The Work of Maxim Kontsevich

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Maxim Kontsevich is known principally for his work on four major problems in geometry. In each case, it is fair to say that Kontsevich's work and his view of the issues has been tremendously influential to subsequent developments. These four problems are:

- Kontsevich presented a proof of a conjecture of Witten to the effect that a certain, natural formal power series whose coefficients are intersection numbers of moduli spaces of complex curves satisfies the Korteweg-de Vries hierarchy of ordinary, differential equations.
- Kontsevich gave a construction for the universal Vassiliev invariant for knots in 3-space, and generalized this construction to give a definition of pertubative Chern-Simons invariants for three dimensional manifolds. In so doing, he introduced the notion of Graph Cohomology which succinctly summarizes the algebraic side of the invariants. His constructions also vastly simplified the analytic aspects of the definitions.
- Kontsevich used the notion of stable maps of complex curves with marked points to compute the number of rational, algebraic curves of a given degree in various complex projective varieties. Moreover, Kontsevich's techniques here have greatly affected this branch of algebraic geometry. Kontsevich's formulation with Manin of the related Mirror Conjecture about Calabi-Yau 3-folds has also proved to be highly influential.
- Kontsevich proved that every Poisson structure can be formally quantized by exhibiting an explicit formula for the quantization.

What follows is a brief introduction for the non-expert to these four areas of Kontsevich's work. Here, I focus almost solely on the contributions of Kontsevich to the essential exclusion of many others; and I ask to be pardonned for my many and glaring omissions.

1 Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function [1]

To start the story, fix integers $g \geq 0$ and $n > 0$ which are constrained so $2g + n \geq 2$. That is, the compact surface of genus g with n punctures has negative Euler characteristic. Introduce the moduli space $M_{g,n}$ of smooth, compact, complex curves of genus g with n distinct marked points. This is to say that a point in $M_{g,n}$ consists of an equivalence class of tuple consisting of a complex structure j on a compact surface C of genus g, together with an ordered set $\Lambda \equiv \{x_1, \dots, x_n\} \subset C$ of n points. The equivalence is under the action of the diffeomorphism group of the surface. This $M_{g,n}$ has a natural compactification (known as the Deligne-Mumford compactification) which will not be notationally distinguished. Suffice it to say that the compactification has a natural fundamental class, as well as an n-tuple of distinguished, complex line bundles. Here, the i'th such line bundle, L_i, at the point $(j, \Lambda) \in M_{g,n}$ is the holomorphic cotangent space at $x_i \in \Lambda$.

With the preceding understood, note that when $\{d_1, \dots, d_n\}$ are non-negative integers which sum to the dimension of $M_{g,n}$ (which is $3g-3+n$). Then, a number is obtained by pairing the cohomology class

$$\prod_{1 \le i \le n} c_i(L_i)^{d_i}$$

with the afore-mentioned fundamental class of $M_{g,n}$. (Think of representing these Chern classes by closed 2-forms and then integrating the appropriate wedge product over the smooth part of $M_{g,n}$.) Using Poincaré duality, such numbers can be viewed as intersection numbers of varieties on $M_{g,n}$ and hence the use of this term in the title of Kontsevich's article.

As g, n and the integers $\{d_1, \dots, d_n\}$ vary, one obtains in this way a slew of intersection numbers from the set of spaces $\{M_{g,n}\}$. In this regard, it proved convenient to keep track of all these numbers with a generating functional. The latter is a formal power series in indeterminants $t_0, t_1, \dots$ which is written schematically as

$$F(t_0, t_1, \dots) = \sum_{(k)} \langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle \prod_{i \ge 0} \frac{t_i^{k_i}}{k_i!}, \tag{1}$$

where, (k) signifies the multi-index $(k_0, k_1, \dots)$ consisting of non-negative integers where only finitely many are non-zero. Here, the expression $\langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle$ is the number which is obtained as follows: Let

$$n = k_1 + k_2 + \dots, \quad \text{and} \quad g = \tfrac{1}{3}(2(k_1 + 2k_2 + 3k_3 + \dots) - n) + 1\,.$$

If g is not a positive integer, set $\langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle = 0$. If g is a positive integer, construct on $M_{g,n}$ the product of $c_1(L_j)$ for $1 \le j \le k_1$ times the product of $c_1(L_j)^2$ for $k_1 + 1 \le j \le k_1 + k_2$ times $\dots$ etc.; and thus construct a form whose dimension is $3g - 3 + n$, which is that of $M_{g,n}$. Finally, pair this class on the fundamental class of $M_{g,n}$ to obtain $\langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle$.

By comparing formal properties of two hypothetical quantum field theories, E. Witten was led to conjecture that the formal series $U \equiv \partial^2 F / \partial t_0^2$ obeys the classical KdV equation,

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}\,. \tag{2}$$

(As U is a formal power series, this last formula can be viewed as a conjectural set of relations among the intersection numbers which appear in the definition of F in (1).)

Kontsevich gave the proof that U obeys this KdV equation. His proof of Equation (2) is remarkable if nothing else then for the fact that he gives what is essentially an explicit calculation of the intersection numbers $\{\langle\tau_0^{k_0}\tau_1^{k_1}\cdots\rangle\}$. To this end, Kontsevich first introduces a model for $M_{g,n}$ based on what he calls ribbon graphs with metrics. (A ribbon graph is obtained from a 3-valent graph by more or less thickening the edges to bands. They are related to Riemann surfaces through the classical theory of quadratic differentials.) With an explicit, almost combinatorial model for $M_{g,n}$ in hand, Kontsevich proceeds to identify the classes $c_1(L_j)$ directly in terms of his model. Moreover, this identification is sufficiently direct to allow for the explicit computation of the integrals for $\{\langle\tau_0^{k_0}\tau_1^{k_1}\cdots\rangle\}$. It should be stressed here that this last step involves some extremely high powered combinatorics. Indeed, many of the steps in this proof exhibit Kontsevich's unique talent for combinatorical calculations. In any event, once the coefficients of U are obtained, the proof ends with an identification of the expression for U with a novel expansion for certain functions which arises in the KdV story. (These are the matrix Airy functions referred to at the very start of this section.)

2 FEYNMAN DIAGRAMS AND LOW DIMENSIONAL TOPOLOGY [2]

From formal quantum field theory arguments, E. Witten suggested that there should exist a family of knot invariants and three manifolds invariants which can be computed via multiple integrals over configuration spaces. Kontsevich gave an essentially complete mathematical definition of these invariants, and his ideas have profoundly affected subsequent developments.

In order to explain, it proves useful to first digress to introduce some basic terminology. First of all, the three dimensional manifolds here will be all taken to be smooth, compact and oriented, or else Euclidean space. A knot in a three manifold is a connected, 1-dimensional submanifold, which is to say, the embedded image of the circle. A link is a finite, disjoint collection of knots. A knot or link invariant is an assignment of some algebraic data to each knot or link (for example, a real number), where the assignments to a pair of knots (or links) agree when one member of the pair is the image of the other under a diffeomorphism of the ambient manifold. (One might also restrict to diffeomorphisms which can be connected by a path of diffeomorphisms to the identity map.)

A simple example is provided by the Gauss linking number an invariant of links with two components which can be computed as follows: Label the components as K_1 and K_2. A point in K_1 together with one in K_2 provides the directed vector from the former to the latter, and thus a point in the 2-sphere. Since both K_1 and K_2 are copies of the circle, this construction provides a map from the 2-torus (the product of two circles) to the 2-sphere. The Gauss linking number is the degree of this map. (The invariance of the degree under homotopies implies that this number is an invariant of the link.) Alternately, one can introduce the standard, oriented volume form ω on the 2-sphere, and then the Gauss linking number is the integral over the $K_1 \times K_2$ of the pull-back of the form ω.

Witten conjectured the existence of a vast number of knot, link and 3-manifold invariants of a form which generalizes this last formula for the Gauss linking number. Independently of Kontsevich, significant work towards constructing these invariants for knots and links had been carried out by Bar-Natan, Birman, Garoufalidis, Lin, and Guadagnini-Martinelli-Mintchev. Meanwhile, Axelrod and Singer had developed a formulation of the three-manifold invariants.

In any event, what follows is a three step sketch of Kontsevich's formulation for an invariant of a three-manifold M with vanishing first Betti number.

Step 1: The invariants in question will land in a certain graded, abelian group which is constructed from graphs. Kontsevich calls these groups "graph cohomology groups." To describe the groups, introduce the set G_0 of pairs consisting of a compact graph Γ with only three-valent vertices and a certain kind of orientation o for Γ. To be precise, o is an orientation for

$$(\bigoplus_{\mathrm{edges}(\Gamma)} \mathbb{R}) \otimes \mathrm{H}^1(\Gamma)\,.$$

Note that isomorphisms between such graphs pull back the given o. Thus, one can think of G_0 as a set of isomorphism classes. Next, think of the elements of G_0 as defining a basis for a vector space over $\mathbb{Z}$ where consistency forces the identification of $(\Gamma, -o)$ with $-(\Gamma, o)$.

One can make a similar definition for graphs where all vertices are three valent save for one four valent vertex. The resulting $\mathbb{Z}$-module is called G_1. In fact, for each $n \geq 0$ there is a $\mathbb{Z}$-module G_n which is constructed from graphs with all vertices being at least 3-valent, and with the sum over the vertices of (valence -3) equal to n.

With the set $\{G_n\}_{n\geq 0}$ more or less understood, remark that there are natural homomorphisms $\partial\colon G_n \to G_{n+1}$ which obey $\partial^2 = 0$. Indeed, ∂ is defined schematically as follows:

$$\partial(\Gamma, o) = \sum_{e\in \mathrm{edges}(\Gamma)} (\Gamma/e,\ \text{induced orientation from } o)\,.$$

Here, Γ/e is the graph which is obtained from Γ by contracting e to a point. The induced orientation is quite natural and left to the reader to work out. In any event, with ∂ in hand, the modules $\{G_n\}$ define a differential complex, whose cohomology groups are

$$GC_* \equiv\ \mathrm{kernel}(\partial\colon\ G_* \to G_{*+1})/\ \mathrm{Image}(\partial\colon\ G_{*-1} \to G_*)\,. \tag{3}$$

This is 'graph cohomology'. For the purpose of defining 3-manifold invariants, only GC_0 is required.

Step 2: Fix a point $p \in M$ and introduce in $M \times M$ the subvariety

$$\Sigma = (p \times M) \cup (M \times p) \cup \Delta\,,$$

where Δ denotes the diagonal. A simple Meyer-Vietoris argument finds closed 2-forms on $M \times M - \Sigma$ which integrate to 1 on any linking 2-sphere of any of the three components of Σ. Moreover, there is such a form ω with $\omega \wedge \omega = 0$ near Σ. In fact, near Σ, this ω can be specified almost canonically with the choice of a framing for the tangent bundle of M. (The tangent bundle of an oriented 3-manifold can always be framed. Furthermore, Atiyah essentially determined a canonical frame for TM.) Away from Σ, the precise details of ω are immaterial. In any event, fix ω using the canonical framing for TM.

With ω chosen, consider a pair (Γ, o) from G_0. Associate to each vertex of Γ a copy of M, and to each oriented edge e of Γ, the copy of $M \times M$ where the first factor of M is labeled by the staring vertex of e, and the second factor by the ending vertex. Associate to this copy of $M \times M$ the form ω, and in this way, the edge e labels a (singular) 2-form ω_e on $\times_{\mathrm{vertices}(\Gamma)} M$.

Step 3: At least away from all versions of the subvariety Σ, the forms $\{\omega_e\}_{e \in \mathrm{edges}(\Gamma)}$ can be wedged together to give a top dimensional form $\prod_{e \in \mathrm{edges}(\Gamma)} \omega_e$, on $\times_{v \in \mathrm{vertices}(\Gamma)} M$. It is a non-trivial task to prove that this form is integrable. In any event, the assignment of this integral to the pair (Γ, o) gives a $\mathbb{Z}$-linear map from G_0 to $\mathbb{R}$. The latter map does not define an invariant of M from the pair (Γ, o) as there are choices involved in the definition of ω, and these choices effect the value of the integral. However, Kontsevich found a Stokes theorem argument which shows that this map from G_0 to $\mathbb{R}$ descends to the kernel of ∂ as an invariant of M. That is, these graph-parameterized integrals define a 3-manifold invariant with values in the dual space $(GC_0)^*$. (A recent paper by Bott and Cattaneo has an exceptionally elegant discussion of these points.)

Kontsevich's construction of 3-manifold invariants completely separates the analytic issues from the algebraic ones. Indeed, the module GC_0 encapsulates all of the algebra; while the analysis, as it were, is confined to issues which surround the integrals over products of M. In particular, much is known about GC_0; for example, it is known to be highly non-trivial.

Kontsevich has a similar story for knots which involves integrals over configuration spaces that consist of points on the knot and points in the ambient space. Here, there is a somewhat more complicated analog of graph cohomology. In the case of knots in 3-sphere, Kontsevich's construction is now known to give all Vassiliev invariant of knots.

In closing this section, it should be said that Kontsevich has a deep understanding of these and related graph cohomology in terms of certain infinite dimensional algebras [3].

3 ENUMERATION OF RATIONAL CURVES VIA TORUS ACTIONS [4]

The general problem here is as follows: Suppose X is a compact, complex algebraic variety in some complex projective space. Fix a 2-dimensional homology class on X and 'count' the number of holomorphic maps from the projective line $\mathbb{P}^1$ into X which represent the given homology class. To make this a well posed problem, maps should be identified when they have the same image in X. The use of quotes around the word count signifies that further restrictions are typically necessary in order to make the problem well posed. For example, a common additional restriction fixes some finite number of points in X and requires the maps in question to hit the given points.

These algebro-geometric enumeration problems were considered very difficult. Indeed, for the case where $X = \mathbb{P}^2$, the answer was well understood prior to Kontsevich's work only for the lowest multiples of the generator of $\mathrm{H}_2(\mathbb{P}^2;\mathbb{Z})$. Kontsevich synthesized an approach to this counting problem which has been quickly adopted by algebraic geometers as the method of choice. Of particular interest are the counts made by Kontsevich for the simplest case of $X = \mathbb{P}^2$ and for the case where X is the zero locus in $\mathbb{P}^4$ of a homogeneous, degree 5 polynomial. (The latter has trivial canonical class which is the characterization of a Calabi-Yau manifold.)

There are two parts to Kontsevich's approach to the counting problem. The first is fairly general and is roughly as follows: Let V be a compact, algebraic variety and let β denote a 2-dimensional homology class on V. Kontsevich introduces a certain space M of triples (C, x, f) where C is a connected, compact, reduced complex curve, while $x = (x_1, \ldots, x_k)$ is a k-tuple of pairwise distinct points on C and $f\colon C \to V$ is a holomorphic map which sends the fundamental class of C to β. Moreover, the associated automorphism group of f is suitably constrained. (Here, k could be zero.) This space M is designed so that its compactification is a reasonable, complex algebraic space with a well defined fundamental class. (This compactification covers, in a sense, the oft used Deligne-Mumford compactification of the space of complex curves with marked points.) The utilization of this space M with its compactification is one key to Kontsevich's approach. In particular, suppose $X \subset V$ is an algebraic subvariety. Under certain circumstances, the problem of counting holomorphic maps from C into X can be computed by translating the latter problem into that of evaluating the pairing of M's fundamental class with certain products of Chern classes on M. The point here is that the condition that a map $f\colon C \to V$ lie in X can be reinterpreted as the condition that the corresponding points in M lie in the zero locus of a certain section of a certain bundle over M.

With these last points understood, Part 2 of Kontsevich's approach exploits the observation that $V = \mathbb{P}^n$ has a non-trivial torus action. Such an action induces one on M and its compactification. Then, in the manner of Ellingsrud and Stromme, Kontsevich uses one of Bott's fixed point formulas to obtain a formula for the appropriate Chern numbers in various interesting examples.

4 DEFORMATION QUANTIZATION OF POISSON MANIFOLDS

This last subject comes from very recent work of Kontsevich, so the discussion here will necessarily be brief. A 'Poisson structure' on a manifold X can be thought of as a bilinear map

$$B_1\colon C^\infty(X) \otimes C^\infty(X) \to C^\infty(X)$$

which gives a Lie algebra structure to $C^\infty(X)$. In particular, B_1 sends a pair (f, g) to $\langle \alpha, \mathrm{d}f \wedge \mathrm{d}g\rangle$ where α is a non-degenerate section of $\Lambda^2 TX$ which satisfies a certain quadratic differential constraint. The problem of quantizing such a Poisson structure can be phrased as follows: Let h be a formal parameter (think Planck's constant). Find a set of bi-differential operators $B_2, B_3, \ldots$ so that

$$f * g \equiv fg + h \cdot B_1(f,g) + h^2 \cdot B_2(f,g) + \ldots$$

defines an associative product taking pairs of functions on X and returning a formal power series with $C^\infty(X)$ valued coefficients. (A bi-differential operator acts as a differential operator on each entry separately.) Kontsevich solves this problem by providing a formula for $\{B_2, B_3, \ldots\}$ in terms of B_1. The solution has the following remarkable form

$$f * g = \sum_{0 \le n \le \infty} h^n \sum_{\Gamma \in G[n]} \omega_\Gamma B_{\Gamma,\alpha}(f,g)\,,$$

where

- $G[n]$ is a certain set of $(n(n+1))^n$ labeled graphs with $n+2$ vertices and n edges.
- $B_{\Gamma,\alpha}$ is a bi-differential operator whose coefficients are constructed from multiple order derivatives of the given α by a rules which come from the graph Γ.
- ω_Γ is a number which is obtained from Γ by integrating a certain Γ-dependent differential form over the configuration space of n distinct points in the upper half plane.

The details can be found in [5].

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