π-BAER RINGS

WITH Y. KARA AND A. TERCAN

NOTATION: $\mathcal{I}(R)$ IS THE SUBRING OF R GEN. BY THE IDEMPOTENT ELEMENTS OF R.

$\underline{\mathcal{S}_{\ell}(R)} := \{e = e^2 \in R \mid re = ere \ \forall\, r \in R\}$.
(LEFT SEMICENTRAL IDEMPOTENTS)

$\underline{\mathcal{S}_{r}(R)} := \{e = e^2 \in R \mid er = ere \ \forall\, r \in R\}$.
(RIGHT SEMICENTRAL IDEMPOTENTS)

DEF. R IS A BAER (QUASI-BAER) RING IF FOR EACH NONEMPTY SET S (IDEAL I) OF R THERE IS AN $e = e^2 \in R$ s.t. $r(S) = eR$ ($r(I) = eR$).

EXAMPLES & PROPERTIES

BAER	QUASI-BAER
1) R IS A DOMAIN iff R IS BAER & $\{0,1\}$ ARE THE ONLY IDEMPOTENTS OF R.	R IS A PRIME RING iff R IS QUASI-BAER & $\{0,1\}$ ARE THE ONLY SEMICENTRAL IDEMPOTENTS OF R.
(CHATTERS - KHURI) 2) R IS RT. NONSING. RT. EXTENDING iff R IS RT. CONONSINGULAR & BAER.	ASSUME R IS SEMIPRIME, THEN R IS RT. FI-EXTENDING iff R IS QUASI-BAER.
3) ASSUME R IS A VNR. THEN BAER $\not\Leftrightarrow$ EXTENDING.	ASSUME R IS A VNR. THEN R IS QUASI-BAER $\Longleftrightarrow$ FI-EXTENDING.

(ARMENDARIZ)

4) ASSUME R IS REDUCED. THEN R IS BAER IFF $R[x]$ IS BAER.	R IS QUASI-BAER IFF $R[x]$ IS QUASI-BAER.
5) R BAER $\not\Leftrightarrow$ $\mathcal{T}_n(R)$ OR $\mathcal{M}_n(R)$ ARE BAER.	R IS QUASI-BAER IFF $\mathcal{T}_n(R)$ AND $\mathcal{M}_n(R)$ ARE QUASI-BAER.

QUESTION: IS THERE A CLASS OF RINGS INTERMEDIATE BETWEEN BAER & QUASI-BAER WHICH HAS IMPORTANT FEATURES OF EACH OF THESE CLASSES?

DEF. A LEFT (RIGHT) IDEAL Y OF R IS CALLED <u>PROJECTION INVARIANT</u> IF $Ye \subseteq Y$ ($eY \subseteq Y$) $\forall\, e = e^2 \in R$.

EXAMPLES: (i) R IS ABELIAN IFF EACH 1-SIDED IDEAL IS PROJECTION INVAR.

(ii) IF $R = \mathcal{B}(R)$, THEN EACH 1-SIDED PROJECTION INVARIANT IDEAL IS A 2-SIDED IDEAL.

(iii) ASSUME A IS AN ABELIAN RING, AND X IS A RT. IDEAL OF A WHICH IS NOT AN IDEAL. LET $R = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$. THEN $\begin{bmatrix} X & X \\ 0 & 0 \end{bmatrix}$ IS A PROJECTION INVAR. RT. IDEAL OF R WHICH IS NOT AN IDEAL OF R.

ALSO, $\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$ IS A RT. IDEAL OF R WHICH IS NOT PROJECTION INVAR.

DEF. R IS A <u>PROJECTION INVARIANT BAER RING</u>, <u>π-BAER</u>, IF FOR EACH PROJECTION INVAR. LEFT IDEAL Y OF R $\exists\ c = c^2 \in R$ s.t. $\underline{r}(Y) = cR$.

NOTE: R IS A DOMAIN IFF R IS π-BAER & $\{0,1\}$ ARE THE ONLY IDEMPOTENTS OF R.

NOTE: BAER, QUASI-BAER AND π-BAER ARE ALL LEFT-RIGHT SYMMETRIC CONDITIONS.

BASIC RESULTS

THM. 1. T.F.A.E.:

(i) R IS π-BAER.

(ii) FOR EACH $\emptyset \neq S \subseteq R$ s.t. $Se \subseteq S$ $\forall\, e = e^2 \in R$, $\exists\, c = c^2 \in R$ s.t. $\underline{r}(S) = cR$.

(iii) EVERY PROJECTION INVAR. RIGHT ANNIHILATOR IS GEN. BY AN IDEMPOTENT.

(iv) FOR EACH PROJECTION INVAR. RIGHT IDEAL I OF R $\exists\, e \in \mathcal{S}_\ell(R)$ s.t. $I \subseteq eR$ AND

$$\underline{\ell}(I) \cap eR = eR(1-e).$$

THM. 2. CONSIDER THE FOLLOWING CONDITIONS:

(i) R IS BAER.

(ii) R IS π-BAER.

(iii) R IS QUASI-BAER.

THEN $(i) \Rightarrow (ii) \Rightarrow (iii)$.

$\qquad\quad \not\Leftarrow \qquad \not\Leftarrow$

$(i) \not\Leftarrow (ii)$ LET $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$, WHERE D IS A DOMAIN, NOT A DIVISION RING.

$(ii) \not\Leftarrow (iii)$ LET R BE A PRIME RING, NOT A DOMAIN, WITH ONLY TRIVIAL IDEMPOTENTS. FOR EXAMPLE:

ZALESSKII – NEROSLAVSKII (SIMPLE NOETHERIAN)

CERTAIN GROUP RINGS (K. A. BROWN).

COR. 3. (i) R IS π-BAER $\Rightarrow$ CEN(R) IS BAER.
(ii) IF R IS ABELIAN, THEN R IS π-BAER $\Leftrightarrow$ R IS BAER.
(iii) IF $R = \mathcal{I}(R)$, THEN R IS π-BAER $\Leftrightarrow$ R IS QUASI-BAER.

DEF. A SUBRING S OF R IS SAID TO BE <u>LEFT (RIGHT) INTRINSIC</u> IN R IF $X \cap S \neq 0$ FOR EACH NONZERO LEFT (RIGHT) IDEAL X OF R (e.g., $\mathbb{Z}$ IS LEFT & RT. INTRINSIC IN $\mathbb{H}$)

THM. 4. LET S BE A SUBRING OF R.
(i) IF $\mathcal{I}(R) \subseteq S$ AND R IS π-BAER, THEN S IS π-BAER.
(ii) IF S IS LEFT AND RIGHT INTRINSIC IN R AND S IS π-BAER, THEN R IS π-BAER.

DEF. R IS <u>RIGHT π-EXTENDING</u> IF EACH PROJECTION INVAR. RIGHT IDEAL IS ESSENTIAL IN AN IDEMPOTENT GEN. RIGHT IDEAL.

THM. 5. LET R BE RIGHT NONSINGULAR. THEN R IS RIGHT π-EXTENDING $\Leftrightarrow$ R IS π-BAER AND $A_R \leq^{ess} \underline{r}\,\underline{\ell}(A)$ FOR EVERY PROJECTION INVAR. RIGHT IDEAL A OF R.

THM. 6. (i) ASSUME R IS RIGHT NONSINGULAR AND RIGHT CONONSINGULAR. THEN R IS π-BAER $\Leftrightarrow$ R IS RIGHT π-EXTENDING.

(ii) ASSUME R IS A SEMIPRIME π-BAER RING s.t. $\partial(R)$ IS RIGHT INTRINSIC IN R. THEN R IS RIGHT π-EXTENDING.

(iii) ASSUME R IS A REGULAR RING. THEN R IS π-BAER $\Leftrightarrow$ R IS RIGHT π-EXTENDING.

THM. 7. ASSUME R IS π-BAER, $e = e^2 \in R$ AND eR IS A PROJECTION INVAR. RIGHT IDEAL OF R. THEN BOTH eRe AND $(1-e)R(1-e)$ ARE π-BAER RINGS AND $e \in \mathcal{S}_\ell(R)$.

MATRIX RINGS.

THM. 8. LET $T = \begin{bmatrix} S & {}_S M_R \\ 0 & R \end{bmatrix}$. T IS π-BAER ⟺

(i) S AND R ARE π-BAER;

(ii) $r_M(I) = r_S(I)M$ ∀ PROJECTION INAR. LEFT IDEALS I OF S; AND

(iii) IF ${}_S N_{\mathcal{S}(R)} \trianglelefteq {}_S M_{\mathcal{S}(R)}$, THEN $r_R(N) = aR$ FOR SOME $a = a^2 \in R$.

COR. 9. LET R BE A π-BAER RING, S A SUBRING OF R s.t. $\mathcal{S}(R) \subseteq S$ AND $M \trianglelefteq R$. THEN T IS π-BAER. IN PARTICULAR, IF R IS A DOMAIN WHICH IS NOT A DIVISION RING AND $0 \neq M$, THEN T IS π-BAER, BUT NOT BAER.

COR. 10. ASSUME R_R AND M_R ARE π-EXTENDING AND NONSINGULAR, AND $S = END(M_R)$. THEN T IS π-BAER AND π-EXTENDING. HOWEVER, IN GENERAL, T IS NEITHER BAER NOR EXTENDING.

EXAMPLE LET R = S BE RIGHT ORE DOMAINS AND $0 \neq M \trianglelefteq R$. THEN T IS π-BAER AND π-EXTENDING BUT NEITHER BAER NOR EXTENDING.

COR. 11. T.F.A.E.

(i) R IS π-BAER.

(ii) $T_n(R)$ IS π-BAER FOR EACH $n \in \mathbb{Z}^+$.

(iii) $T_n(R)$ IS π-BAER FOR SOME $n \geq 1$.

PROP. 12. IF R IS π-BAER, THEN $\mathcal{M}_n(R)$ IS π-BAER $\forall\ n \in \mathbb{Z}^+$.

π - BAER POLYNOMIAL EXTENSIONS

THM. 13. LET R BE A π-BAER RING. THEN THE FOLLOWING EXTENSION RINGS ARE π-BAER, WHERE X IS AN ARBITRARY NONEMPTY SET OF INDETERMINATES, α IS A RING AUTOMORPHISM OF R AND σ IS AN α DERIVATION OF R:

(i) $R[X]$,
(ii) $R[[X]]$,
(iii) $R[x;\alpha,\sigma]$ (i.e., THE ORE EXTENSION),
(iv) $R[[x;\alpha]]$,
(v) $R[x,x^{-1};\alpha]$,
(vi) $R[[x,x^{-1};\alpha]]$.

EXAMPLE (P.M. COHN) LET $R = \mathcal{M}_2(\mathbb{Z})$. THEN R IS A BAER RING. BY THM. 13 $R[x]$ IS π-BAER. NOTE THAT $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x = \begin{bmatrix} 2 & 0 \\ x & 0 \end{bmatrix} \in R[x]$. BUT

$\ell_{R[x]}\left(\begin{bmatrix} 2 & 0 \\ x & 0 \end{bmatrix}\right)$ IS NOT GEN. BY AN IDEMPOTENT.

SO, $R[x]$ IS NOT BAER.

LEM. 14 LET $e(x) \in R[x]$ (resp. $R[[x]]$), WHERE $e(x) = e_0 + e_1 x + \cdots + e_m x^m$ (resp. $e(x) = e_0 + e_1 x + \cdots$).

ASSUME $e(x) = (e(x))^2$. THEN

(i) $e_j \in \mathcal{J}(R)$ $\forall j \geq 0$.

(ii) $e_j, e_0 e_j, e_j e_0 \in \mathcal{N}(R)$ $\forall j \geq 1$, WHERE $N(R)$ IS THE SUBRING OF (NOT NECESSARILY WITH 1) GEN. BY THE NILPOTENT ELEMENTS OF R.

THM. 15. T.F.A.E.

(i) R IS π-BAER.

(ii) $R[x]$ IS π-BAER.

(iii) $R[[x]]$ IS π-BAER.

THANK YOU !!!