Symmetries and Killing vectors

1 Symmetries

Consider 2-d flat space with metric

$$ds^2 = dx^2 + dy^2 \tag{1}$$

It is clear that this 2-d plane has translation symmetry; i.e., if we shift $x \to x + \epsilon$, things 'remain the same'. Similarly, we have translation symmetry in y. Now consider a 2-d space with a different metric: one which is in the shape of a hat; i.e., it has a 'bump' in the middle of the plane. This time the metric is *not* invariant under translations: if we make a shift $x' = x + \epsilon^x$ then the bump will move to a different location.

Our goal is to arrive at a general formulation of symmetries of the metric. Thus given a metric $g_{\mu\nu}(\xi)$, can we check if a given infinitesimal shift is a symmetry? (Finite shifts of coordinates can usually be built up from a succession of infinitesimal shifts, so it is convenient to ask the question about infinitesimal shifts. We will talk later about the case of *discrete* symmetries, where we cannot start from infinitesimal shifts.)

Returning to the 2-d flat plane, we see that these translation symmetries exist because the metric coefficients do not depend on $\{x, y\}$. Thus if we write

$$x' = x + \epsilon^x, \quad y' = y \quad (\epsilon^x = constant) \tag{2}$$

then $dx' = dx,\ dy' = dy$, and the metric looks the same as (1):

$$ds^2 = dx'^2 + dy'^2 \tag{3}$$

On the other hand if the metric had the form

$$ds^2 = x^2[dx^2 + dy^2] \tag{4}$$

then we would get

$$ds^2 = (x' - \epsilon^x)^2[dx'^2 + dy'^2] \approx (x'^2 - 2x'\epsilon^x)[dx'^2 + dy'^2] \tag{5}$$

This is not the same functional form as (4); the metric coefficients are altered at order ϵ^x, which was the order of the coordinate shift that we made. Thus the coordinate shift (2) is not a symmetry of the metric (4).

We know that in the 2-d plane there is also a rotational symmetry around the origin, but this is not clear from form of the metric (1). The rotational symmetry does become clear if we write the metric of the flat plane in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{6}$$

Now we see that a change of coordinates

$$r' = r, \quad \theta' = \theta + \epsilon^\theta \quad (\epsilon^\theta = constant) \tag{7}$$

leaves the metric unchanged.

2 Killing vectors

Let us now address the problem of symmetry in general terms. Start with a manifold $\mathcal{M}$, with coordinates ξ^μ. Let the metric in these coordinates be $g_{\mu\nu}(\xi)$. Suppose we make an infinitesimal change of coordinates

$$\xi'^\mu = \xi^\mu + \epsilon^\mu(\xi) \tag{8}$$

The metric will change to

$$g'_{\mu'\nu'}(\xi') = \frac{\partial \xi^\mu}{\partial \xi'^{\mu'}} \frac{\partial \xi^\nu}{\partial \xi'^{\nu'}} g_{\mu\nu}(\xi) \tag{9}$$

Note that the LHS has the new metric *at the same physical point on the manifold*, but at a different *numerical value* for the coordinate. For our present purposes, we wish to know if the new metric has the *same functional form* as the old metric; i.e. we wish to know if the new metric coefficients have the same numerical value as the old metric components, if we look at the *same* numerical value of the coordinates. Thus we expand

$$g'_{\mu'\nu'}(\xi') = g'_{\mu'\nu'}(\xi) + g'_{\mu'\nu',\lambda}(\xi)\epsilon^\lambda \tag{10}$$

The functional form of the metric will be unchanged under the coordinate change (8) if

$$g'_{\mu'\nu'}(\xi) = g_{\mu'\nu'}(\xi) \tag{11}$$

for each set of indices (μ', ν'). This is the condition that we are after; all we have to do now is to use the above relations to write the condition in a more compact form.

First, we note from (8)

$$\frac{\partial \xi'^{\mu'}}{\partial \xi^\mu} = \delta^{\mu'}_\mu + \epsilon^{\mu'}{}_{,\mu} \tag{12}$$

What we need though are the derivatives $\frac{\partial \xi^\mu}{\partial \xi'^{\mu'}}$. For these we write

$$\xi^\mu = \xi'^\mu - \epsilon^\mu(\xi) \approx \xi'^\mu - \epsilon^\mu(\xi') \tag{13}$$

where in the second step we have replaced the argument ξ by ξ'. We can make this replacement because the two coordinate systems differ only by order ϵ, and the correction to our term would be $O(\epsilon^2)$:

$$\epsilon^\mu(\xi) = \epsilon^\mu(\xi') - \epsilon^\mu{}_{,\nu}\epsilon^\nu = \epsilon^\mu(\xi') + O(\epsilon^2) \tag{14}$$

From (13) we now get

$$\frac{\partial \xi^\mu}{\partial \xi'^{\mu'}} = \delta^\mu_{\mu'} - \epsilon^\mu{}_{,\mu'} \tag{15}$$

Using this in (9) we get

$$\begin{aligned} g'_{\mu'\nu'}(\xi') &= [\delta^\mu_{\mu'} - \epsilon^\mu{}_{,\mu'}][\delta^\nu_{\nu'} - \epsilon^\nu{}_{,\nu'}]g_{\mu\nu}(\xi) \\ &= g_{\mu'\nu'}(\xi) - \epsilon^\mu{}_{,\mu'}g_{\mu\nu'}(\xi) - \epsilon^\nu{}_{,\nu'}g_{\mu'\nu}(\xi) \end{aligned} \tag{16}$$

From (10) we get

$$\begin{aligned} g'_{\mu'\nu'}(\xi) &= g'_{\mu'\nu'}(\xi') - g'_{\mu'\nu',\lambda}(\xi)\epsilon^\lambda \\ &= g_{\mu'\nu'}(\xi) - \epsilon^{\mu'}{}_{,\mu}g_{\mu\nu'}(\xi) - \epsilon^{\nu'}{}_{,\nu}g_{\mu'\nu}(\xi) - g'_{\mu'\nu',\lambda}(\xi)\epsilon^\lambda \\ &= g_{\mu'\nu'}(\xi) - \epsilon^{\mu'}{}_{,\mu}g_{\mu\nu'}(\xi) - \epsilon^{\nu'}{}_{,\nu}g_{\mu'\nu}(\xi) - g_{\mu'\nu',\lambda}(\xi)\epsilon^\lambda \end{aligned} \tag{17}$$

where in the last step we have replaced the metric g' with g since the difference would yield a term that is higher order in ϵ. The condition (11) then gives

$$\epsilon^\mu{}_{,\mu'}g_{\mu\nu'} + \epsilon^\nu{}_{,\nu'}g_{\mu'\nu} + g_{\mu'\nu',\lambda}\epsilon^\lambda = 0 \tag{18}$$

This condition can be written more elegantly if we use the vector ϵ^μ with indices lowered. Thus we have

$$\left(\epsilon^\mu g_{\mu\nu'}\right)_{,\mu'} + \left(\epsilon^\nu g_{\mu'\nu}\right)_{,\nu'} - \left[g_{\mu\nu',\mu'}\epsilon^\mu + g_{\mu'\nu,\nu'}\epsilon^\nu - g_{\mu'\nu',\lambda}\epsilon^\lambda\right] = 0 \tag{19}$$

A little index manipulation shows that this is

$$\epsilon_{\nu',\mu'} + \epsilon_{\mu',\nu'} - 2\Gamma^{\kappa'}_{\mu'\nu'}\epsilon_{\kappa'} = 0 \tag{20}$$

which is seen to be equivalent to the equation

$$\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} = 0 \tag{21}$$

where we have dropped the primes on the indices for simplicity. Vectors satisfying (21) are called Killing vectors.

Looking at this final relation, we note something remarkable: this is a purely geometric condition on the vector ϵ, and thus independent of any coordinate system that we may have used to describe the metric. To summarize, suppose we have a vector field ϵ^μ, which satisfies the relation (21). Then we can take each point of our spacetime and move it by an infinitesimal number times this vector. This 'shift' will then be a symmetry of the manifold. In this way we will be able to express both the translation and rotation symmetries of the 2-d flat plane by giving vector fields appropriate to the two translations and the rotation; we would not have to worry about the fact that cartesian coordinates were needed to manifest the translation symmetries and polar coordinates were needed to manifest the rotational symmetries.