1 Small oscillations

Start with a Lagrangian

$$L[q_i, \dot{q}_i] \tag{1}$$

Suppose there is an equilibrium point for this system, i.e.

$$q_i = q_{i0}, \quad \dot{q}_i = 0 \tag{2}$$

Then we can consider small deformations of the system around this equilibrium point. We write

$$q'_i \equiv q_i - q_{i0} \tag{3}$$

and consider q'_i as small. We get

$$\dot{q}'_i = \dot{q}_i \tag{4}$$

and we will also regard $\dot{q}'_i$ as being small, of the same order as q'_i. We expand L as

$$\begin{aligned}
L[q_i, \dot{q}_i] &= L[q_{i,0}, 0] + \{\frac{\partial L}{\partial q_i}[q_{i,0}, 0]\}q'_i + \{\frac{\partial L}{\partial \dot{q}_i}[q_{i,0}, 0]\}\dot{q}'_i \\
&+ \frac{1}{2}\{\frac{\partial^2 L}{\partial q_i \partial q_j}[q_{i,0}, 0]\}q'_i q'_j + \frac{1}{2}\{\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}[q_{i,0}, 0]\}\dot{q}'_i \dot{q}'_j + \{\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}[q_{i,0}, 0]\}q'_i \dot{q}'_j \\
&+ \dots
\end{aligned} \tag{5}$$

Now we note the following points:

(a) The first term is a constant, so it can be dropped from the lagrangian.

(b) The third term is a total derivative, so we can drop it as well.

(c) Working to first order in the perturbation, let us write the Lagrangian equations. To this order we get

$$\{\frac{\partial L}{\partial q_i}[q_{i,0}, 0]\} = 0 \tag{6}$$

so we must be at a point q_{i0} which satisfies this condition.

(d) The fourth and fifth terms have coefficients that are symmetric

$$\begin{aligned}
\{\frac{\partial^2 L}{\partial q_i \partial q_j}[q_{i,0}, 0]\} \equiv V_{ij} &= V_{ji} \\
\{\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}[q_{i,0}, 0]\} \equiv T_{ij} &= T_{ji}
\end{aligned} \tag{7}$$

We have chosen the symbols V_{ij}, T_{ij} here since if we had a Lagrangian of the simple form

$$L = T - V \tag{8}$$

then we would have

$$V_{ij} = \frac{\partial^2 V}{q_i q_j}[q_{i,0}, 0], \quad T_{ij} = \frac{\partial^2 T}{\dot{q}_i \dot{q}_j}[q_{i,0}, 0] \tag{9}$$

(e) The sixth term would be absent in a simple lagrangian of the form (8). Let us assume for now that it vanishes, and then return to its consideration later.

1.1 The lagrangian (8)

We have

$$L = \frac{1}{2} T_{ij} \dot{q}'_i \dot{q}'_j - \frac{1}{2} V_{ij} q'_i q'_j \tag{10}$$

The equation of motion for q'_i is

$$T_{ij} \ddot{q}'_i + V_{ij} q'_j = 0 \tag{11}$$

Let us write

$$q'_i = s_i e^{-i\omega t} \tag{12}$$

Then we get

$$-\omega^2 T_{ij} s_j + V_{ij} s_j = 0 \tag{13}$$

or in matrix form

$$[V - \omega^2 T] S = 0 \tag{14}$$

This is almost in the form of a standard eigenvalue problem. We could make it look like an eigenvalue problem by writing

$$(T^{-1} V) S = \omega^2 S \tag{15}$$

but note that $(T^{-1} V)$ would not be symmetric in general

$$(T^{-1} V)^T = V^T T^{-1T} = V T^{-1} \neq T^{-1} V \tag{16}$$

so we cannot use the nice symmetry properties $T^T = T, V^T = V$. Thus to see the implications of these symmetries for ω, let us follow the usual arguments used in eigenvalue problems from first principles.

(a) First let us ask if ω^2 has to be real or if it can be complex. Starting with

$$VS = \omega^2 TS \tag{17}$$

write

$$S^\dagger V S = \omega^2 S^\dagger T S \tag{18}$$

Note that the matrices V, T are real and symmetric, so that

$$V^\dagger = V, \quad T^\dagger = T \tag{19}$$

Thus

$$\begin{aligned} (S^\dagger V S)^* &= S^\dagger V^\dagger S = S^\dagger V S \\ (T^\dagger V T)^* &= T^\dagger V^\dagger T = T^\dagger V T \end{aligned} \tag{20}$$

and we find that

$$\omega^2 = \frac{S^\dagger V S}{S^\dagger T S} \tag{21}$$

is real. Thus we conclude that either ω is real (if ω^2 is positive) or ω is pure imaginary (if ω^2 is negative).

(b) The kinetic energy is positive on physical grounds, so for any $\dot{q}'_i$ we should have

$$\dot{q}'_i T_{ij} \dot{q}_j > 0 \tag{22}$$

Thus for real vectors S we will have

$$S^T T S > 0 \tag{23}$$

and for complex vectors S we will have

$$S^\dagger T S = (S_R - iS_I)^T T (S_R + iS_I) = S_R^T T S_R + S_I^T T S_I > 0 \tag{24}$$

From (6) we see that we are perturbing around a point which is an extremum of V (i.e. the forces vanish there). This equilibrium position need not be a minimum of V: in general it could be a minimum, a maximum, or a saddle point. But for most physical applications we perturb around a minimum. Then we will have

$$\frac{\partial^2 V}{q_i q_j}[q_{i,0}, 0] q'_i q'_j = q'_i V_{ij} q_j > 0 \tag{25}$$

and V will also be a positive definite matrix. In this case we see from (21) that

$$\omega^2 > 0 \tag{26}$$

and so the actual frequencies ω will be real.

1.2 Solving the equations of motion

Let us return to our dynamical equation (14)

$$[V - \omega^2 T]S = 0 \tag{27}$$

For the matrix $(V - \omega^2 T)$ to have a null eigenvector S, it will need to have a determinant zero. (The determinant is the product of all the eigenvalues of the matrix, and a single vanishing eigenvalue will make the determinant vanish.) Thus we solve

$$|V - \omega^2 T| = 0 \tag{28}$$

to find the allowed frequencies ω^2. We then obtain the eigenvectors S_{ω^2} from (27). The full solution of the system is then

$$q_i = A_k e^{-i\omega_k t}(S_{\omega_k^2})_i + \text{(complex conjugate)} \tag{29}$$

where A_k are arbitrary complex constants.

1.3 The term $\{\frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}[q_{i,0}, 0]\}q'_i\dot{q}'_j$

Now let us ask when the sixth term in the expansion (5) can be nonzero, and what the small oscillations problem looks like in that case. This term has the form

$$C_{ij}q'_i\dot{q}'_j \tag{30}$$

but there is no reason to have any symmetry of C:

$$C_{ij} \neq C_{ji} \tag{31}$$

By subtracting a total derivative from L we can flip the time derivative from q'_j to q'_i

$$C_{ij}q'_i\dot{q}'_j = -C_{ij}\dot{q}'_i q'_j + \frac{d}{dt}C_{ij}q'_i q'_j \tag{32}$$

Dropping the total derivative, we can thus write this term in the Lagrangian as

$$\frac{1}{2}[C_{ij} - C_{ji}]q'_i\dot{q}'_j \tag{33}$$

Thus if C_{ij} happens to be symmetric then this term will vanish. So we can just keep the antisymmetric part of C. Where does such a term arise? Consider the $x - y$ plane, with a uniform magnetic field B in the z direction. Let a point particle of mass m and charge q move in this plane. The Lagrangian is

$$L = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2] + q[A_x\dot{x} + A_y\dot{y}] \tag{34}$$

Here

$$C_{xx} = A_{x,x}, \quad C_{xy} = A_{x.y}, \quad C_{yx} = A_{y,x}, \quad C_{yy} = A_{y,y} \tag{35}$$

The fact that we can keep just the antisymmetric part of C means that the only relevant part of C is

$$C_{xy} - C_{yx} = A_{x,y} - A_{y,x} = B \tag{36}$$

so we see that only the physical magnetic field shows up in the dynamics, not the different possible vector potentials A which can give rise to the same B.

The x equations of motion is

$$\begin{aligned} 0 &= \frac{d}{dt}[m\dot{x} + qA_x] - q\frac{\partial}{\partial x}[A_x\dot{x} + A_y\dot{y}] \\ &= [m\ddot{x} + qA_{x,x}\dot{x} + qA_{x,y}\dot{y}] - q[A_{x,x}\dot{x} + A_{y,x}\dot{y}] \\ &= m\ddot{x} - qB\dot{y} \end{aligned} \tag{37}$$

and the y equation is

$$0 = m\ddot{y} + qB\dot{x} \tag{38}$$

Clearly, $x = y = 0$ is an equilibrium point, since this satisfies the equation of motion. Note that the equations of small perturbations are not of the form (6), since we have a first derivative term. But since the equations are linear in the perturbation, we can still solve them by writing

$$x = Re[ae^{-i\omega t}], \quad y = Re[be^{-i\omega t}] \tag{39}$$

which gives

$$-m\omega^2 a + i\omega qBb = 0, \qquad - m\omega^2 b - i\omega qBa = 0 \tag{40}$$

One solution is $\omega = 0$, with (a, b) arbitrary; this corresponds to placing the particle at rest with an arbitrary displacement. The other solution is

$$\omega^2 = \frac{qB}{m} \tag{41}$$

which gives the eigenvalues and eigenvectors

$$\omega = \pm\sqrt{\frac{qM}{m}}, \quad a = \pm ib \tag{42}$$

Taking the real parts of the motion to get x, y we get circular orbits with the cyclotron frequency.