TORUS-INVARIANT PRIME IDEALS AS INVARIANTS OF NONCOMMUTATIVE ALGEBRAS

Ken Goodearl

33rd Ohio State–Denison Mathematics Conference

14 May 2016

1. THE INVARIANT

$A = \bigcup_{i=0}^{\infty} A_i$ = fin. gen. noeth. filtered algebra, $\dim A_i < \infty$

some A_n generates A

$\operatorname{fAut} A := \{\text{filtered automs. of } A\} \hookrightarrow GL(A_n)$

= affine algebraic group

H = a maximal torus of $\operatorname{fAut} A$

$H\text{-spec}\, A := \{H\text{-prime ideals of } A\}$, = poset w.r.t. $\subseteq$

Invariance: H' = another maximal torus of $\operatorname{fAut} A$

$\Longrightarrow H' = \alpha H \alpha^{-1}$, some $\alpha \in \operatorname{fAut} A$

$\Longrightarrow H'\text{-spec}\, A \cong H\text{-spec}\, A$

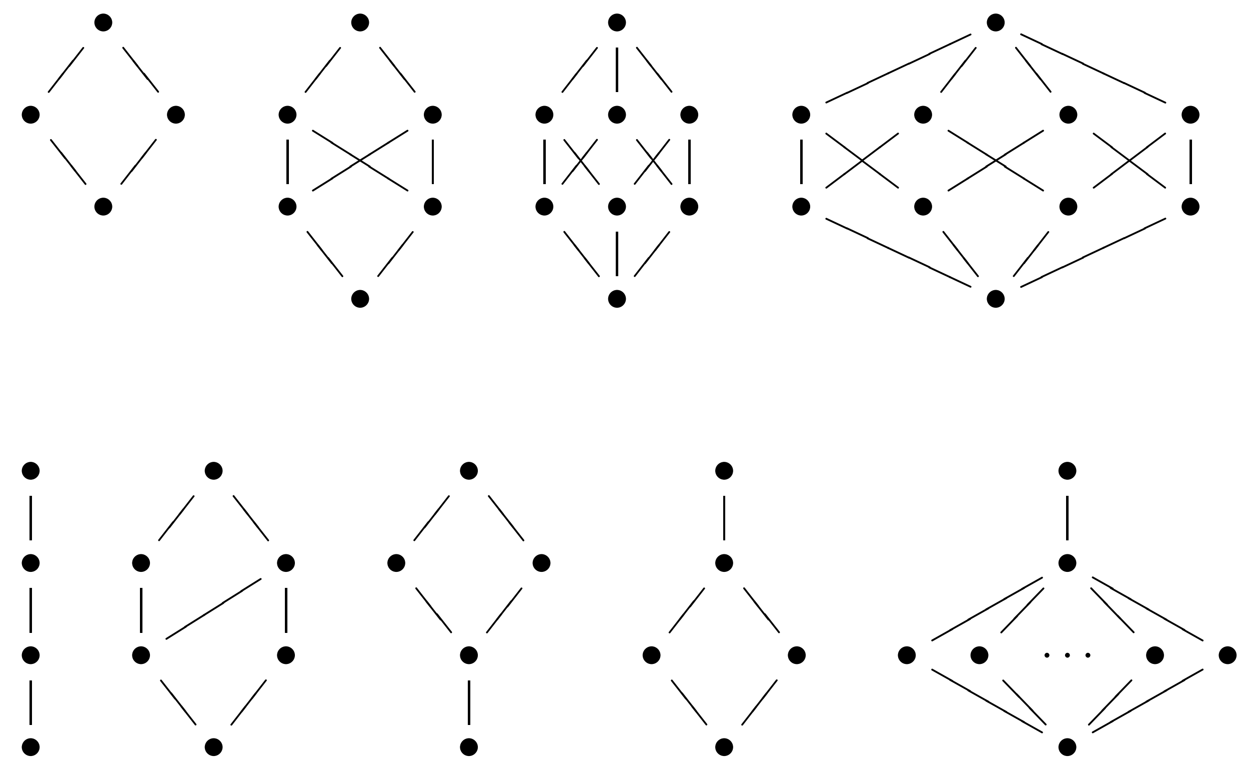
Common Generic Situation:

$\exists$ "natural" torus $H \subseteq \operatorname{fAut} A$ and $H\text{-spec}\, A$ finite

$H \subseteq \widetilde{H}$ = maximal torus

then $H\text{-spec}\, A = \widetilde{H}\text{-spec}\, A$

Some H-spec examples



2. BRUHAT INTERVALS

Coxeter group W: generators $s \in S$

relations $s^2 = 1$ and $(st)^{m(s,t)} = 1 \ \forall\, s \neq t$

$m(s,t) = m(t,s) \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$

reduced expression: $w = s_1 s_2 \cdots s_l$, $s_i \in S$, l minimal

Bruhat order: $v \leq w \iff$ some subword of $s_1 s_2 \cdots s_l$ is a reduced expression for v

Bruhat interval: $[v, w] := \{x \in W \mid v \leq x \leq w\}$

Fix base field k, $\quad 0 \neq q \in k, \quad q \neq \sqrt[\bullet]{1}$

Quantized Schubert cell algebras

$\mathfrak{g} = \operatorname{Lie} G$ semisimple, $\quad w \in W =$ Weyl group (Coxeter)

W generated by reflections $s_i \sim$ simple roots α_i

$w = s_{i_1} s_{i_2} \cdots s_{i_l}$ reduced

$\beta_1 := \alpha_{i_1}$, $\beta_2 := s_{i_1}(\alpha_{i_2})$, $\ldots$, $\beta_l := s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l})$

$U_q^+[w] :=$ subalgebra of $U_q(\mathfrak{g})^+$ generated by

quantum root vectors $\sim \beta_1, \ldots, \beta_l$

$=$ quantized coord. ring of Schubert cell $B \cdot wB \subseteq G/B$

$H =$ maximal torus of G acts naturally on $U_q^+[w]$

Thm [Yakimov] $\quad H\text{-spec}\, U_q^+[w] \cong [\mathrm{id}, w]$

Qns [Nowlin] $\quad A =$ a generic quantized coord. ring with

$H\text{-spec}\, A$ finite

Is $H\text{-spec}\, A \cong$ a Bruhat interval?

In what Coxeter group?

3. QUANTUM NILPOTENT ALGEBRAS

CGL extension of k :

$R = k[x_1][x_2;\sigma_2,\delta_2]\cdots[x_N;\sigma_N,\delta_N]$

with rational action of a k-torus H such that

$x_i = H$-eigenvectors; δ_i locally nilpotent;

$\sigma_k = (h_k\cdot)$ for some $h_k \in H$ with

h_k-eigenvalue of $x_k \neq \sqrt[\bullet]{1}$

E.G. Quantum 2×2 matrices:

$R = \mathcal{O}_q(M_2)$ generated by $X_{11}, X_{12}, X_{21}, X_{22}$ with

$X_{i1}X_{i2} = qX_{i2}X_{i1}$; $X_{1j}X_{2j} = qX_{2j}X_{1j}$

$X_{12}X_{21} = X_{21}X_{12}$;

$X_{11}X_{22} - X_{22}X_{11} = (q-q^{-1})X_{12}X_{21}$

$H = (k^\times)^4$ acts with $(\alpha_1,\alpha_2,\beta_1,\beta_2)\cdot X_{ij} = \alpha_i\beta_j X_{ij}$

Thm [$\subseteq$ Lusztig; De Concini–Kac–Procesi]

All $U_q^+[w]$ are CGL extensions

Thm [Letzter-KG] $R =$ CGL, length $N \implies \left|H\text{-spec}\, R\right| \le 2^N$

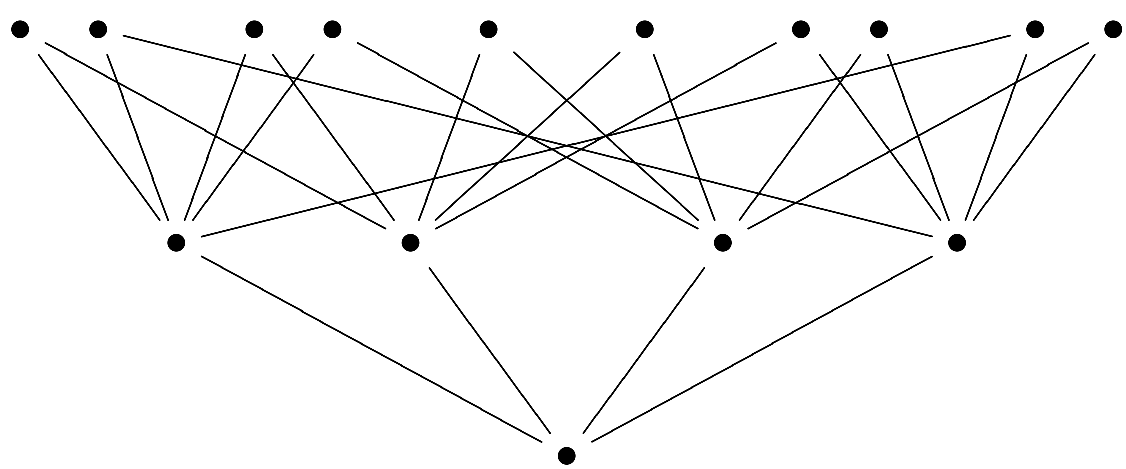
E.G. R-matrix of type $D_4 \longrightarrow$

F-R-T algebra A with generators X_{ij}, $1 \le i, j \le 4$

[Nowlin] : algebra $\mathbb{X}$ covering $k\langle X_{1j}, X_{2j} \mid 1 \le j \le 4 \rangle \quad \ni$

- $\mathbb{X} =$ CGL extension of length 8
- $\big| H\text{-spec}\, \mathbb{X} \big| = 112$
- $H\text{-spec}\, \mathbb{X} \not\cong [\mathrm{id}, w] \subseteq$ any finite Coxeter group
  $\Longrightarrow$ $\mathbb{X} \not\cong$ any $U_q^+[w]$
- $H\text{-spec}\, \mathbb{X}$ is $\cong [\mathrm{id}, w] \subseteq$ an infinite Coxeter group

H-SPEC $\mathbb{X}$, UP TO HEIGHT 2



Qn [Nowlin] Is H-spec$\,(CGL) \cong$ a Bruhat interval?

Example

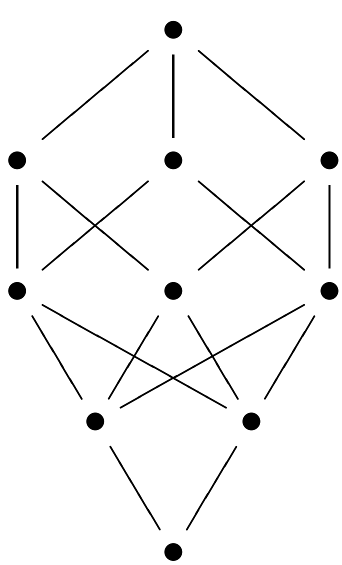
$R = k[x_1, x_2][x_3; \sigma_3, \delta_3][x_4; \sigma_4, \delta_4]$ with

$$x_2 x_1 = x_1 x_2 \quad x_3 x_1 = x_1 x_3 \quad x_3 x_2 = q x_2 x_3 + x_1^2$$
$$x_4 x_1 = x_1 x_4 \quad x_4 x_2 = q^{-1} x_2 x_4 + x_1^2 \quad x_4 x_3 = q x_3 x_4$$

and action of $H = (k^\times)^2$ such that

$$(\alpha, \beta) \cdot x_1 = \alpha x_1 \qquad (\alpha, \beta) \cdot x_2 = \beta x_2$$
$$(\alpha, \beta) \cdot x_3 = \alpha^2 \beta^{-1} x_3 \qquad (\alpha, \beta) \cdot x_4 = \alpha^2 \beta^{-1} x_4$$

H-spec$\,R =$



H-spec$\,R \ncong$ Bruhat interval in any Coxeter group

Known "height 2 property" in Bruhat intervals:

Any element of height 2 dominates exactly two elements of height 1

Thm [Lenagan-KG] H-spec$\,(CGL)$ has height 2 property

Qns For H-spec of CGLs or quantized coordinate rings:

- What finite posets appear?
- Are they ranked (= catenary)?
- $\exists$ maximum element?

Open Problem

Characterize Bruhat intervals among finite posets