1 Integration by parts

Suppose we have an integral

$$\int_D d^2z \partial_z f(z, \bar{z}) \tag{1}$$

where the integral ranges over a rectangular region called D:

$$D: \quad a < x_1 < b, \quad c < x_2 < d \tag{2}$$

We should be able to integrate this by parts and write the result as an integral over the boundary of D. Let

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2 \tag{3}$$

We have

$$\partial_z = \frac{1}{2}[\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}], \quad \partial_{\bar{z}} = \frac{1}{2}[\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}] \tag{4}$$

Our integral is

$$\int_a^b dx_1 \int_c^d dx_2 \frac{1}{2}[\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}] f(x_1, x_2) \tag{5}$$

The first term in then brackets in x_1 can be integrated to give contributions from the two boundaries at the two ends of the range of x_1

$$\frac{1}{2}\int_c^d dx_2[f(b, x_2) - f(a, x_2)] \tag{6}$$

while the second term will give

$$-i\frac{1}{2}\int_a^b dx_1[f(x_1, d) - f(x_1, c)] \tag{7}$$

We can now see that all these contributions can be written as a boundary integral over $\bar{z}$. We will let the integral run anticlockwise along the boundary of D. We have

$$\begin{aligned}
x_1 = b: &\qquad \frac{1}{2}\int_c^d dx_2 f = \frac{1}{2}\frac{1}{(-i)}\int_c^d d\bar{z} f \to \frac{i}{2}\int_{\partial D} d\bar{z} f \\
x_2 = d: &\qquad -i\frac{1}{2}\int_a^b dx_1 f = -i\frac{1}{2}(-)\int_b^a d\bar{z} f \to \frac{i}{2}\int_{\partial D} d\bar{z} f \\
x_1 = a: &\qquad -\frac{1}{2}\int_c^d dx_2 f = -\frac{1}{2}\frac{1}{(-i)}(-)\int_d^c d\bar{z} f \to \frac{i}{2}\int_{\partial D} d\bar{z} f \\
x_2 = c: &\qquad i\frac{1}{2}\int_a^b dx_1 f = -\frac{1}{2}\int_a^b d\bar{z} f \to \frac{i}{2}\int_{\partial D} d\bar{z} f
\end{aligned} \tag{8}$$

Thus we see that

$$\int_D d^2z \partial_z f = \frac{i}{2}\int_{\partial D} d\bar{z} f \tag{9}$$

Similarly,

$$\int_D d^2z \partial_{\bar{z}} f = -\frac{i}{2}\int_{\partial D} dz f \tag{10}$$

To derive these results in brief we can use

$$dx_1 \wedge dx_2 = \frac{i}{2} dz \wedge d\bar{z} \tag{11}$$

2 The delta function

Consider the expression

$$\partial_{\bar{z}} \frac{1}{z} \tag{12}$$

If $z \neq 0$, this vanishes. But at $z = 0$ there is a singularity, so it is not clear what we should write. If there is a contribution from $z = 0$ then we should write

$$\partial_{\bar{z}} \frac{1}{z} = \alpha \delta^2(z, \bar{z}) \tag{13}$$

where

$$\int d^2 z \delta^2(z, \bar{z}) = 1 \tag{14}$$

To compute α, let us integrate both sides of (13) over a disc D of radius R. The RHS gives α. The LHS gives

$$\int_D d^2 z \partial_{\bar{z}} \frac{1}{z} = -\frac{i}{2} \int_{\partial D} dz \frac{1}{z} \tag{15}$$

Writing for z on the boundary of the disc

$$z = Re^{i\theta} \tag{16}$$

we get

$$\int \frac{dz}{z} = \int \frac{Rie^{i\theta} d\theta}{Re^{i\theta}} = i \int d\theta = 2\pi i \tag{17}$$

Thus

$$-\frac{i}{2}(2\pi i) = \alpha, \quad \alpha = \pi \tag{18}$$

Thus we have

$$\partial_{\bar{z}} \frac{1}{z} = \pi \delta^2(z, \bar{z}) \tag{19}$$

3 Conformal transformations

A conformal transformation is a change of coordinates

$$z' = z'(z) \tag{20}$$

where $z'(z)$ is an analytic function of z. The only function of z that is regular everywhere, including at infinity, is a constant. Thus we will typically have singularities at some points z. A simple example of a transformation that will be useful to us is

$$w = \log z, \quad z = e^w \tag{21}$$

Write

$$z = re^{i\theta} \tag{22}$$

Then writing $w = w_R + i w_I$ we get

$$w = \log[re^{i\theta}] = \log r + i\theta \tag{23}$$

Thus

$$-\infty < w_R < \infty \tag{24}$$

and

$$0 < w_I < 2\pi \tag{25}$$

where we see that the two ends of the range of w_I are identified as the same point. Thus in the w coordinates we get a cylinder. The metric on the plane was

$$ds^2 = dx_1^2 + dx_2^2 = dzd\bar{z} \tag{26}$$

On the cylinder this will be

$$ds^2 = dzd\bar{z} = \frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}dwd\bar{w} = [e^w e^{\bar{w}}]dwd\bar{w} \tag{27}$$

If we make a conformal transformation we can change the metric by a conformal factor

$$ds'^2 = [e^w e^{\bar{w}}]^{-1} ds^2 \tag{28}$$

and the new metric will be

$$ds'^2 = dwd\bar{w} = dw_R^2 + dw_I^2 \tag{29}$$

so we get the usual flat metric on the cylinder.

4 Conformal primary operators

Suppose we make a change of coordinates

$$z' = z'(z) \tag{30}$$

Then an operator $O(z)$ will change to some operator $O'(z')$. The location of the operator has changed, but in general the form of the operator will change as well. Let us see how this happens.

The simplest operator was $X(z)$, for the free scalar field. But we had sen that this was not a good conformal operator, and did not have a well defined conformal scaling dimension. The next simplest operator was $\partial_z X(z)$. Under the change of coordinates we will have

$$\partial_z X(z) = (\frac{\partial z'}{\partial z})\partial_{z'} X(z') \tag{31}$$

Thus not only has the location of the operator changed, it also got multiplied by a number $(\frac{\partial z'(z)}{\partial z})$. If we have a transformation like the one above with

$$O'(z', \bar{z}') = [\frac{\partial z'}{\partial z}]^{\Delta}[\frac{\partial \bar{z}'}{\partial \bar{z}}]^{\bar{\Delta}} O(z, \bar{z}) \tag{32}$$

Then we say that O is a conformal primary operator with scaling dimensions

$$(\Delta, \bar{\Delta}) \tag{33}$$

Thus $\partial_z X$ is a primary with dimensions $(1, 0)$.

Not all operators will transform as primaries; in fact most will not, so a primary is a very special kind of operator. Consider

$$\partial_z^2 X \tag{34}$$

Under the change of coordinates this will be

$$\partial_z[(\frac{\partial z'}{\partial z})\partial_{z'}X(z')] = (\frac{\partial z'}{\partial z})^2\partial_{z'}^2 X(z') + (\frac{\partial^2 z'}{\partial z^2})\partial_{z'}X(z') \tag{35}$$

The first part on the RHS suggests that this operator has scaling dimensions $(2, 0)$, but the second term is not of the form that would give the scaling behavior required of a primary. Thus $\partial_z^2 X$ is not a primary operator.

5 Conformal transformations on correlation functions of primary operators

Suppose we have the operator $\partial_z X$. On the plane z the 2-point functions is

$$< \partial_z X(z_1)\partial_z X(z_2) >= -\frac{1}{2}\frac{1}{(z_1 - z_2)^2} \tag{36}$$

Let us make the coordinate change to the cylinder

$$w = \ln z \tag{37}$$

Then we have

$$\partial_z X(z_1) = \frac{dw}{dz}(z_1)\partial_w X(w_1) = \frac{1}{z_1}\partial_w X(w_1) = e^{-w_1}\partial_w X(w_1) \tag{38}$$

$$\partial_z X(z_2) = \frac{dw}{dz}(z_2)\partial_w X(w_2) = \frac{1}{z_2}\partial_w X(w_2) = e^{-w_2}\partial_w X(w_2) \tag{39}$$

Thus

$$< \partial_z X(z_1)\partial_z X(z_2) >= e^{-(w_1+w_2)} < \partial_w X(w_1)\partial_w X(w_2) > \tag{40}$$

But

$$< \partial_z X(z_1)\partial_z X(z_2) >= -\frac{1}{2}\frac{1}{(z_1 - z_2)^2} = -\frac{1}{2}\frac{1}{(e^{w_1} - e^{w_2})^2} \tag{41}$$

Thus

$$< \partial_w X(w_1)\partial_w X(w_2) > -\frac{e^{(w_1+w_2)}}{2(e^{w_1} - e^{w_2})^2} \tag{42}$$

Thus we have been able to compute the 2-point function of ∂X on the cylinder.

6 The metric and its variations

The metric on flat space is

$$ds^2 = dzd\bar{z} \tag{43}$$

which gives

$$g_{z\bar{z}} = \frac{1}{2} \tag{44}$$

with other components zero.

Let us see some properties of the variation of the metric. We have

$$g_{ab}g^{bc} = \delta^c_a \tag{45}$$

If we write

$$g_{ab} = \bar{g}_{ab} + \delta g_{ab} \tag{46}$$

$$g^{ab} = \bar{g}^{ab} + \delta g^{ab} \tag{47}$$

Then we find that

$$[\bar{g}_{ab} + \delta g_{ab}][\bar{g}^{bc} + \delta g^{bc}] = \delta^c_a + \delta g_{ab}\bar{g}^{bc} + \bar{g}_{ab}\delta g^{bc} \tag{48}$$

Thus we need

$$\delta g_{ab}\bar{g}^{bc} + \bar{g}_{ab}\delta g^{bc} = 0 \tag{49}$$

or contracting both sides with $\bar{g}^{ad}$

$$\delta g^{dc} = -\delta_{ab}\bar{g}^{bc}\bar{g}^{ad} \tag{50}$$

Thus the variation of the inverse metric is the negative of what we get by raising the indices of the variation of the metric itself.

Now let us look at the effect of diffeomorphisms. The proper distance between points does not change under a coordinate change. Thus we have

$$ds^2 = g_{ab}dx^a dx^b = g'_{a'b'}dx'^{a'}dx'^{b'} \tag{51}$$

Writing

$$dx^a = \frac{\partial x^a}{\partial x'^{a'}}dx'^{a'}, \quad dx^b = \frac{\partial x^b}{\partial x'^{b'}}dx'^{b'} \tag{52}$$

we have

$$g_{ab}\frac{\partial x^a}{\partial x'^{a'}}\frac{\partial x^b}{\partial x'^{b'}}dx'^{a'}dx'^{b'} = g'_{a'b'}dx'^{a'}dx'^{b'} \tag{53}$$

or comparing coefficients of $dx'^{a'}dx'^{b'}$

$$g'_{a'b'} = g_{ab}\frac{\partial x^a}{\partial x'^{a'}}\frac{\partial x^b}{\partial x'^{b'}} \tag{54}$$

Suppose we have an infinitesimal transformation

$$x'^a = x^a + \epsilon^a(x) \tag{55}$$

Then

$$x^a = x'^a - \epsilon^a(x) \tag{56}$$

$$\frac{\partial x^a}{\partial x'^{a'}} \approx \delta^a_{a'} - \epsilon^a_{,a'} \tag{57}$$

where the approximation arises from the fact that we do not distinguish derivatives with respect to x and x' in the last term. Then we find

$$g'_{a'b'} = g_{ab}[\delta^a_{a'} - \epsilon^a_{,a'}][\delta^b_{b'} - \epsilon^b_{,b'}] \approx g_{a'b'} - g_{ab'}\epsilon^a_{,a'} - g_{a'b}\epsilon^b_{,b'} \tag{58}$$

Thus

$$\delta g_{a'b'} = g'_{a'b'} - g_{a'b'} = -g_{ab'}\epsilon^a_{,a'} - g_{a'b}\epsilon^b_{,b'} \tag{59}$$

If the components of g are constant at leading order then we can write

$$\delta g_{a'b'} = -\epsilon_{b',a'} - \epsilon_{a',b'} \tag{60}$$

For the metric (44) we will therefore have

$$\delta g_{zz} = -2\epsilon_{z,z} \tag{61}$$

Note that

$$\epsilon_z = g_{z\bar{z}}\epsilon^{\bar{z}} = \frac{1}{2}\epsilon^{\bar{z}} \tag{62}$$

Thus

$$\delta g_{zz} = -2\epsilon_{z,z} = -\partial_z \epsilon^{\bar{z}} \tag{63}$$

We also have

$$\delta g^{zz} = -g^{z\bar{z}}g^{z\bar{z}}\delta g_{\bar{z}\bar{z}} = 4\partial_{\bar{z}}\epsilon^z \tag{64}$$

7 Making variations of the metric

Suppose we have computed the path integral with the insertion of some operators

$$Z_O = \int D[X]e^{-S(g,X)}O_1(z_1)\ldots O_k(z_k) =< O_1(z_1\ldots O_k(z_k) >_g \tag{65}$$

We have noted the fact that the expressions all depend upon the metric g of the 2-d base space on which our field theory is defined. Now suppose we make a localized change in the metric

$$g_{ab} \to g_{ab} + \delta g_{ab} \tag{66}$$

where $\delta g_{ab} \neq 0$ only in a small region which does not overlap with the locations z_i. Ignoring for the moment any change in the measure $D[X]$ we have

$$\delta Z_O = \int D[X]e^{-S(g,X)}[-\int d^2z \frac{\delta S}{\delta g^{ab}(z)}\delta g^{ab}(z)]O_1(z_1)\ldots O_k(z_k) \tag{67}$$

But we will write

$$2\frac{\delta S}{\delta g^{ab}} = T_{ab} \tag{68}$$

Thus we have

$$\delta Z_O = \int D[X]e^{-S(g,X)}[-\int d^2z \frac{1}{2}T_{ab}(z)\delta g^{ab}(z)]O_1(z_1)\ldots O_k(z_k) \tag{69}$$

Imagine making the variation of g at just one point as follows

$$\delta g^{zz} = \delta^2(z - z_0) \tag{70}$$

Then we get

$$\delta Z_O = -\frac{1}{2}\int D[X]e^{-S(g,X)}T_{ab}(z_0)O_1(z_1)\ldots O_k(z_k) = -\frac{1}{2} < T_{ab}(z_0)O_1(z_1)\ldots O_k(z_k) > \tag{71}$$

Thus we get a correlator with T inserted along with the other operators.

What we want now is a way to relate this correlator to one without the T insertion. This will be possible because of a Ward identity. The argument goes as follows. All we have done is make a small localized change in the metric, and looked at the change in the correlator. Let us see if there is some other way to make the same change in the metric. We can change metrics by making diffeomorphisms. We know that

$$\delta g^{zz} = 4\partial_{\bar{z}}\epsilon^z \tag{72}$$

If we want

$$\delta g^{zz} = \delta^2(z - z_0) \tag{73}$$

then because

$$\partial_{\bar{z}}\frac{1}{z} = \pi\delta \tag{74}$$

we should take

$$4\partial_{\bar{z}}\epsilon^z = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} \tag{75}$$

From this we see that we need the diffeomorphism

$$\epsilon^z = \frac{1}{4\pi}\frac{1}{z - z_0} \tag{76}$$

Thus we see that we have a holomorphic diffeomorphism everywhere, except at $z = z_0$ where there is a singularity. Thus all we have to do is start with the original correlator (wothout the T insertion) and see what the change is if we make this diffeomorphism. Thus we write

$$\delta Z_O =< \delta_\epsilon O_1 O_2 \ldots O_k > + \ldots < O_1 O_2 \ldots \delta_\epsilon O_k > \tag{77}$$

where $\delta_\epsilon O$ is the change in the operator O under the diffeomorphism ϵ. At this level of generality we cannot say more, but now let us assume that the O_i are conformal primaries. Then we have one contribution to the change from just the shift of the location of O

$$\delta_\epsilon O_i = \epsilon_z \partial_z O_i = \frac{1}{4\pi}\frac{1}{z_i - z_0}\partial_z O(z_i) \tag{78}$$

The other part of the change comes because of the change in local scaling

$$O_i(z_i) \to O_i(z') = (\frac{\partial z}{\partial z'})^{\Delta_i}(z_i)O_i(z_i) \tag{79}$$

But

$$z' = z + \epsilon^z \tag{80}$$

Thus

$$\frac{\partial z}{\partial z'} = 1 + \partial_z\epsilon^z = 1 + \frac{1}{4\pi}\frac{1}{(z - z_0)^2} \tag{81}$$

Thus we have

$$O_i(z_i) \to [1 + \frac{1}{4\pi}\frac{1}{(z_i - z_0)^2}]^{\Delta_i} O_i(z_i) \tag{82}$$

which gives a change

$$\delta O_i = \frac{1}{4\pi}\frac{\Delta_i}{(z_i - z_0)^2} \tag{83}$$

Thus the total change in the correlator is

$$\delta Z_O = \frac{1}{4\pi}\sum_i < O_1 \ldots [\frac{\Delta_i}{(z_i - z_0)^2} O_i + \frac{1}{(z_i - z_0)}\partial_z O(z_i)] \ldots O_k > \tag{84}$$

Comparing to the change δZ when we inserted T, we get

$$-\frac{1}{2} < T_{ab}(z_0) O_1(z_1) \ldots O_k(z_k) >= \frac{1}{4\pi}\sum_i < O_1 \ldots [\frac{\Delta_i}{(z_i - z_0)^2} O_i - \frac{1}{(z_i - z_0)}\partial_z O(z_i)] \ldots O_k > \tag{85}$$

which gives

$$< T_{ab}(z_0) O_1(z_1) \ldots O_k(z_k) >= -\frac{1}{2\pi}\sum_i < O_1 \ldots [\frac{\Delta_i}{(z_i - z_0)^2} O_i - \frac{1}{(z_i - z_0)}\partial_z O(z_i)] \ldots O_k > \tag{86}$$

In particular if we bring T close to one of the primary operators O_i then we see the OPE

$$T(z) O(z') = -\frac{1}{2\pi}[\frac{\Delta_i}{(z - z')^2} O(z') + \frac{1}{z - z'}\partial_z O(z')] + \ldots \tag{87}$$

where we have written only the singular terms.

8 The free scalar field

For the free scalar field we have the action

$$S = \frac{1}{2\pi\alpha'}\int d^2\xi \frac{1}{2}\partial_a X \partial_b X g^{ab} = \frac{1}{2\pi\alpha'}\int d^2\xi \frac{1}{2}[\partial_z X \partial_z X g^{zz} + \partial_{\bar z} X \partial_{\bar z} X g^{\bar z \bar z} + 2\partial_z X \partial_{\bar z} X g^{z\bar z}] \tag{88}$$

The stress tensor is defined as

$$T_{ab} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} \tag{89}$$

Thus we have

$$T_{zz} = \frac{1}{2\pi\alpha'}\partial_z X \partial_z X \tag{90}$$

Consider the primary $\partial_z X$, and look at the OPE

$$T_{zz}(z)\partial_z X(z') \sim \frac{1}{2\pi\alpha'} : \partial_z X \partial_z X(z) : \partial_z X(z') \tag{91}$$

The leading term comes if we contract one of the two ∂X factors at z with the one at z'. We set $\alpha' = 1$. This gives

$$\frac{1}{2\pi}(2)(-\frac{1}{2}\frac{1}{(z - z')^2})\partial_z X(z) = -\frac{1}{2\pi}\frac{1}{(z - z')^2}\partial_z X(z) \tag{92}$$

Note that we still have the operator $\partial_z X$ at the location z. We can write it at the location z' with a correction term

$$\partial_z X(z) = \partial_z X(z') + (z - z')\partial_z[\partial_z X] + \ldots \tag{93}$$

Thus the OPE gives

$$T_{zz}(z)\partial_z X(z') \sim -\frac{1}{2\pi}\frac{1}{(z-z')^2}[\partial_z X(z') + (z-z')\partial_z[\partial_z X]] = -\frac{1}{2\pi}[\frac{1}{(z-z')^2}\partial_z X(z') + \frac{1}{(z-z')}\partial_z[\partial_z X]] + \ldots \tag{94}$$

which agrees with the general result in the above section when we note that $\Delta_z = 1$ for the operator $\partial_z X$.

In what follows we will redefine T with a scaling so that we do not get the factor $-\frac{1}{2\pi}$; this is more conventional.

9 The algebra of analytic diffemorphisms

Consider a change of coordinates

$$T_m: \quad z' = z + \epsilon z^{m+1} \tag{95}$$

where ϵ is small. Let us follow this up with another transfirmation

$$T_n: \quad z'' = z' + \epsilon' z'^{n+1} \tag{96}$$

We have

$$\begin{aligned} z'' &= z + \epsilon z^{m+1} + \epsilon'(z + \epsilon z^{m+1})^{n+1} \\ &= z + \epsilon z^{m+1} + \epsilon' z^{n+1}[1 + \epsilon z^m]^{n+1} \\ &\approx z + \epsilon z^{m+1} + \epsilon' z^{n+1}[1 + (n+1)\epsilon z^m] \\ &\approx z + \epsilon z^{m+1} + \epsilon' z^{n+1} + (n+1)\epsilon\epsilon' z^{m+n+1} \end{aligned} \tag{97}$$

Thus

$$T_n T_m: \quad z'' \approx \epsilon z^{m+1} + \epsilon' z^{n+1} + (n+1)\epsilon\epsilon' z^{m+n+1} \tag{98}$$

If we had done these transformations in the reverse order then we would get

$$T_m T_n: \quad z'' \approx z + \epsilon z^{m+1} + \epsilon' z^{n+1} + (m+1)\epsilon\epsilon' z^{m+n+1} \tag{99}$$

Thus the commutator is

$$[T_n, T_m]: \quad z'' = z + \epsilon\epsilon'(n-m)z^{m+n+1} \tag{100}$$

This is again a transformation of the same type as before. Thus we define the infintesimal generators as

$$T_m = I + \epsilon L_m \tag{101}$$

and then we find

$$[L_n, L_m] = (n-m)L_{n+m} \tag{102}$$

So we get the Virasoro algebra of diffeomorphisms.

This is the way coordinate changes behave. But now we have to think about objects that live on the z plane. There are field operators $\phi, \partial\phi$ etc, and they will be affected by the diffeomorphims. Some operators will just chnage their locatiobn, some will scale in addition, and some will distort and become completely different operators. Is there a sense in which we should write an algebra like L_n which can derive the action on these fields?

10 Conformal transformations

Let us start with the idea of quantum states. Thus imagine that we are on the cylinder, and we have a particular state ψ. This state is defined on the circle at some $\tau = \tau_0$. Now we make the diffeomorphism T_m. We now have our spatial circle as a different, distorted one, and are asking what the state of the system looks like on this new circle. Suppose the change of the state is given by the operator $\hat{L}_m$

$$|\psi'\rangle = |\psi\rangle + \epsilon \hat{L}_m |\psi\rangle \tag{103}$$

We might think tht these operators should satisfy the algebra (102). But this turns out to not be strictly true; in fact we get the algebra with a central extension

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0} \tag{104}$$

Let us see how this happens. We proceed in the folowing steps.

(A) Let us start with a cylinder with the usual flat metric on it. Let the coordinate be called $w = \tau + i\sigma$. Let us make the slice S

$$\tau = \tau_0, \quad 0 \le \sigma < 2\pi \tag{105}$$

On this slice we take a state $|\psi\rangle$.

(B) Consider the new slice S' which is constructed as follows. Each point on the cylinder is moved to a new point

$$w \to w + \epsilon f(w) \tag{106}$$

where $f(w)$ is an analytic function in the vicinity of S.

This new slice is thus a new curve on the cylinder. It is not 'straight', and is in fact longer than the initial slice S. But since it cuts across the entire cylinder, we should be able to specify the state of the system on this slice. Let this state be $|\psi'\rangle$.

Now comes the main point. In principle the space of states H made on S have no relation to the states H' made on S'. The two slices are different, and so the space of states on them are different.

We can still make a map

$$H \leftrightarrow H' \tag{107}$$

which maps states on S to states on S'. This is possible because we are describing the same system in two ways. A state is just an assignment of weights w_i to classical configurations C_i

$$|\psi\rangle = \sum_i w_i C_i \tag{108}$$

The weights w_i are determined by using the path integral, as we did for the Ising model. The configurations are different for S', but we can again do the path integral to get

$$|\psi'\rangle = \sum_i w'_i C'_i \tag{109}$$

So for a given situation we can make each of the states $|\psi\rangle, |\psi'\rangle$, and so there will be a map like (107).

(C) Now we try to do something stronger. We try to see if we can think of the state $\psi'\rangle$ as a state in the space H itself. Why should that be possible? The length of S' is different from that of S, and so there is no way to map the configurations C'_i rigorously to the configurations C_i. In a general field theory we in fact cannot make such a map. But a conformal field theory is special, so we may be able to do something.

Let us make a change of coordinates which makes S' look like S. The points on S' were

$$\tau_0 + \epsilon Re[f(w)], \quad \sigma + \epsilon Im[f(w)] \tag{110}$$

So we make a put new coordinates on the cylinder

$$w' = w - \epsilon f(w) \tag{111}$$

This makes the points on S' have the values

$$(\tau = \tau_0, \quad \sigma) \tag{112}$$

so the slice S' again seems to have a length 2π. But this is of course not true; a coordinate change cannot change the actual length of a slice. We have to note that the metric was initially

$$g_{ab} = \delta_{ab} \tag{113}$$

and now will be

$$g'_{ab} = e^{2\rho}\delta_{ab} \tag{114}$$

where

$$e^{2\rho} = (\frac{df(w)}{dw})(\frac{d\bar{f}(\bar{w})}{d\bar{w}}) \tag{115}$$

So even though the slice S' may look like S, we have to do our CFT with a different metric. But now we use the fact that we have a conformal theory. Thus we should be able to change the metric by a conformal factor

$$g'_{ab} \to e^{-2\rho} g'_{ab} = \delta_{ab} \tag{116}$$

and correlation functions should not change. If we can do this, then we have really come back to the slice S and the original field theory, so all we have to ask is what state we got on S. It will of course not be the same state as $|\psi\rangle$. To see what kinds of changes are expected, let us first ignore the change in τ in our diffeomorphism, and look at the chage in σ alone. The shift from S to S' took a point σ to $\sigma + \delta\sigma(\sigma)$, and the change of coordinates relabelled this point as σ. Thus we are left with a real shift of the point by $\delta\sigma(\sigma)$, even though the coordinates of S' look just like the coordinates on S. So a function $\phi(\sigma)$ will change to a function

$$\phi(\sigma) = \phi(\sigma) + \delta\sigma(\sigma)\partial_\sigma\phi \tag{117}$$

Thus there is a change of state becaue the classical configurations change in the expected way. But because of the change in metric, there will be an additional change that we must look at. The path integral is different, and in particular we do not know how to make the enw measure. this is because to do the path integral we have to first take a lattice of points and then do an integral $\int d\phi_i$ at each of these points. Suppose we took a regular square grid of points. After the conformal transformation the metric was different, so we have to ask what is the new lattice of points. The square grid does not look square now, so we have to add or remove points from places where the density has become lower or higher than before. Thus the overall path integral will have a normalization that will be different from the one before the conformal transformation. This means that we cannot get the correct overall normalization of the new state easily, though we can find the relative contributions of the classical configurations C_i in the state. Thus we have an additional correction from the measure changes

$$|\psi\rangle \to \alpha|\psi\rangle \tag{118}$$

So strictly speaking we should write the change in configuration under the diffeomorphism, and also note the change in the measure. If we ignore the latter, we can still write the state but upto an overall normalization. It turns out that this normalization change gives the central extension of the conformal algebra. Thus we will get not a true representation of the diffeomorphism algebra, but a 'projective representation', which is the one with the central charge.

11 Making diffeomorphisms with T_{zz}

Let us make a change of coordinates

$$z' = z + \epsilon(z) \tag{119}$$

Then we will get a change of metric

$$\delta g^{zz} = 4\partial_{\bar{z}}\epsilon^z \tag{120}$$

and the change in the path integral is

$$\delta Z = -\int d^2z \frac{\delta S}{\delta g^{zz}}\delta g^{zz} = -\int d^2z \frac{1}{2}T_{zz}4\partial_{\bar{z}}\epsilon^z \tag{121}$$

We can integrate this by parts, getting

$$\delta Z = -\int_\partial dz \frac{(-i)\pi}{2}2T_{zz} = i\pi\int_\partial dz T_{zz}\epsilon^z \tag{122}$$

Suppose the variation of confined to a bounded region D. Suppose we took two different contours C_1, C_2, both of which enclose D. Then on each of them we will get

$$\int_{C_1} dz T_{zz}\epsilon^z = \int_{C_2} dz T_{zz}\epsilon^z \tag{123}$$

Suppose we take

$$\epsilon^z = z^{m+1} \tag{124}$$

Then we have the diffeomorphism for L_m. Thus suppose we want to change a state at the origin by L_m. We should use the operator

$$\int_C dz T_{zz} \epsilon^{m+1} \tag{125}$$

12 OPE of TT

If the the operator T had a good scaling dimension, then it would be 2. We can see this from the example of the free field, or more generally from the variation

$$\delta Z =< -\int d^2 z T_{zz} \partial_{\bar{z}} \epsilon^z > \tag{126}$$

The LHS has no units, d^2z has units L^2, $\partial_{\bar{z}}$ has units $\frac{1}{L}$ and ϵ^z has units L. Thus the OPE of two T operators must have the form

$$T(z')T(z) \sim \frac{\alpha}{(z'-z)^4} + \frac{2T(z)}{('z-z)^2} + \frac{\partial_z T(z)}{(z -' z)} + \dots \tag{127}$$

There can be no cubic term because the OPE must be symmetric in the two T operators. Here α is a constant which we will now determine.

13 Algebra of diffeomorphisms

Let us see how such operators commute. First we apply L_m, then L_n. Let us define

$$\frac{1}{2\pi i}\int_C dz \equiv \int'_C \tag{128}$$

Let us consider the following operator

$$T_m : \quad \int'_{C_1} dz T_{zz}(z) z^{m+1} \tag{129}$$

We follow this up with another operator

$$T_n : \quad \int'_{C_2} dz' T_{zz}(z') z'^{m+1} \tag{130}$$

where C_2 is outside C_1. Thus we have

$$\int'_{C_2} dz' z'^{n+1} T_{zz}(z') \int'_{C_1} dz z^{m+1} T_{zz}(z) \tag{131}$$

where C_1 is inside C_2. If we do the operations in the other order, we get

$$\int'_{C'_2} dz' z'^{n+1} T_{zz}(z') \int'_{C_1} dz z^{m+1} T_{zz}(z) \tag{132}$$

where C'_2 is inside C_1. The difference is equal to keeping z fixed and letting z' circle anticlockwise around z in a contour C

$$T_n T_m - T_m T_n : \quad \int'_{C} dz' z'^{n+1} T_{zz}(z') \int'_{C_1} dz z^{m+1} T_{zz}(z) \tag{133}$$

Let us do the C integral first. We expand the TT in an OPE

$$T_{zz}(z') T_{zz}(z) \sim \frac{\alpha}{(z'-z)^4} + \frac{2T_{zz}(z)}{(z'-z)^2} + \frac{\partial T(z)}{(z'-z)} + \ldots \tag{134}$$

Let us deal with the first term. We have

$$\int'_{C} \frac{\alpha}{(z'-z)^4} z'^{n+1} z^{m+1} = \int'_{C} (n+1)n(n-1)\alpha z^{m+n-1} = \frac{\alpha}{6}(n^3-n) z^{n+m-1} \tag{135}$$

The next integral gives

$$\int'_{C_1} dz \frac{\alpha}{6}(n^3-n) z^{n+m-1} = \frac{\alpha}{6}(n^3-n)\delta_{m+n,0} \tag{136}$$

Now let us look at the next term. We have

$$\int'_{C} dz' \frac{2T(z)}{(z'-z)^2} z'^{n+1} z^{m+1} = 2(n+1) T_{zz}(z) z^{m+n+1} \tag{137}$$

The next integral gives

$$2(n+1) T_{n+m} \tag{138}$$

Now look at the last term. We have

$$\int'_{C} \partial T(z) \frac{1}{(z'-z)} z'^{n+1} z^{m+1} = \partial T_{zz}(z) z^{n+m+2} \tag{139}$$

In the next integral we can integrate by parts to get

$$\int_{C_1} dz \partial T_{zz}(z) z^{n+m+2} = -\int_{C_1} dz T_{zz}(z)(n+m+2) z^{n+m+1} = (n+m+2) T_{n+m} \tag{140}$$

Combining the last two contributions we get

$$(2n+2-n-m-2) T_{n+m} = (n-m) T_{n+m} \tag{141}$$

Thus we get the algebra

$$[T_n, T_m] = (n-m) T_{n+m} + \frac{\alpha}{6}\delta_{n+m,0} \tag{142}$$

To make this agree with the Virasoro algebra, we see that we need to define

$$\alpha = \frac{c}{2} \tag{143}$$

14 Computing the curvature

Let the metric be

$$g_{z\bar{z}} = \frac{1}{2}e^{2\rho}, \quad g^{z\bar{z}} = 2e^{2\rho} \tag{144}$$

The connection components are

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}[g_{db,c} + g_{dc,b} - g_{bc,d}] \tag{145}$$

We then get

$$\Gamma^z_{zz} = \frac{1}{2}g^{z\bar{z}}[g_{\bar{z}z,z} + g_{\bar{z}z,z} - g_{zz,\bar{z}}] = \frac{1}{2}2e^{-2\rho}[\frac{1}{2}e^{2\rho}2\rho_{,z} + \frac{1}{2}e^{2\rho}2\rho_{,z}] = 2\rho_{,z} \tag{146}$$

$$\Gamma^{\bar{z}}_{zz} = \frac{1}{2}g^{\bar{z}z}[g_{zz,z}] = 0 \tag{147}$$

$$\Gamma^z_{\bar{z}z} = \frac{1}{2}g^{z\bar{z}}[g_{\bar{z}z,\bar{z}} - g_{\bar{z}z,\bar{z}}] = 0 \tag{148}$$

So there are no mixed components of the connection. The only nonvanishing components are

$$\Gamma^z_{zz} = 2\rho_{,z}, \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\rho_{,\bar{z}} \tag{149}$$

The curvature components are

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{cf}\Gamma^f_{bd} - \Gamma^a_{df}\Gamma^f_{bc} \tag{150}$$

If we lower the index on R, then we will have antisymmetry in the first pair of indices, and antisymmetry in the second pair. Thus we can only have the nonvanishing component

$$R_{\bar{z}z\bar{z}z} = g_{\bar{z}z}R^z{}_{z\bar{z}z} \tag{151}$$

Thus we compute

$$R^z{}_{z\bar{z}z} = \Gamma^z_{zz,\bar{z}} - \Gamma^z_{z\bar{z},z} + \Gamma^z_{\bar{z}f}\Gamma^f_{zz} - \Gamma^z_{zf}\Gamma^f_{z\bar{z}} = \Gamma^z_{zz,\bar{z}} = 2\rho_{,z\bar{z}} \tag{152}$$

Thus

$$R^z{}_{z\bar{z}z} = -R^z{}_{zz\bar{z}} = -2\rho_{,z\bar{z}} \tag{153}$$

Thus

$$R_{z\bar{z}} = -2\rho_{,z\bar{z}} \tag{154}$$

$$R = g^{z\bar{z}}R_{z\bar{z}} + g^{\bar{z}z}R_{\bar{z}z} = 2(2)e^{-2\rho}(-2\rho_{,z\bar{z}}) = -8e^{-2\rho}\rho_{,z\bar{z}} \tag{155}$$

Note that

$$\rho_{,z\bar{z}} = \frac{1}{4}\hat{\triangle}\rho \tag{156}$$

where

$$\hat{\triangle}\rho = 4\rho_{,z\bar{z}} \tag{157}$$

is the laplacian evaluated using the flat metric on the z plane. Thus

$$R = -2e^{-2\rho}\hat{\triangle}\rho \tag{158}$$

15 The stress tensor

Let

$$\hat{T} = \hat{T}_{z\bar{z}} g^{z\bar{z}} + \hat{T}_{\bar{z}z} g^{\bar{z}z} = 4\hat{T}_{z\bar{z}} e^{-2\rho} = \mu R \tag{159}$$

Then

$$\hat{T}_{z\bar{z}} = \frac{\mu}{4} R e^{2\rho} = \frac{\mu}{4}(-8e^{-2\rho}\rho_{,z\bar{z}})e^{2\rho} = -2\mu\rho_{,z\bar{z}} \tag{160}$$

Now look at the conservation equation. We have

$$\hat{T}_{z\bar{z};}{}^{\bar{z}} + \hat{T}_{zz;}{}^{z} = 0 \tag{161}$$

We can write this as

$$\hat{T}_{z\bar{z};z} g^{z\bar{z}} + \hat{T}_{zz;\bar{z}} g^{\bar{z}z} = 0 \tag{162}$$

which gives

$$\hat{T}_{z\bar{z};z} + \hat{T}_{zz;\bar{z}} = 0 \tag{163}$$

Using the fact that the connection has only z terms or only $\bar{z}$ terms, we get

$$\hat{T}_{z\bar{z},z} - \Gamma^z_{zz}\hat{T}_{z\bar{z}} + \hat{T}_{zz,\bar{z}} = 0 \tag{164}$$

Thus

$$\hat{T}_{zz,\bar{z}} = -(-2\mu\rho_{,z\bar{z}})_{,z} + 2\rho_{,z}(-2\mu\rho_{,z\bar{z}}) = 2\mu\rho_{,zz\bar{z}} - 4\mu\rho_{,z}\rho_{,z\bar{z}} \tag{165}$$

This can be rewritten as

$$\hat{T}_{zz,\bar{z}} = 2\mu[\rho_{zz} - (\rho_{,z})^2]_{,\bar{z}} \tag{166}$$

Thus we see that

$$T_{zz} \equiv \hat{T}_{zz} - 2\mu[\rho_{zz} - (\rho_{,z})^2] \tag{167}$$

satisfies

$$T_{zz,\bar{z}} = 0 \tag{168}$$

We will call this as the holomorphic stess tensor, and use this in all our constructions in the CFT. But we should note that while $\hat{T}_{zz}$ was a component of a tensor, T_{zz} is not. Instead, it will transform with an inhomogeneous term under coordinate transformations.

Now suppose we obtained this ρ from an analytic transformation. Then we would have

$$ds^2 = dzd\bar{z} = dz'd\bar{z}'\frac{dz}{dz'}\frac{d\bar{z}}{d\bar{z}'} \tag{169}$$

Thus if the new coordinate is given by

$$z = f(z') \tag{170}$$

Then the new metric has

$$e^{2\rho} = \frac{df}{dz}\frac{d\bar{f}}{d\bar{z}} \tag{171}$$

Thus

$$\rho = \frac{1}{2}[\log f' + \log \bar{f}'] \tag{172}$$

$$\rho_{,z} = \frac{1}{2}\frac{f''}{f'} \tag{173}$$

$$\rho_{,zz} = \frac{1}{2}(\frac{f'''}{f'} - (\frac{f''}{f'})^2) \tag{174}$$

$$T_{zz} = -2\mu[\rho_{zz} - (\rho_{,z})^2] = -2\mu[\frac{1}{2}(\frac{f'''}{f'} - (\frac{f''}{f'})^2) - \frac{1}{4}(\frac{f''}{f'})^2] = -\mu[\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2] \tag{175}$$

The expression

$$\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 \tag{176}$$

is called the Schwarzian derivative of f with respect to z.

16 The TT OPE

We have seen that inserting ϵT at the origin is equivalent to a diffeomorphism

$$z = z' + \frac{\epsilon}{z} \tag{177}$$

Thus

$$f' = 1 - \frac{\epsilon}{z^2} \tag{178}$$

$$f'' = \frac{2\epsilon}{z^3} \tag{179}$$

$$f''' = -\frac{6\epsilon}{z^4} \tag{180}$$

Thus

$$T(z) \to -\mu\frac{f'''}{f'} = \frac{6\mu\epsilon}{z^4} \tag{181}$$

where we have noted that the other term in the Schwarzian is $O(\epsilon^2)$.

We had earlier written the OPE as

$$T(0)T(z) \sim \frac{\alpha}{z^4} + \dots \tag{182}$$

Thus we see that

$$\alpha = 6\mu \tag{183}$$

We will later find it convenient to write

$$\alpha = \frac{c}{2} \tag{184}$$

in which case we will get

$$\mu = \frac{c}{12} \tag{185}$$

Thus we find that

$$T_{z\bar{z}} = -\frac{c}{6}\rho_{z\bar{z}} \tag{186}$$

17 Computing T on the cylinder

We have

$$z = e^w \tag{187}$$

Thus

$$f(w) = e^w \tag{188}$$

$$\frac{f'''}{f'} = 1 \tag{189}$$

$$(\frac{f''}{f'})^2 = 1 \tag{190}$$

$$\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2 = -\frac{1}{2} \tag{191}$$

Thus

$$< T_{zz} >= -\frac{1}{2}\mu = -\frac{c}{24} \tag{192}$$

We get this value also from the Casimir energy computation.

18 The effect of curvature

We have found that

$$< 2\frac{\delta S}{\delta g^{z\bar{z}}} >=< T_{z\bar{z}} >= -2\mu\rho_{z\bar{z}} = -\frac{\mu}{2}\hat{\triangle}\rho = -\frac{c}{24}\hat{\triangle}\rho \tag{193}$$

where

$$\hat{\triangle}\rho = 4\rho_{,z\bar{z}} \tag{194}$$

is the laplacian evaluated using the flat metric on the z plane. But

$$\frac{\delta S}{\delta g^{z\bar{z}}} = \frac{\delta S}{\delta \rho}[\frac{\delta g^{z\bar{z}}}{\delta \rho}]^{-1} = \frac{\delta S}{\delta \rho}[-4e^{-2\rho}]^{-1} = -\frac{1}{4}e^{2\rho}\frac{\delta S}{\delta \rho} \tag{195}$$

Thus

$$\frac{\delta S}{\delta \rho} = \frac{c}{12}e^{-2\rho}\hat{\triangle}\rho = \frac{c}{12}\triangle\rho \tag{196}$$

Thus

$$S = \frac{c}{24}\int \rho\triangle\rho \tag{197}$$

where all integrals are evaluated with the correct metric on the plane (as opposed to the flat metric). Noting that

$$R = -2e^{-2\rho}\hat{\triangle}\rho = -2\triangle\rho \tag{198}$$

we get that

$$\rho = -\frac{1}{2}\triangle^{-1}R \tag{199}$$

and

$$S = \frac{c}{48}\int R\triangle^{-1}R \tag{200}$$

where again all integrals are evaluated with the correct metric on the surface.