

# Area and Volume in curvilinear coordinates

## 1 Overview

There are several reasons why we need to have a way of measuring areas and volumes in general relativity. The gravitational field of a point mass  $m$  will be singular at the location of the mass, so we typically compute the field of a mass *density*; i.e., the mass contained in a unit *volume*. If we are dealing with the Gauss law of electrodynamics (and there will be a similar law for gravity) then we need to compute the flux through an *area*. How should areas and volumes be defined in general metrics?

Let us start in flat 3-dimensional space with Cartesian coordinates. Consider two vectors  $\vec{V}, \vec{W}$ . These vectors describe the sides of a parallelogram, whose area can be written as

$$\vec{A} = \vec{V} \times \vec{W} \quad (1)$$

The cross-product has magnitude

$$|\vec{A}| = |\vec{V}||\vec{W}| \sin \theta \quad (2)$$

and is seen to equal the area of the parallelogram. The direction of the cross product also serves a useful purpose: it gives the normal direction to the area spanned by the vectors. Note that there are *two* normal directions to the area – pointing above the plane and below the plane – and the ‘right hand rule’ for cross products is a convention that picks out one of these over the other. Thus there is a choice of convention in the choice of normal; we will see this fact again later.

The cross product can be written in terms of a determinant

$$\vec{V} \times \vec{W} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ V^x & V^y & V^z \\ W^x & W^y & W^z \end{pmatrix} \quad (3)$$

A determinant is computed by computing a product of numbers, taking one number from each row, while making sure that we also take only one number from each column. Taking the rows sequentially from the top, we take this product with a positive sign for even permutations of the columns, and a negative sign for odd permutations. We can encode these rules by introducing the *Livi-Civita symbol*  $\epsilon_{ijk}$  where the indices take values 1, 2, 3. We set  $\epsilon_{123} = 1$ , and all other values are determined by requiring that  $\epsilon_{ijk}$  be totally antisymmetric in its indices. Thus  $\epsilon_{112} = 0$ , while  $\epsilon_{213} = -1$  etc. We then find that

$$A_i = \epsilon_{ijk} V^j W^k \quad (4)$$

Since we are in Cartesian coordinates in flat Euclidean space, there is no difference between upper and lower indices. But we have written the indices in a way that anticipates their general structure: the vectors carry upper indices, and the indices of  $\epsilon$  that contract with these are all lower indices (they all have the same kind if the notion of antisymmetrization is to make sense). We then find that the ‘area vector’  $A_i$  is comes out as a *covariant* vector.

Now consider a third vector  $\vec{X}$ . The three vectors  $\vec{V}, \vec{W}, \vec{X}$  form a paralelepiped, whose volume is given by

$$v = \vec{X} \cdot (\vec{V} \times \vec{W}) \quad (5)$$

In terms of the Levi-Civita symbol we get

$$v = \epsilon_{ijk} X^i V^j W^k \quad (6)$$

Strictly speaking,  $|v|$  is the volume of the paralelepiped; the quantity  $v$  itself can have either sign, and this sign depends on our convention of the cross product (which is now embodied in the choice  $\epsilon_{123} = +1$ ) and on the order of the three edge vectors  $\vec{X}, \vec{Y}, \vec{Z}$  that we take to define the paralelepiped.

## 2 Curvilinear coordinates

Let us now place curvilinear coordinates  $\xi^i, i = 1, 2, 3$  on our 3d Euclidean space. The vectors will have components  $V^{i'} = \frac{\partial \xi^{i'}}{\partial x^j} V^j$  etc. The volume of our paralelepiped is just a number, and we ask that it remain the same. Then we would need new values for the Levi-Civita symbol in our new coordinates,

$$\epsilon'_{i'j'k'} X^{i'} V^{j'} W^{k'} = \epsilon_{ijk} X^i V^j W^k \quad (7)$$

This equality will hold only if the components of  $\epsilon_{ijk}$  transform as a covariant tensor of rank 3

$$\epsilon'_{i'j'k'} = \frac{\partial x^i}{\partial \xi^{i'}} \frac{\partial x^j}{\partial \xi^{j'}} \frac{\partial x^k}{\partial \xi^{k'}} \epsilon_{ijk} \quad (8)$$

While this relation is true for any rank 3 covariant tensor, in the present case  $\epsilon_{ijk}$  has particularly simple values in the starting Cartesian frame. We find

$$\epsilon'_{123} = \det M \quad (9)$$

where  $M$  is the matrix of partial derivatives describing the change of coordinates

$$M = \left\{ \frac{\partial x^i}{\partial \xi^j} \right\} = \begin{pmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} & \frac{\partial x^1}{\partial \xi^3} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} & \frac{\partial x^2}{\partial \xi^3} \\ \frac{\partial x^3}{\partial \xi^1} & \frac{\partial x^3}{\partial \xi^2} & \frac{\partial x^3}{\partial \xi^3} \end{pmatrix} \quad (10)$$

and the other components are again related by antisymmetry; i.e.,  $\epsilon'_{213} = -\det M$ ,  $\epsilon'_{112} = 0$  etc.

We can go one step further by relating  $\det M$  to the metric. In the starting Cartesian coordinates, we have  $g_{ij} = \delta_{ij}$ , while in the curvilinear coordinates  $\xi^k$  we will have

$$g'_{i'j'} = \frac{\partial x^i}{\partial \xi^{i'}} \frac{\partial x^j}{\partial \xi^{j'}} \delta_{ij} = M_{i'i} \delta_{ij} M_{jj'}^T = (MM^T)_{i'j'} \quad (11)$$

where we have used the matrix of partial derivatives defined in (10). We have

$$g \equiv \det\{g_{ij}\} = \det\{\delta_{ij}\} = 1 \quad (12)$$

Let us compute  $g' \equiv \det\{g'_{i'j'}\}$ . From (11) we find

$$g' = (\det M)^2 \quad (13)$$

Returning to (9) we find

$$\epsilon'_{123} = \sqrt{g'} \quad (14)$$

Thus in any set of curvilinear coordinates we can find the values of the Levi-Civita symbol in terms of the determinant of the metric.

### 3 Generalizations

It is immediately clear how the above analysis extends to higher dimensions. In  $n$  dimensional space with Euclidean signature, we have

$$\epsilon_{i_1 i_2 \dots i_n} = \pm \sqrt{g} \quad (15)$$

if all the indices are different, and zero if two indices happen to be the same. The choice of sign is determined if we choose it for one ordering of indices. This choice is arbitrary; to fix a definite choice, we proceed by ordering our coordinates in a particular order  $\{\xi^1, \dots, \xi^n\}$ , and then we set  $\epsilon_{1,2,\dots,n} = \sqrt{g}$ .

In spacetime with Lorentzian signature the analog of (11) is

$$g'_{i'j'} = \frac{\partial x^i}{\partial \xi^{i'}} \frac{\partial x^j}{\partial \xi^{j'}} \eta_{ij} = M_{i'i} \eta_{ij} M_{jj'}^T = (M\eta M^T)_{i'j'} \quad (16)$$

We have  $g = \det\{\eta_{ij}\} = -1$ . From (16) we get

$$g' = -(\det M)^2 \quad (17)$$

Thus we can write

$$\epsilon_{i_1 i_2 \dots i_n} = \pm \sqrt{-g} \quad (18)$$

with the choice of signs again being determined from  $\epsilon_{12\dots n} = \sqrt{-g}$  with the positive sign of the squareroot.

## 4 Integration over a volume

We can now set up the structure for integration on a general curved manifold of dimension  $n$ :

(i) Consider a scalar function  $f(\xi)$  on the manifold. We wish to integrate this over some volume  $\mathcal{R}$ .

(ii) Cut up the region  $\mathcal{R}$  into small hypercubes. The integral  $I$  will be given by multiplying the volume of each hypercube by the value of  $f$  taken, for example, at the center of the hypercube.

(iii) Each hypercube has edges given by a set of  $n$  vectors  $V_{(1)}^i, \dots, V_{(n)}^i$ . We choose any ordering of these vectors, but then keep the same ordering as we move to other hypercubes (using ‘continuity’ to determine which ordering is the ‘same’ in neighboring hypercubes). The volume of the hypercube is given by

$$dv = \epsilon_{i_1 \dots i_n} V_{(1)}^{i_1} \dots V_{(n)}^{i_n} \quad (19)$$

(iv) A simple way to cut the space into hypercubes is to make use of the coordinate system  $\xi$ . At a point  $\xi$  in the manifold consider the vectors

$$\begin{aligned} V_{(1)} &= \{d\xi^1, 0, 0, \dots, 0\} \\ \dots &= \dots \\ V_{(n)} &= \{0, 0, 0, \dots, d\xi^n\} \end{aligned} \quad (20)$$

for infinitesimal numbers  $d\xi^1, \dots, d\xi^n$ . Then we get

$$dv = \sqrt{g} d\xi^1 \dots d\xi^n \quad (21)$$

and the integral is

$$I = \int_{\mathcal{R}} d\xi^1 \dots d\xi^n \sqrt{g} f(\xi) \quad (22)$$

For a spacetime with Lorentzian signature, we would write

$$I = \int_{\mathcal{R}} d\xi^1 \dots d\xi^n \sqrt{-g} f(\xi) \quad (23)$$

Note that the overall sign of the integral is still a matter of convention, since the ordering of the vectors making the edges of our hypercubes was arbitrary. For most purposes we choose the ordering so that the volume of the region is positive (i.e.  $\int d\xi^1 \dots d\xi^n \sqrt{g} > 0$  for Euclidean signature and  $\int d\xi^1 \dots d\xi^n \sqrt{-g} > 0$  for Lorentzian signature). Note that it does not make sense to integrate a function all over a Mobius band, since there is no way of choosing a consistent ordering of the edges of the hypercube in a continuous way.

## 5 Integration over an area

There are two main situations where we may wish to integrate over a  $n - 1$  dimensional hypersurface instead of an  $n$  dimensional volume.

### 5.1 Integrating a vector field over a hypersurface

We are given a vector field  $W^i$ , and we wish to compute the flux through a  $n - 1$  dimensional region  $\mathcal{A}$  of a hypersurface. We proceed as follows:

(i) We cut the hypersurface into small  $n - 1$  dimensional hypercubes. Suppose a hypercube has edges  $V_{(1)}^i \dots V_{(n-1)}^i$ . Then we can define the covariant vector

$$d\Sigma_i = \epsilon_{ii_1 \dots i_{n-1}} V_{(1)}^{i_1} \dots V_{(n-1)}^{i_{n-1}} \quad (24)$$

This quantity has the correct magnitude to be the ‘area’ of the small  $n - 1$  dimensional hypercube, and a direction which is ‘normal’ to this ‘area’.

(ii) We take the inner product of  $W^i$  with  $d\Sigma_i$ , getting the ‘flux through the small area segment’

$$W^i d\Sigma_i = \epsilon_{ii_1 \dots i_{n-1}} W^i V_{(1)}^{i_1} \dots V_{(n-1)}^{i_{n-1}} \quad (25)$$

Again, a convenient way to do this is to put coordinates  $\chi^i, i = 1, \dots, n - 1$  on the hypersurface. The edges of the hypercubes are then given by the vectors

$$\begin{aligned} V_{(1)} &= \left\{ \frac{\partial \xi^1}{\partial \chi^1} d\chi^1, \frac{\partial \xi^2}{\partial \chi^1} d\chi^1, \dots, \frac{\partial \xi^n}{\partial \chi^1} d\chi^1 \right\}, \\ &\dots = \dots \\ V_{(n-1)} &= \left\{ \frac{\partial \xi^1}{\partial \chi^{n-1}} d\chi^{n-1}, \frac{\partial \xi^2}{\partial \chi^{n-1}} d\chi^{n-1}, \dots, \frac{\partial \xi^n}{\partial \chi^{n-1}} d\chi^{n-1} \right\} \end{aligned} \quad (26)$$

and we get

$$d\Sigma_i = \epsilon_{ii_1 \dots i_{n-1}} \frac{\partial \xi^{i_1}}{\partial \chi^1} \dots \frac{\partial \xi^{i_{n-1}}}{\partial \chi^{n-1}} d\chi^1 \dots d\chi^{n-1} \quad (27)$$

(iii) We add over all hypercubes, getting

$$I = \int_{\mathcal{A}} W^i d\Sigma_i = \epsilon_{ii_1 \dots i_{n-1}} W^i \frac{\partial \xi^{i_1}}{\partial \chi^1} \dots \frac{\partial \xi^{i_{n-1}}}{\partial \chi^{n-1}} d\chi^1 \dots d\chi^{n-1} \quad (28)$$

## 5.2 Integrating a scalar over a hypersurface

The other situation is where we are given a scalar function and we wish to integrate it over a hypersurface. One example is the integration of the function ‘unity’; i.e., we wish to find the ‘area’ of the hypersurface.

Note that the ‘area element’  $d\Sigma_i$  is a vector, with direction normal to the hypersurface. Let us call  $n^i(\xi)$  the ‘unit normal’ to the hypersurface at the point  $\xi$  in the hypersurface. Thus  $n_i n^i = 1$ , and  $n^i W_i = 0$  for any vector  $W$  at the point  $\xi$  tangent to the hypersurface. We can then make a scalar  $n^i d\Sigma_i$  and integrate this over the hypersurface

$$I = \int_{\mathcal{A}} n^i d\Sigma_i \quad (29)$$

This gives the ‘area’ of the hypersurface. Note that the area can have either sign depending on which of the two directions of the normal we choose; the physical area will thus be the absolute value of the above quantity.

Similarly, we can integrate a scalar function  $f$  over the hypersurface. Writing this out explicitly in terms of coordinates  $\chi^i$  on the hypersurface,

$$I = \int_{\mathcal{A}} n^i d\Sigma_i f = \epsilon_{i_1 \dots i_{n-1}} n^i \frac{\partial \xi^{i_1}}{\partial \chi^1} \dots \frac{\partial \xi^{i_{n-1}}}{\partial \chi^{n-1}} d\chi^1 \dots d\chi^{n-1} f(\chi) \quad (30)$$

## 6 Curvature of 2-d manifolds

Consider a 2-dimensional manifold with Euclidean signature. The metric has 3 independent components, and diffeomorphisms allow us to fix 2 of these. Thus the metric is described (at least in a coordinate patch) by 1 function. One choice is to choose coordinates so that the metric is brought to *conformal gauge*

$$g_{ab} = e^{\rho(\xi)} \delta_{ab} \quad (31)$$

Thus the metric is given by a ‘scale factor’  $e^{\rho(\xi)}$  times the flat 2-d metric. In general we cannot obtain this form everywhere, but we can do it in a coordinate patch, and a separate coordinate transformation brings us to conformal gauge in a new patch.

The connection is

$$\begin{aligned} \Gamma_{bc}^a &= \frac{1}{2} g^{ad} [g_{db,c} + g_{dc,b} - g_{bc,d}] \\ &= \frac{1}{2} \delta^{ad} [\delta_{db} \rho_{,c} + \delta_{dc} \rho_{,b} - \delta_{bc} \rho_{,d}] \\ &= \frac{1}{2} [\delta^a_{\ b} \rho_{,c} + \delta^a_{\ c} \rho_{,b} - \delta_{bc} \rho^{,a}] \end{aligned} \quad (32)$$

where the raising of the index on the last term is done with the *flat* metric  $\delta^{ab}$ . In fact we will not use the curved metric  $e^{\rho} \delta_{ab}$  any more, referring all quantities to the flat metric  $\delta_{ab}$ ; the effect of the curvature is captured by the function  $\rho(\xi)$ .

We find

$$\begin{aligned}
R^a{}_{bcd} &= \Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{fc}^a \Gamma_{bd}^f - \Gamma_{fd}^a \Gamma_{bc}^f \\
&= \frac{1}{2} [\delta^a{}_{b\rho,dc} + \delta^a{}_{d\rho,bc} - \delta_{bd}\rho^a{}_c] - \frac{1}{2} [\delta^a{}_{b\rho,cd} + \delta^a{}_{c\rho,bd} - \delta_{bc}\rho^a{}_d] \\
&+ \frac{1}{4} [\delta^a{}_{f\rho,c} + \delta^a{}_{c\rho,f} - \delta_{fc}\rho^a] [\delta^f{}_{b\rho,d} + \delta^f{}_{d\rho,b} - \delta_{bd}\rho^f] \\
&- \frac{1}{4} [\delta^a{}_{f\rho,d} + \delta^a{}_{d\rho,f} - \delta_{fd}\rho^a] [\delta^f{}_{b\rho,c} + \delta^f{}_{c\rho,b} - \delta_{bc}\rho^f] \\
&= \frac{1}{2} [\delta^a{}_{d\rho,bc} - \delta_{bd}\rho^a{}_c - \delta^a{}_{c\rho,bd} + \delta_{bc}\rho^a{}_d] \\
&+ \frac{1}{4} [-\delta^a{}_{d\rho,c\rho,b} + \delta^a{}_{c\rho,b\rho,d} - \delta_c^a \delta_{bd}\rho_{,f}\rho^f + \delta_d^a \delta_{bc}\rho_{,f}\rho^f - \delta_{bc}\rho^a{}_{\rho,d} + \delta_{bd}\rho^a{}_{\rho,c}]
\end{aligned} \tag{33}$$

The Ricci tensor is

$$R_{bd} = -\frac{1}{2}\delta_{bd}\rho^a{}_a \tag{34}$$

The Ricci scalar is

$$R = -e^{-\rho}\rho^a{}_a \tag{35}$$

If we integrate the Ricci scalar over a region of the 2-d surface, we get

$$\int_{\mathcal{R}} \sqrt{g} R d^2\xi = - \int_{\mathcal{R}} \rho^a{}_a d^2\xi \tag{36}$$

Recalling that up and down indices are the same here since raising and lowering is done by the identity metric, we see that

$$\int_{\mathcal{R}} \sqrt{g} R d^2\xi = - \int_{\mathcal{R}} \Delta \rho d^2\xi \tag{37}$$

where  $\Delta$  is the Laplacian in *flat* 2-d space (i.e. the space with metric  $\delta_{ab}$ ).

By an integration by parts, we have

$$\int_{\mathcal{R}} \sqrt{g} R d^2\xi = - \int dl n^a \rho_{,a} \tag{38}$$

where  $dl$  measures the length of the boundary path (in the flat metric  $\delta_{ab}$ ),  $n^a$  is the unit outward normal to the boundary (again with length measured in the flat metric).

Suppose we hold fixed the function  $\rho$  and its first derivative at the boundary of  $\mathcal{R}$ , and change  $\rho$  only in the interior of  $\mathcal{R}$ . Since the quantity above was written in terms of a boundary integral, it remains unchanged. Thus we conclude that if we make a ‘bump’ in some region of a 2-d metric, we will not change the value of  $\int_{\mathcal{R}} \sqrt{g} R d^2\xi$ .

Now consider a closed surface, like the surface of a sphere. For the sphere with its spherically symmetric metric and radius  $a$ , we find

$$\int_{S^2} \sqrt{g} R d^2\xi = \frac{2}{a^2} \int \sqrt{g} d^2\xi = \frac{2}{a^2} (4\pi a^2) = 8\pi \quad (39)$$

But now we can divide the sphere into regions, in each of which we can write the metric in conformal gauge, and using the above argument, conclude that a change in the metric in the interior of that region does not change the value of  $\int \sqrt{g} R d^2\xi$  in that region. Thus we find that

$$\int_{S^2} \sqrt{g} R d^2\xi = 8\pi \quad (40)$$

for *all* metrics on a surface with the topology of a 2-sphere.

## 7 The gravitational action

Let us take the scalar  $R$  and integrate it over a region  $\mathcal{R}$

$$I = \int_{\mathcal{R}} d^d\xi \sqrt{-g} R = \int_{\mathcal{R}} d^d\xi \sqrt{-g} g^{ab} R_{ab} \quad (41)$$

Consider an infinitesimal variation  $g^{ab} \rightarrow g^{ab} + \delta g^{ab}$ . Then we have

$$\delta I = \int_{\mathcal{R}} d^d\xi [\delta \sqrt{-g} g^{ab} R_{ab} + \sqrt{-g} \delta g^{ab} R_{ab} + \sqrt{-g} g^{ab} \delta R_{ab}] \quad (42)$$

We have

$$\delta \sqrt{-g} = \frac{1}{2\sqrt{-g}} g g_{ab} \delta g^{ab} = -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab} \quad (43)$$

To find  $\delta R_{ab}$  we go to a local coordinate system where  $g_{ab} = \eta_{ab}$ ,  $g_{ab,c} = 0$ . Then

$$\delta R_{ab} = \delta R^c{}_{acb} = \delta \Gamma^c{}_{ab,c} - \delta \Gamma^c{}_{ac,b} \quad (44)$$

We note that while  $\Gamma^a{}_{bc}$  is not a tensor, its variations  $\delta \Gamma^a{}_{bc}$  are. Thus we can write

$$g^{ab} \delta R_{ab} = g^{ab} \delta \Gamma^c{}_{ab;c} - g^{ab} \delta \Gamma^c{}_{ac;b} = [g^{ab} \delta \Gamma^c{}_{ab}]_{;c} - \delta \Gamma^c{}_{ac};{}^a \equiv V^a{}_{;a} \quad (45)$$

where we have defined

$$V^a = g^{bc} \delta \Gamma^a{}_{bc} - g^{ab} \delta \Gamma^c{}_{bc} \quad (46)$$

Thus

$$\int_{\mathcal{R}} d^d\xi \sqrt{-g} g^{ab} R_{ab} = \int_{\mathcal{R}} d^d\xi \sqrt{-g} V^a{}_{;a} \quad (47)$$

is a total divergence, and vanishes if we keep all variations to vanish at the boundary. Thus we get

$$\delta I = \int_{\mathcal{R}} d^d \xi [R_{ab} - \frac{1}{2}g_{ab}R] \delta g^{ab} \quad (48)$$

Thus

$$\frac{\delta I}{\delta g^{ab}} = R_{ab} - \frac{1}{2}g_{ab}R \quad (49)$$

We write the complete action as

$$S = S_{grav} + S_{matter} = \frac{1}{16\pi G} \int_{\mathcal{R}} d^d \xi \sqrt{-g} R + S_{matter} \quad (50)$$

We write

$$\delta S_{matter} = \int d^d \xi \sqrt{-g} \frac{1}{2} T_{ab} \delta g^{ab} \quad (51)$$

Then we get

$$\delta S = [\frac{1}{16\pi G} G_{ab} - \frac{1}{2} T_{ab}] \delta g^{ab} \quad (52)$$

Setting the variation to zero, we get the field equations

$$G_{ab} = 8\pi G T_{ab} \quad (53)$$