1 Hermitian conjugation

Recall that the operators α_n were defined by

$$\alpha_n^\mu = \frac{\hat{a}_n^\mu}{\sqrt{n}}, \quad n > 0 \tag{1}$$

$$\alpha_{-n}^\mu = \frac{(\hat{a}^\mu)^\dagger}{\sqrt{n}} \tag{2}$$

where $\hat{a}_n, \hat{a}_n^\dagger$ are annihilation and creation operators for the mode n. Thus we have

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger \tag{3}$$

Now look at the Virasoro operators. We have

$$L_n^\dagger = (\alpha_{n-m}\alpha_m)^\dagger = \alpha_m^\dagger \alpha_{n-m}^\dagger = \alpha_{-m}\alpha_{-n+m} = L_{-n} \tag{4}$$

While these rules for Hermitian conjugation look natural, we will have some trouble in defining our Hilbert space if we do not place additional restrictions on our theory. These issues stem from the fact that by Lorentz covariance we must write

$$[\hat{a}^\mu, (\hat{a}^\nu)^\dagger] = \eta^{\mu\nu} \tag{5}$$

Thus for the timelike direction we have a commutation relation with a sign opposite to the one that we are used to

$$[\hat{a}_n^0, (\hat{a}_n^0)^\dagger] = -1 \tag{6}$$

This causes the following problem. Again by covariance, we define the vacuum by

$$\hat{a}_n^\mu |0\rangle = 0 \tag{7}$$

First consider the state

$$|\psi\rangle = (\hat{a}_n^i)^\dagger |0\rangle \tag{8}$$

where i is a direction other than the timelike direction, so $\eta^{ii} = 1$. The norm of this state is

$$\langle\psi|\psi\rangle = \langle 0|\hat{a}_n^i (\hat{a}_n^i)^\dagger|0\rangle = \langle 0|\hat{a}_n^i (\hat{a}_n^i)^\dagger|0\rangle + \langle 0|[\hat{a}_n^i, (\hat{a}_n^i)^\dagger|0\rangle = 1 \tag{9}$$

This is positive, so there is no problem. But now consider a similar state but using the oscillators in the timelike direction

$$\langle\psi|\psi\rangle = \langle 0|\hat{a}_n^0 (\hat{a}_n^0)^\dagger|0\rangle = \langle 0|\hat{a}_n^0 (\hat{a}_n^0)^\dagger|0\rangle + \langle 0|[\hat{a}_n^0, (\hat{a}_n^0)^\dagger|0\rangle = -1 \tag{10}$$

This state has *negative* norm. Thus we do not have a good Hilbert space. We will have to remove the negative norm states from our consideration. Let us investigate the norms of states a little more, and then see how we solve this problem.

2 Some basic relations

The mass formula of a state had been derived earlier in the classical limit. We have also advanced some arguments as to why the ground state of the string has an oscillator level -1 because of quantum effects related to the Casimir energy. We will return to this fact later in more detail, but we accept it for now and write the mass formula that we will use. We will not take any dimensions compact for now, so we will have

$$p_L^\mu = p_L^\mu - \frac{p^\mu}{2} \tag{11}$$

We have

$$\alpha_0^\mu = \sqrt{2\alpha'} p_L^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \tag{12}$$

The mass formula is

$$m^2 = -p^2 = 8\pi T(N-1) = \frac{4}{\alpha'}(N-1) \tag{13}$$

where N is the oscillator level of the right movers. A similar relation holds for the left movers.

2.1 States at level one

The vacuum is normalized to have

$$\langle 0|0\rangle = 1 \tag{14}$$

We will use the oscillators α_n instead of the operators $\hat{a}_n, \hat{a}_n^\dagger$ for convenience. At the first excited level we have the states

$$|\psi\rangle = C_\mu \alpha_{-1}^\mu |0\rangle \tag{15}$$

where C_n are complex constants. The norm of such a state is

$$\langle\psi|\psi\rangle = C_\mu^* C_\nu \langle 0|\alpha_1^\mu \alpha_{-1}^\nu|0\rangle \tag{16}$$

Using the basic relation

$$[\alpha_n^\mu, \alpha_m^\nu] = \eta^{\mu\nu}\delta_{m+n,0} \tag{17}$$

we get

$$\langle\psi|\psi\rangle = C_\mu^* C_\nu \eta^{\mu\nu} = C_\mu^* C^\mu \equiv |C|^2 \tag{18}$$

Thus this norm can be positive, negative or zero, because C can be spacelike, null or timelike. But recall that not all states in the Hilbert space are supposed to be physical. We have to impose the conditions

$$L_n|\psi\rangle = 0, \quad n > 0 \tag{19}$$

In the present case it is clear that

$$L_n|\psi\rangle = 0, \quad n \geq 2 \tag{20}$$

This follows because such an L_n make the total level of the state positive, and thus every term will have to contain an oscillator α_m with $m > 0$, which will annihilate the vacuum. Thus the condition that we should check is

$$L_1|\psi\rangle = 0 \tag{21}$$

We have

$$L_1 = \frac{1}{2}\alpha^\mu_{1-m}\alpha_{\mu,m} = \alpha^\mu_0\alpha_{\mu,1} + \alpha^\mu_{-1}\alpha_{\mu,2} + \alpha^\mu_{-2}\alpha_{\mu,3} + \dots \tag{22}$$

Then we get

$$\begin{aligned} L_1|\psi\rangle &= \left(\alpha^\mu_0\alpha_{\mu,1} + \alpha^\mu_{-1}\alpha_{\mu,2} + \alpha^\mu_{-2}\alpha_{\mu,3} + \dots\right) C_\nu \alpha^\nu_{-1}|0\rangle \\ &= \alpha^\mu_0\alpha_{\mu,1} C_\nu \alpha^\nu_{-1}|0\rangle \\ &= C_\nu \delta^\nu_\mu \alpha^\mu_0|0\rangle \\ &= \sqrt{\frac{\alpha'}{2}} C_\mu p^\mu|0\rangle \end{aligned} \tag{23}$$

where we have assumed that the vacuum state has no oscillator excitations but carries a momentum p^μ

$$\alpha^\mu_0|0\rangle = \sqrt{\frac{\alpha'}{2}}\hat{p}^\mu|0\rangle = \sqrt{\frac{\alpha'}{2}} p^\mu|0\rangle \tag{24}$$

From now on we will set

$$\alpha' = 1 \tag{25}$$

and thus we have

$$L_1|0\rangle = \frac{1}{\sqrt{2}} C_\mu p^\mu |0\rangle \tag{26}$$

Thus we see that there is one constraint on the C_μ. From the mass formula for string states we see that for a state with one oscillator excitation we will have

$$m^2 = 8\pi T(N-1) = 0 \tag{27}$$

Thus this is a massless particle state, and by Lorentz invariance we can go to a frame where

$$p^\mu = (1, 1, 0, \dots 0) \tag{28}$$

In this frame our constraint becomes

$$C_0 + C_1 = 0 \tag{29}$$

Thus we find

$$|C|^2 = -|C_0|^2 + |C_1|^2 + |C_i|^2 = |C_i|^2 > 0 \tag{30}$$

so that C is spacelike. But in this case the norm of the state is positive

$$\langle\psi|\psi\rangle = |C|^2 > 0 \tag{31}$$

and we see that we have a well defined Hilbert space. We have eliminated the 'ghost' state which ahd negative norm by imposing the constraints $L_n > 0$ for $n > 0$.

Now the question is whether we can keep doing something similar for all states which have negative norm. If we can, then we have a good theory, and can use it to describe the first quantized string. We wil see that we can in fact achive this, but only if we have a ceratin dimension D and the choice that

$$(L_0 - 1)|\psi\rangle = 0 \tag{32}$$

This condition can be thought of as a member of the conditions $L_n > 0$ for $n > 0$. The mode L_0 is in-between the positive and negative modes of the L_n, so it was not immediately clear what we should do with this mode as far as the constraints were concerned. Recall that only for this mode L_0 there is a normal ordering issue and so it is not a priori clear what the value of L_0 on the vacuum should be.

3 Gauge modes

We have seen that physical states must satisfy

$$L_n|\psi\rangle = 0, \quad (n > 0), \qquad (L_0 - 1)|\psi\rangle = 0 \tag{33}$$

These physicality conditions removed the negative norm states from out Hilbert space. Are the states satisfying the physical state conditions all positive norm states? The $L_1|\psi\rangle = 0$ condition told us that

$$C_\mu p^\mu = 0 \tag{34}$$

and the $(L_0 - 1)|\psi\rangle = 0$ condition told us that p was null so it had the form

$$p^\mu = (q, q, 0, 0 \ldots 0) \tag{35}$$

Suppose we choose

$$C^0 = C^1 = c \tag{36}$$

Then this state satisfies the physicality conditions, but has norm zero

$$|\psi|^2 = |C^2| = 0 \tag{37}$$

Thus physical states can have zero norm. Note that this state can be generated in the following form

$$L_{-1}|0\rangle = [\alpha_0^\mu \alpha_{\mu,-1} + \ldots]|0\rangle = \frac{1}{\sqrt{2}} p^\mu \alpha_{\mu,-1}|0\rangle \tag{38}$$

so that the state has $C^\mu \propto p^\mu$.

This state is a pure gauge state, since it is a massless state with polarization proportional to the momentum. In gauge theory we make a pure gauge state by

$$A_\mu(x) = \partial_\mu \Lambda(x) \quad \Rightarrow \quad A_\mu(p) \sim p_\mu \tag{39}$$

4 Extracting a general principle

We define descendent states by

$$|\chi\rangle = L_{n_k} \ldots L_{n_1}|\lambda\rangle, \quad n_i > 0 \tag{40}$$

These states are orthogonal to all states that satisfy

$$L_n|\psi\rangle = 0, \quad n > 0 \tag{41}$$

This follows from

$$\langle\chi|\psi\rangle = \langle\lambda|L_{n_k} \ldots L_{n_1}|\psi\rangle = 0 \tag{42}$$

What happens if a state $|\chi\rangle$ is a descendent and also a state that is annihilated by the $L_n, n > 0$? Then we get

$$\langle\chi|\chi\rangle = \langle\lambda|L_{n_k} \ldots L_{n_1}|\chi\rangle = 0 \tag{43}$$

so the state has zero norm.

Suppose that the state also satisfies

$$(L_0 - 1)|\chi\rangle = 0 \tag{44}$$

Then we must regard it as a physical state because it satisfies

$$L_n|\chi\rangle = 0, \quad n > 0, \qquad (L_0 - 1)|\chi\rangle = 0 \tag{45}$$

But it is also a descendent since it is of the form

$$|\chi\rangle = L_{-n_k} \dots L_{n_1}|\lambda\rangle \tag{46}$$

So it will be null. We will regard these states as pure gauge states.

5 States of level 2

Now let us look at states of level 2. We can make such states in two ways

$$\alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle \tag{47}$$

and

$$\alpha^\mu_{-2}|0\rangle \tag{48}$$

This time we will have to consider two conditions

$$L_1|0\rangle = 0, \quad L_2|0\rangle = 0 \tag{49}$$

Actually even if had a state of level 3 or higher, we would still have to consider only these two conditions. The reason is that we have

$$[L_1, L_2] = -L_3 \tag{50}$$

so if L_1, L_2 annihilate a state then L_3 will also do so automatically. More generally

$$[L_1, L_n] = (1-n)L_{1+n} \tag{51}$$

so all the L_n with $n > 3$ will automatically annihilate the state if L_1, L_2 do so.

We write the state at level 2 as

$$\left(C_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + D_\mu\alpha^\mu_{-2}\right)|0\rangle \tag{52}$$

The relevant part of L_1 is

$$L_1 = \alpha^\mu_0\alpha_{\mu,1} + \alpha^\mu_{-1}\alpha_{\mu,2} + \dots \tag{53}$$

The action of L_1 on $C_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle$ gives

$$(\alpha^\mu_0\alpha_{\mu,1})\, C_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle = \frac{2}{\sqrt{2}}p^\mu C_{\mu\nu}\alpha^\nu_{-1}|0\rangle \tag{54}$$

The action of L_{-1} on $D_\mu \alpha^\mu_{-2}|0\rangle$ gives

$$\left(\alpha^\mu_{-1}\alpha_{\mu,2}\right) D_\mu \alpha^\mu_{-2}|0\rangle = 2D_\mu \alpha^\mu_{-1}|0\rangle \tag{55}$$

Thus we need

$$\frac{1}{\sqrt{2}} C_{\mu\nu} p^\nu + D_\mu = 0 \tag{56}$$

Now consider the action of L_2. The relevant part of L_2 is

$$L_2 = \alpha^\mu_0 \alpha_{\mu,2} + \frac{1}{2}\alpha^\mu_1 \alpha_{\mu,1} + \ldots \tag{57}$$

The action of L_2 on $C_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle$ gives

$$\left(\frac{1}{2}\alpha^\lambda_1 \alpha_{\lambda,1}\right) C_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0\rangle = C^\mu_\mu |0\rangle \tag{58}$$

The action of L_2 on $D_\mu \alpha^\mu_{-2}$ gives

$$\left(\alpha^\lambda_0 \alpha_{\lambda,2}\right) D_\mu \alpha^\mu_{-2}|0\rangle = \frac{2}{\sqrt{2}} p^\mu D_\mu |0\rangle \tag{59}$$

Thus we get the condition

$$C^\mu_\mu + \sqrt{2} p^\mu D_\mu = 0 \tag{60}$$

Let us now see what these conditions give us. From the mass formula we have

$$-p^2 = m^2 = 8\pi T(N-1) = 8\pi T = \frac{4}{\alpha'} = 4 \tag{61}$$

Thus we can go to the rest frame where

$$p^\mu = (2, 0, 0, \ldots 0) \tag{62}$$

In this frame we get the conditions

$$\sqrt{2} C_{0\mu} + D_\mu = 0 \tag{63}$$

$$C^\mu_\mu + 2\sqrt{2} D_0 = 0 \tag{64}$$

The second condition gives

$$D_0 = -\frac{1}{2\sqrt{2}} C^\mu_\mu = -\frac{1}{2\sqrt{2}}(-C_{00} + \sum_i C_{ii}) \tag{65}$$

In the first condition we can set $\mu = 0$ to get

$$D_0 = -\sqrt{2} C_{00} \tag{66}$$

Using this in the equation just before we get

$$C_{00} = \frac{1}{5}\sum_i C_{ii} \tag{67}$$

Putting $\mu = i$ in (63) we get

$$D_i = -\sqrt{2}C_{0i} \tag{68}$$

Thus the D_μ are compeltely fixed by the $C_{\mu\nu}$, and the $C_{\mu\nu}$ satisfy the additional condition (67).

Now let us see if once we impose the L_n conditions whether we get positive norm states. Thus we should compute the norm of our state (52).

$$|\psi|^2 = \langle 0| \left(C^*_{\mu'\nu'}\alpha_1^{\mu'}\alpha_1^{\nu'} + D_{\mu'}\alpha_2^{\mu'} \right) \left(C_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu} + D_\mu \alpha_{-2}^{\mu} \right) |0\rangle \tag{69}$$

There are no cross terms between the two kinds of terms, and we find

$$|\psi|^2 = 2\left(C^*_{\mu\nu}C^{\mu\nu} + D^*_\mu D^\mu \right) \tag{70}$$

Let us set $C_{0i} = 0$. This sets $D_i = 0$. Let us take all the $C_{ii} = 1$. Then we have

$$C_{00} = \frac{1}{5}(D-1) \tag{71}$$

$$D_0 = -\sqrt{2}C_{00} = -\frac{\sqrt{2}}{5}(D-1) \tag{72}$$

We have

$$C^*_{\mu\nu}C^{\mu\nu} + D^*_\mu D^\mu = C_{00}^2 + (D-1) - D_0^2 = \frac{(D-1)^2}{25} + (D-1) - \frac{2(D-1)^2}{25} = -\frac{(D-1)(D-26)}{25} \tag{73}$$

Thus if $D > 26$ then we will get a negative norm state. It will turn out that we can get a consistent theory when $D = 26$, where this is a zero norm state.

Let us now consider the termns C_{0i}, D_i which we had set to zero above. These contribute to the norm squared

$$2\left(-2\sum_i C_{0i}C_{0i} + \sum_i D_i D_i \right) = 0 \tag{74}$$

where we have used (68).

6 More on zero norm states

Suppose that we ask that there be physical states of a special form

$$|\chi\rangle = L_{-1}|\lambda\rangle \tag{75}$$

where we ask that $|\lambda\rangle$ be such that

$$L_n|\lambda\rangle = 0, \quad n > 0 \tag{76}$$

The physical state $|\chi\rangle$ must be annihilated by all $L_n, n > 0$, so in particular

$$0 = L_1|\chi\rangle = L_1L_{-1}|\lambda\rangle = 2L_0|\lambda\rangle \tag{77}$$

Suppose we did not know the normal ordering constant in L_0, so we just ask that physical states satisfy

$$(L_0 - a)|\psi\rangle = 0 \tag{78}$$

Then we will have

$$L_0|\lambda\rangle = a - 1 \tag{79}$$

and we find from (77)

$$a = 1 \tag{80}$$

Thus this value of a that we have selected on physical grounds provides physical states of type (75), which, being also descendants, are null states. Thus at $a = 1$ we get extra null states of the kind (75).

Now let us try the same with two levels of excitations created by the L_{-n}. Thus we take

$$|\chi\rangle = (L_{-2} + \gamma L_{-1}^2)|\lambda\rangle \tag{81}$$

with

$$L_n|\lambda\rangle = 0, \quad n > 0 \tag{82}$$

and

$$(L_0 - 1)|\chi\rangle = 0 \tag{83}$$

The latter relation implies that

$$L_0|\lambda\rangle = -1 \tag{84}$$

We have

$$0 = L_1|\chi\rangle = (3L_{-1} + \gamma[2L_{-1}L_0 + 2L_0L_{-1}|\lambda\rangle = (3 - 2\gamma)L_{-1}|\lambda\rangle \tag{85}$$

so that we find

$$\gamma = \frac{3}{2} \tag{86}$$

Now we look at

$$0 = L_2|\chi\rangle = L_2(L_{-2} + \frac{3}{2}L_{-1}^2)|\lambda\rangle = [(-4 + \frac{D}{2}) + \frac{3}{2}(3L_1L_{-1} + 6L_0)|\lambda\rangle = (-13 + \frac{D}{2})|\lambda\rangle \tag{87}$$

Thus we get null states of this form for $D = 26$.

Thus we see that at $D = 26$ and $a = 1$ there are additional null states of a special kind.

7 Exercises

(A) Consider a complex fermion

$$d = d^{(1)} + id^{(2)} \tag{88}$$

We have

$$(d_n^{(1)})^\dagger = d_{-n}^{(1)}, \quad (d_n^{(2)})^\dagger = d_{-n}^{(2)} \tag{89}$$

The commutation relations are

$$\{d_n^{(1)}, d_m^{(1)}\} = \delta_{n+m,0}, \quad \{d_n^{(2)}, d_m^{(2)}\} = \delta_{n+m,0} \tag{90}$$

We make the currents

$$J_n = i d_m^\dagger d_{n+m} \tag{91}$$

Find the commutation relations of these currents, together with the anomaly term.

(B) Analyse the Hilbert space at level three, following the analysis of levels one and two above.