1 The virial theorem

Consider a collection of particles with masses m_i, $i = 1, 2, \dots N$. Let the complete system be in a 'steady state', where the individual particles move around but the overall description of the system does not change qualitatively; i.e., its macroscopic parameters remain within certain bounds. Then we can obtain a relation between the kinetic and potential energies of the system.

The equations of motion for the ith particle are

$$\dot{p}_i = F_i \tag{1}$$

Write

$$G = \sum_i p_i \cdot r_i \tag{2}$$

Then

$$\dot{G} = \sum_i \dot{p}_i \cdot r_i + \sum_i p_i \cdot v_i = \sum_i F_i \cdot r_i + 2T \tag{3}$$

Let us compute the time average of each quantity over time τ. The time average of a quantity Q is given by

$$\bar{Q} = \frac{1}{\tau} \int_{t=0}^{\tau} dt Q(t) \tag{4}$$

Computing these time averages we find

$$\frac{1}{\tau} \int_0^{\tau} \dot{G} dt = \overline{2T} + \overline{\sum_i F_i \cdot r_i} \tag{5}$$

In a steady state, the difference $G(\tau) - G(0)$ will remain finite, so if we take the large τ limit we will get

$$\frac{1}{\tau} \int_0^{\tau} \dot{G} dt = \frac{1}{\tau}[G(\tau) - G(0)] \to 0 \tag{6}$$

So we find that in steady state

$$\overline{T} = -\frac{1}{2} \overline{\sum_i F_i \cdot r_i} \tag{7}$$

where the time averages are now assumed to be taken with the limit $\tau \to \infty$.

The RHS of the above equation does not make much physical sense as it stands, but we will now evaluate it for a specific force law. Let us consider a 2-body central force, given by a potential V

$$V = \frac{1}{2} \sum_{j \neq i} \alpha_{ij} (r_{ij})^n \tag{8}$$

where

$$r_{ij} = |\vec{r}_i - \vec{r}_j| \tag{9}$$

is the distance between particles i and j. Then the force on the kth particle is obtained by taking the gradient with respect to $\vec{r}_k$ (with a negative sign)

$$\vec{F}_k = -\frac{1}{2}\vec{\nabla}_k \sum_{j\neq i} \alpha_{ij} r_{ij}^n \tag{10}$$

The variable $\vec{r}_k$ appears in two ways in the expression above:

$$\vec{F}_k = -\frac{1}{2}\vec{\nabla}_k \sum_{j\neq k} \alpha_{kj} r_{kj}^n - \frac{1}{2}\vec{\nabla}_k \sum_{j\neq k} \alpha_{jk} r_{jk}^n \tag{11}$$

We have

$$\vec{\nabla}_k r_{kj} = \vec{\nabla}_k [(\vec{r}_k - \vec{r}_j)\cdot(\vec{r}_k - \vec{r}_j)]^{\frac{1}{2}} = \frac{1}{r_{kj}}(\vec{r}_k - \vec{r}_j) \tag{12}$$

$$\vec{\nabla}_k r_{jk} = \vec{\nabla}_k [(\vec{r}_j - \vec{r}_k)\cdot(\vec{r}_j - \vec{r}_k)]^{\frac{1}{2}} = -\frac{1}{r_{jk}}(\vec{r}_j - \vec{r}_k) = \frac{1}{r_{kj}}(\vec{r}_k - \vec{r}_j) \tag{13}$$

So we get

$$\vec{F}_k = -\sum_{j\neq k} \alpha_{kj} n r_{kj}^{n-1} \frac{1}{r_{kj}}(\vec{r}_k - \vec{r}_j) \tag{14}$$

Now we compute our quantity of interest

$$\sum_k \vec{F}_k \cdot \vec{r}_k = -\sum_{j\neq k} \alpha_{kj} n r_{kj}^{n-1} \frac{1}{r_{kj}}(\vec{r}_k - \vec{r}_j)\cdot\vec{r}_k \tag{15}$$

Note that $\alpha_{jk} = \alpha_{kj}$, and $r_{jk} = r_{kj}$. Interchanging the dummy labels j, k we can also write

$$\sum_k \vec{F}_k \cdot \vec{r}_k = -\sum_{j\neq k} \alpha_{kj} n r_{kj}^{n-1} \frac{1}{r_{kj}}(\vec{r}_j - \vec{r}_k)\cdot\vec{r}_j \tag{16}$$

Adding the above two expressions for $\sum_k \vec{F}_k \cdot \vec{r}_k$ and dividing by 2, we get

$$\sum_k \vec{F}_k \cdot \vec{r}_k = -\frac{1}{2}\sum_{j\neq k} \alpha_{kj} n r_{kj}^{n-1} \frac{1}{r_{kj}}(\vec{r}_k - \vec{r}_j)\cdot(\vec{r}_k - \vec{r}_j) = -\frac{1}{2}\sum_{k\neq j} \alpha_{kj} n r_{kj}^n = -nV \tag{17}$$

Thus we have found that

$$\overline{T} = -\frac{1}{2}\sum_k \vec{F}_k \cdot \vec{r}_k = \frac{n}{2}\overline{V} \tag{18}$$

For the Kepler potential we have $n = -1$ and we get

$$\bar{T} = -\frac{1}{2}\bar{V} \tag{19}$$