The commutative core of a Leavitt path algebra

Cristóbal Gil Canto

(Joint work with Alireza Nasr-Isfahani)

University of Málaga

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Outline

The corresponding analytic result for graph C^*-algebras was given in:

G. NAGY, S. REZNIKOFF, Abelian core of graph algebras, *J. London Math. Soc.* **85 (2)**,(2012), 889–908.

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Introduction

Leavitt path algebras are natural generalizations of the Leavitt algebras $L(1,n)$ of type $(1,n)$:

$$L(1,n) =< x_1, \ldots, x_n, y_1, \ldots, y_n \mid x_i y_j = \delta_{ij}, \sum_{i=1}^{n} y_i x_i = 1 > .$$

They are originally investigated by Leavitt in

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Graphs I

Let E be a directed graph: $E = (E^0, E^1, r, s)$ where E^0, E^1 are two sets and $r, s : E^1 \to E^0$ maps. The elements of E^0 are called vertices and the elements of E^1, edges.

$$\bullet^{s(e)} \xrightarrow{\ e\ } \bullet^{r(e)}$$

$s(e)$ is the source of e and $r(e)$ is the range.

v is called a regular vertex if $s^{-1}(v)$ is a finite non-empty set.

For each $e \in E^1$, we call e^* a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$:

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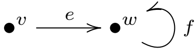
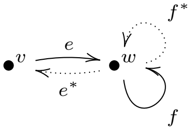
For 

we have 

Graphs II

A path μ, of length $n \geq 1$, in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$.

- μ is closed if $r(e_n) = s(e_1)$.
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- e is an exit for μ if $s(e) = s(e_i)$ for some i and $e \neq e_i$:

Given two paths α and β, we say $\alpha \leq \beta$ if $\beta = \alpha\alpha'$ for some path α'.

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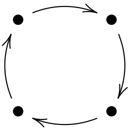

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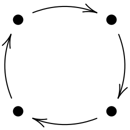

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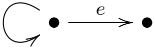

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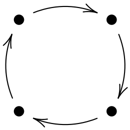

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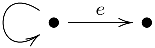

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Leavitt path algebras

Definition

Given an arbitrary graph E and a commutative ring R with unit, the Leavitt path R-algebra $L_R(E)$ is defined to be the R-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.

(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.

(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

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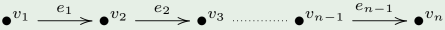
Some well-known examples

Examples

- The matrix algebra $M_n(R)$ is the Leavitt path algebra of a line graph given by:



- The ring of Laurent polynomials $R[x, x^{-1}]$ is the Leavitt path algebra of the graph R_1:

- Classical Leavitt algebras $L(1, n)$ for $n \geq 2$: consider the graph R_n and then $L(1, n) \cong L_R(R_n)$:

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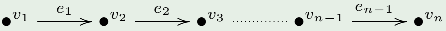

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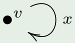

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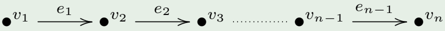

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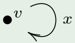

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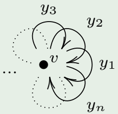

Basic properties of Leavitt path algebras

- There exists an involution $x \mapsto \overline{x}$ defined on the generators by $\overline{v} = v$, $\overline{e} = e^*$, $\overline{e^*} = e$ for every $v \in E^0, e \in E^1$.
- Monomials of the type pq^*, where p and q are paths in E.

$$L_R(E) = \mathrm{span}_R(G_E)$$

where $G_E = \{pq^* : p, q \in \mathrm{Path}(E) \text{ and } r(p) = r(q)\}$ and $rv \neq 0$ for all $v \in E$ and all $r \in R \setminus \{0\}$.

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Distinguished paths

Let us consider a path $\alpha = e_1 \dots e_n$ such that $r(\alpha)$ is visited by the cycle without exits λ_α and:

- if $l(\alpha) = 0$ then α is simply a vertex v and the cycle λ_α starts and ends at v;
- if $l(\alpha) \geq 1$ then λ_α is a cycle that starts and ends at $r(\alpha) = r(e_n)$ but does not visit any of the vertices $s(e_1), \dots, s(e_n)$.

Definition

The path α above described will be called a distinguished path.

Example

$\alpha = e_1 e_2$ is a distinguished path:

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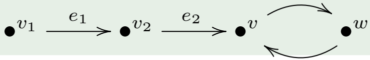

Normal elements

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An element $x \in L_R(E)$ is said to be normal if $xx^* = x^*x$.

Example

$x = ef^* + fe^*$ is normal since:

- we have $x^* = (ef^* + fe^*)^* = fe^* + ef^* = x$;
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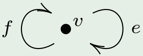

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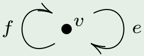

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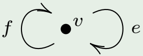

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The commutative core

Proposition

Let $\alpha, \beta \in$ Path(E) with $r(\alpha) = r(\beta)$. The generator $\alpha\beta^* \in G_E$ is a normal element in $L_R(E)$ if and only if one of the following holds:

1. $\alpha = \beta$;
2. $\beta \leq \alpha$ and β is a distinguished path: $\alpha = \beta\lambda_\beta$ and λ_β a cycle without exits;
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Denote by G_E^M the set of all such normal generators, then the R-algebra $M_R(E)$ generated by G_E^M, $M_R(E) = \langle G_E^M \rangle \subseteq L_R(E)$ is commutative.

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The subalgebra $M_R(E)$ is called the commutative core of $L_R(E)$.

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Main theorem

The diagonal subalgebra is $\Delta(E) = \langle G_E^{\Delta} \rangle$ where $G_E^{\Delta} = \{\alpha\alpha^* : \alpha \in \mathsf{Path}(E)\}$. Notice that $\Delta(E) \subseteq M_R(E)$.

Theorem

Let E be a graph and R a commutative ring with unit. Consider the commutative core $M_R(E) \subseteq L_R(E)$. Then

$$M_R(E) = \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\}.$$

Furthermore, $M_R(E)$ is a maximal commutative subalgebra of $L_R(E)$.

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Furthermore, $M_R(E)$ is a maximal commutative subalgebra of $L_R(E)$.

Some ideas in the proof I

- $\subseteq$ Clearly $M_R(E) \subseteq \{x \in L_R(E) : xd = dx$ for every $d \in \Delta(E)\}$.
- $\supseteq$ The other containment is the hardest part in the proof:
  - The tools come from C^*-algebras, but can be interpreted for rings.
  - The idea is to define a map $\mathcal{E}_M : L_R(E) \to M_R(E)$ satisfying certain conditions: in particular one of these says that $\mathrm{Im}(\mathcal{E}_M) = M_R(E)$. These maps are called conditional expectations.
  - In the end if $x' \in \{x \in L_R(E) : xd = dx$ for every $d \in \Delta(E)\}$, then it is proved that

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Some ideas in the proof II

To see $M_R(E)$ is a maximal commutative subalgebra inside $L_R(E)$: consider $\mathcal{C}$ a commutative subalgebra of $L_R(E)$ such that $M_R(E) \subseteq \mathcal{C}$. We have $\Delta(E) \subseteq M_R(E) \subseteq \mathcal{C}$; then in particular

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So necessarily $\mathcal{C} = M_R(E)$.

Remark: $M_R(E)$ might not be the unique maximal commutative subalgebra of $L_R(E)$.

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Remark: $M_R(E)$ might not be the unique maximal commutative subalgebra of $L_R(E)$.

The center of a Leavitt path algebra

Corollary

The center of the Leavitt path algebra $L_R(E)$, that is

$$Z(L_R(E)) = \{x \in L_R(E) : xy = yx \text{ for every } y \in L_R(E)\}$$

satisfies that $Z(L_R(E)) \subseteq M_R(E)$.

Generalized Cuntz-Krieger uniqueness theorem

Theorem

Let E be a graph and R a commutative ring with unit. Consider $\Phi : L_R(E) \to \mathcal{A}$ a ring homomorphism. Then the following conditions are equivalent:

(i) Φ *is injective;*

(ii) *the restriction of* Φ *to* $M_R(E)$ *is injective;*

(iii) *both these conditions are satisfied:*

(a) $\Phi(rv) \neq 0$, *for all* $v \in E^0$ *and for all* $r \in R \setminus \{0\}$*;*

(b) *for every distinguished path* α *denote* $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$, *where* λ_α *is the cycle without exits that starts and ends at* $s(\alpha)$*. Then the* $*$ R*-algebra* $< \Phi(\omega_\alpha) >$ *generated by* $\Phi(\omega_\alpha)$ *is* $*$*-isomorphic to* $R[x, x^{-1}]$*; i.e.,*

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★ Important result used in the proof: "Reduction theorem"

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Cuntz-Krieger uniqueness theorem

Condition (L)

Every cycle in E has an exit.

Corollary

Let E be a graph satisfying Condition (L) and let R be a commutative ring with unit. Suppose $\Phi : L_R(E) \to \mathcal{A}$ is a ring homomorphism. Then the following conditions are equivalent:

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THANK YOU!