimensional Diagonalization Theorem

For any K -vector space V and any closed subalgebra $A \subseteq \text{End}(V)$, the following are equivalent:

- (a) A is diagonalizable
- (b) $A \cong K^\Omega$ as topological algebras for some cardinal $\Omega \leq \dim(V)$
- (c) (when $K = \overline{K}$) A is K -pseudocompact and Jacobson semisimple
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Proof: (a) \Rightarrow (b): A diagonalizable w.r.t. basis \mathcal{B} implies $A \subseteq K^{\mathcal{B}}$, closed inside function algebra. So $A \cong K^\Omega$ with $\Omega \leq |\mathcal{B}| = \dim(V)$.

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(b) \Rightarrow (a): $A \cong K^\Omega$ generated by orthogonal idempotents e_i whose net of finite sums converges to 1. In $\text{End}(V)$, these project onto subspaces V_i with $V = \bigoplus V_i$. The V_i are simultaneous eigenspaces, so A is diagonalizable. (Already discussed (b) \Leftrightarrow (c) \Leftrightarrow (d).)

Question: Which operators $T \in \text{End}(V)$ can be “approximated” by diagonalizable operators?

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$$\begin{aligned} H(T) &:= \{v \in V \mid p(T)v = 0 \text{ for some } p \in K[x]\} \\ &= \sum (\text{f.d. } T\text{-invariant subspaces}) \end{aligned}$$

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Theorem: (1) If K is an infinite field, then $T \in \overline{\mathcal{D}}$ if and only if its restriction to $H(T)$ is diagonalizable. (2) If K is finite, then $\overline{\mathcal{D}} = \mathcal{D}$.

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Ex: The right-shift operator $S(v_n) = v_{n+1}$ on $V = \bigoplus K v_n$ has $H(S) = 0$. It is in $\overline{\mathcal{D}}$ when K is infinite, but not in $\overline{\mathcal{D}}$ when K is finite.

Thank you!