1 Fermions

Consider the string world sheet. We have bosons $X^\mu(\sigma,\tau)$ on this world sheet. We will now also put $\psi^\mu(\sigma,\tau)$ on the world sheet. These fermions are spin $\frac{1}{2}$ objects on the worldsheet.

In higher dimensions, we can take a local orthonormal frame and spin around in a complete circle. In 3 space dimensions for example, this changes the fermion as

$$e^{\theta\frac{1}{4}[\sigma_1,\sigma_2]} = e^{i\frac{\theta}{2}\sigma_3} \tag{1}$$

Thus with $\theta = 2\pi$ each component of the spinor changes sign. Thus ψ and $-\psi$ define the same fermion wavefunction.

We will assume the same behavior for the 1+1 case we have now. In the Euclidean case we can rotate in the 2-d worldsheet and obtain this property immediately.

Now consider what boundary conditions are appropriate for the fermion. Suppose we are on the cylinder, and we go around $\sigma \to \sigma + 2\pi$. We want to come back to the same configuration, but we do not know if we should represent this by ψ or $-\psi$. Thus there are two possibilities

$$\psi(\sigma+2\pi) = \psi(\sigma) \quad (Ramond = R) \tag{2}$$

$$\psi(\sigma+2\pi) = -\psi(\sigma) \quad (Neveu-Schwarz = NS) \tag{3}$$

In the first case we get the modes

$$\psi = \sum_{n=-\infty}^{\infty} \psi_n e^{in\sigma} \quad (R) \tag{4}$$

In the second case we get

$$\psi = \sum_{n=-\infty}^{\infty} \psi_{n+\frac{1}{2}} e^{i(n+\frac{1}{2})\sigma} \quad (NS) \tag{5}$$

2 Zero point energy

For bosons the Hamiltonian is

$$H = \frac{1}{2}\omega(a^\dagger a + a a^\dagger) = \omega a^\dagger a + \frac{1}{2}\omega \tag{6}$$

For fermions we have

$$H = \frac{1}{2}\omega(b^\dagger b - b b^\dagger) = \omega b^\dagger b - \frac{1}{2}\omega \tag{7}$$

where we have noted that

$$[a, a^\dagger] = 1, \quad [b, b^\dagger]_+ = 1 \tag{8}$$

Thus the zero point energy of the bosons gave us

$$\frac{1}{2}[1+2+\ldots] = \frac{1}{2}(-\frac{1}{12}) = -\frac{1}{24} \tag{9}$$

where we have used that
$$S = 1 + 2 + \ldots = -\frac{1}{12} \tag{10}$$
which we had proved earlier. If we have the R sector, we see that the fermions give us
$$-\frac{1}{2}(1 + 2 + \ldots) = \frac{1}{24} \tag{11}$$
If our theory is supersymmetric on the world sheet, then we will have as many bosons as fermions, so the total ground state energy is zero in the R sector.

If we have the NS sector then we get
$$-\frac{1}{2}(\frac{1}{2} + \frac{3}{2} + \ldots) \tag{12}$$
Thus we wish to compute
$$T = \frac{1}{2} + \frac{3}{2} + \ldots \tag{13}$$
Suppose we have regularized this sum just like we did for S. Then we can write
$$2T = 1 + 3 + \ldots \tag{14}$$
Adding in the even integers can be done by writing
$$2S = 2 + 4 + \ldots \tag{15}$$
Thus
$$S = 1 + 2 + 3 + 4 + \ldots = 2T + 2S \tag{16}$$
Thus
$$T = -\frac{1}{2}S = \frac{1}{24} \tag{17}$$
If we have supersymmetry on the worldsheet then we have as many bosons and fermions. One boson and one fermion would give
$$-\frac{1}{24} - \frac{1}{48} = -\frac{1}{16} \tag{18}$$
Thus the lowest state in the NS sector will be tachyonic. Let there be d spacetime dimensions transverse to the string world sheet. Then the vacuum energy of the NS sector will be
$$-\frac{d}{16} \tag{19}$$
The lowest excitation that we can make is $\frac{1}{2}$ unit above this level. Suppose we require that this be massless. Then we have
$$-\frac{d}{16} + \frac{1}{2} = 0 \tag{20}$$
which gives
$$d = 8 \tag{21}$$
Adding the two dimensions along the string worldsheet, we get the total spacetime dimension as
$$D = d + 2 = 10 \tag{22}$$

3 The 1-loop partition function

Note: The discussion below follows Polchinski chapter 10.

We have 10 bosons and 10 fermions. Because two directions are fixed along the world sheet, we have 8 transverse bosons and 8 transverse fermions. We do have the momentum from the remaining two directions however.

3.1 Bosons

Let us first write the contribution of a boson. Overall our partition function is

$$Z = V_{10} \int \frac{d^{10}k}{(2\pi)^{10}} q^{-\frac{1}{24}} \bar{q}^{-\frac{1}{24}} \sum_{H^\perp} q^{\alpha' \frac{k^2}{4} + N} \bar{q}^{\alpha' \frac{k^2}{4} + \tilde{N}} \tag{23}$$

where

$$q = e^{2\pi i \tau} \tag{24}$$

For one boson, we get

$$\int \frac{dk}{2\pi} e^{-\alpha' \pi \tau_2 k^2} = \sqrt{\frac{\pi}{\pi \tau_2 \alpha'}} \frac{1}{2\pi} = [4\pi^2 \tau_2 \alpha']^{-\frac{1}{2}} \tag{25}$$

For the oscillators we have

$$|q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}|^2 = |\eta|^{-2} \tag{26}$$

Thus overall for one boson we get

$$L[4\pi^2 \tau_2 \alpha']^{-\frac{1}{2}} |\eta|^{-2} \tag{27}$$

where L is the length of the circle on which the boson is compactified, and we are assuming that $L \to \infty$ at the end. We also have two zero mode integrals from the directons along the string worldsheet, so we get from the bosons

$$Z_X = V_{10} [4\pi^2 \tau_2 \alpha']^{-1} [|\eta|^{-2} [4\pi^2 \tau_2 \alpha']^{-\frac{1}{2}}]^8 \tag{28}$$

3.2 The theta functions

We write

$$z = e^{2\pi i \nu} \tag{29}$$

The theta functions of interest to us are

$$\theta_{00}(\nu, \tau) = \prod_{m=1}^{\infty} (1-q^m)(1+zq^{m-\frac{1}{2}})(1+z^{-1}q^{m-\frac{1}{2}}) \tag{30}$$

$$\theta_{01}(\nu,\tau) = \prod_{m=1}^{\infty}(1-q^m)(1-zq^{m-\frac{1}{2}})(1-z^{-1}q^{m-\frac{1}{2}}) \tag{31}$$

$$\theta_{10} = 2e^{\pi i\tau/4}\cos(\pi\nu)\prod_{m=1}^{\infty}(1-q^m)(1+zq^m)(1+z^{-1}q^m) \tag{32}$$

$$\theta_{11}(\nu,\tau) = -2e^{\pi i\tau/4}\sin(\pi\nu)\prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m) \tag{33}$$

Focus on the holomorphic fermions. Take 2 fermions at a time. On the σ circle, we can have NS or R sectors in the path integral. In the time direction, we also have two choices. If we have periodic fermions in τ, then we must insert $(-1)^F$. If we have antiperiodic fermions, then we insert nothing. We thus get 4 path integrals, and we can add them in some way to get a modular invariant; this will define a theory.

3.3 Periodic and antiperiodic fermions

Consider a path integral for fermions on a line τ, which is latticized to 4 points. The path integral is

$$\int dc_1 dc_2 dc_3 dc_4 e^{-i(c_1c_2+c_2c_3+c_3c_4+c_4c_1)} \tag{34}$$

where we have taken periodicity across the τ circle. There are two contributions:

$$\int dc_4 dc_3 dc_2 dc_1(-ic_1c_2)(-ic_3c_4) = (-1) \tag{35}$$

and

$$\int dc_1 dc_2 dc_3 dc_4(-ic_2c_3)(-ic_4c_1) = 1 \tag{36}$$

so that the total Z vanishes. In the Hamiltonian description, we have two states, $|0\rangle, b^\dagger|0\rangle$. We count these with $(-1)^F$, with $F=0$ for the first, and $F=1$ for the second, to get

$$Z = tr(-1)^F e^{-\tau H} = 1-1 = 0 \tag{37}$$

If the fermions were antiperiodic, then the path integral does not vanish, and we have

$$Z = tre^{-\tau H} \tag{38}$$

3.4 The fermionic oscillators

In the NS sector we have

$$\psi = \frac{1}{\sqrt{2\pi}}\sum_{r=Z+\frac{1}{2}} d_r e^{ir(\tau+\sigma)} \tag{39}$$

Thus

$$[\psi(\sigma),\psi(\sigma')]_+ = \frac{1}{2\pi}\sum_{r,s} e^{i(r+s)\tau}[d_r,d_s]_+ e^{ir\sigma+is\sigma'} \tag{40}$$

Setting

$$[d_r, d_s]_+ = \delta_{r+s,0} \tag{41}$$

we have

$$[\psi(\sigma), \psi(\sigma')]_+ = \frac{1}{2\pi}\sum_{r,s} e^{i(r+s)\tau}\delta_{r+s,0}e^{ir\sigma+is\sigma'} = \frac{1}{2\pi}\sum_{r} e^{ir(\sigma-\sigma')} \tag{42}$$

$$= e^{i\frac{1}{2}(\sigma-\sigma')}\frac{1}{2\pi}\sum_n e^{in(\sigma-\sigma')} = e^{i\frac{1}{2}(\sigma-\sigma')}\delta(\sigma-\sigma') = \delta(\sigma-\sigma') \tag{43}$$

Similarly, in the R sector

$$\psi = \frac{1}{\sqrt{2\pi}}\sum_n d_n e^{in(\tau+\sigma)} \tag{44}$$

with

$$[d_m, d_n]_+ = \delta_{m+n,0} \tag{45}$$

Note that the oscillators carry another index $\mu = 0, 1, \ldots D-1$. Thus we actually have

$$[d^\mu_m, d^\nu_n]_+ = \eta^{\mu\nu}\delta_{m+n,0} \tag{46}$$

The zero modes have the following behavior

$$[d^\mu_0, d^\nu_0]_+ = \eta^{\mu\nu} \tag{47}$$

This is like the clifford algebra for dirac matrices, and we must find its representation in a similar way. Group the fermions in pairs. Consider any two fermions, d^1, d^2, and make the combinations

$$d^+ = \frac{1}{\sqrt{2}}(d^1 + id^2), \quad d^- = \frac{1}{\sqrt{2}}(d^1 - id^2) \tag{48}$$

Thus

$$[d^+, d^+]_+ = 2(d^+)^2 = 0, \quad [d^-, d^-]_+ = 2(d^-)^2 = 0, \quad [d^+, d^-] = 1 \tag{49}$$

Thus we can use D^+, D^- as raising and lowering fermion operators. We can define a vacuum by

$$d^-_i|0\rangle = 0, \quad i = 1, 2, \ldots 5 \tag{50}$$

and use d^+_i as raising operators. Since we can apply each raising operator at most once, we will get $2^5 = 32$ states. Half of these will have an even number of d^+ applications, and half will have an odd number. These will give the two weyl components of the 32 dimensional spinor in 10-D.

3.5 Partition functions of fermions

(a) First consider the NS sector. Let the fermions be antiperiodic across the τ circle as well, so we are in NS-NS. Then we should just take a trace with no insertion of $(-1)^F$. For one fermion mode, we get

$$(1 + q^{n-\frac{1}{2}}) \tag{51}$$

For two fermions, and counting all the modes we get

$$\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})(1+q^{n-\frac{1}{2}}) \tag{52}$$

We wish to make this into a theta function. We multiply and divide by

$$\eta = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \tag{53}$$

We also had a vacuum energy $-\frac{1}{48}$ for each fermion. Thus the total vacuum energy contribution was

$$q^{-\frac{1}{48}}q^{-\frac{1}{48}} = q^{-\frac{1}{24}} \tag{54}$$

This cancels against the contribution of the η, and we get

$$\frac{1}{\eta}\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})(1+q^{n-\frac{1}{2}}) = \frac{\theta_{00}(0,\tau)}{\eta(\tau)} \equiv Z_{00}(\tau) \tag{55}$$

Overall we will have 8 transverse fermions, so 4 such pairs of fermions, so we will get the contribution

$$Z_{00}^4 \tag{56}$$

(b) Now still take the NS sector on the σ cycle, but take R in the τ direction. Thus we must now insert $(-1)^F$, which says that for each fermion mode we get

$$(1-q^{n-\frac{1}{2}}) \tag{57}$$

The rest is the same as before, and we get

$$\frac{1}{\eta}\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})(1-q^{n-\frac{1}{2}}) = \frac{\theta_{01}(0,\tau)}{\eta(\tau)} \equiv Z_{01}(\tau) \tag{58}$$

and with 8 fermions we will get

$$Z_{01}^4 \tag{59}$$

(c) Now take the R sector in the σ direction, and NS in the τ direction. Thus the vacuum energy is now zero if we consider one boson and one fermion, but this means that for each fermion we have $q^{\frac{1}{24}}$. The two fermions give $q^{\frac{1}{12}}$, and we get another $q^{\frac{1}{24}}$ from η, so overall we will have

$$q^{\frac{1}{8}} = e^{2\pi i\tau/8} = e^{i\pi\tau/4} \tag{60}$$

Also note that for the R sector we will have two ground states, since we have $|0\rangle$ and $d^+|0\rangle$. Thus we will get another factor of 2 from these zero modes. The rest of the fermionic oscillators will give

$$\prod_{n=1}^{\infty}(1+q^n)(1+q^n) \tag{61}$$

So overall we get

$$\frac{1}{\eta}2e^{i\pi\tau/4}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)(1+q^n)=\frac{\theta_{10}(0,\tau)}{\eta(\tau)}\equiv Z_{10} \tag{62}$$

and with 8 fermions we will get

$$Z_{10}^4 \tag{63}$$

(d) Now we take R in both directions. We have 2 ground states as before, but with $(-1)^F$ these give $1-1=0$. Thus the result vanishes. we can write this as

$$(\frac{\theta_{11}(0,\tau)}{\eta(\tau)})^4\equiv Z_{11}^4=0 \tag{64}$$

since $\theta_{11}(0,\tau)=0$.

4 Combining the sectors

Consider the NS sector along the σ cycle. In this sector the lowest state has energy $-\frac{1}{2}$. This is a tachyon. The next level is obtained by applying $\psi^i_{-\frac{1}{2}}$, so it is at level zero. Thus these two states differ by a half integer.

Recall that for string states we must have level matching between the left and right movers

$$L_0=\bar{L}_0 \tag{65}$$

If we kept all states on the left, and all states on the right, then we will get cases where from the oscillator levels we would get $L_0^{oscillator}-\tilde{L}_0^{oscillator}$ is a half integer. The difference needs to be made up from the unequal p_L, p_R. But recall that for noncompact directions, $p_L=p_R$, and for compact directions

$$p_L=\frac{2\pi n_p}{L}-Tn_wL, \quad p_R=\frac{2\pi n_p}{L}+Tn_wL \tag{66}$$

Thus

$$p_L^2-p_R^2=4(2\pi)\frac{1}{2\pi\alpha'}n_pn_w=\frac{4}{\alpha' n_pn_w} \tag{67}$$

We have

$$p_L^2+8\pi TN_L=p_R^2+8\pi TN_R \tag{68}$$

$$N_L-N_R=\frac{1}{8\pi T}(p_L^2-p_R^2)=\frac{2\pi\alpha'}{8\pi}\frac{4}{\alpha'}n_pn_w=n_pn_w \tag{69}$$

which is an integer. Thus we *cannot* have arbitrary matches of left and right oscillator levels since half integer differences are not allowed for N_L-N_R.

Thus we must separate out the odd and even levels. Note that only the fermion has half integer modings. Suppose we say that each time we apply a fermion we change the fermion number by unity. Let the vacuum of the NS sector have fermion number 1. The fermion number of defined only mod 2. The next level above the vacuum will be

$$\psi^i_{-\frac{1}{2}}|0\rangle_{NS} \tag{70}$$

with $m^2 = 0$. For the open string we have just one sector, so we see that we get 8 massless quanta. In 10-D, these give the physical degrees of freedom of a photon. We might be confined to a smaller dimension by having the ends of the open string lie on a D-brane. Suppose we have a p-brane. Then the directions i normal to the brane will give transverse vibrations of the brane, while those along the brane will give a gauge field. Thus we learn that D-branes can vibrate, and that they carry a gauge field on their worldvolume.

Returning to the string, we can separate the off and even levels by using the projection operators

$$P_{\pm} = \frac{1}{2}(1 \pm (-1)F) \tag{71}$$

What states should we keep? The bosonic states of the string will come from the NS-NS and RR sectors. The fermions will come from the RR sectors. The RR sector has integer level states. To match onto the other sector, we will have to keep the integer level states from the NS sector. Thus we discard the odd level states by using the projection operator

$$\frac{1}{2}(1 - (-1)^F) \tag{72}$$

In particular this removes the tachyon from the spectrum. It also tells us that we should combine the first two terms in the path integral (a) and (b) as (a)-(b).

In the R sector, we keep one of the two chiralities of the fermion. Each time we apply a d_i^+, we go from odd to even fermion number, but we also change chiralities, So if we are to keep even or odd fermion number, then we have to keep one chirality of the spinor from the ground states. For the left and right sectors, we can either keep the same chirality or opposite chiralities, In the first case we get IIB string theory, and in the second case we get IIA string theory. Thus we should use

$$\frac{1}{2}(1 \pm (-1)^F) \tag{73}$$

in the R sector as well.

There is one last sign that we must understand. In field theory, if we have a fermion loop, then we have to include an extra minus sign. We do this by assigning one sector – the R sector - a minus sign. Then for spacetime bosons which are NSNS or RR there is no sign, while for NSR and RNS we will have a minus sign.

Thus overall we see that we must write

$$\frac{1}{2}[Z_{00}^4 - Z_{01}^4 - Z_{10}^4 \mp Z_{11}^4] \tag{74}$$

The last term is zero, and the first three vanish because of the Jacobi abstruse identity

$$\theta_{00}^4 - \theta_{01}^4 - \theta_{10}^4 \tag{75}$$

This vanishing tells us that we have no vacuum energy for the superstring in flat space.

5 Excitations of a D-brane

We have seen that an open string can end on a D-brane. An open string has only one sector, not L R sectors. The NS sector will have a GSO projection, and so the tachyon state will be removed. The Next level is made from

$$\psi^i_{-\frac{1}{2}}|p\rangle \tag{76}$$

We have

$$m^2 = -p^2 = \frac{1}{\alpha'}(N - \frac{1}{2}) = 0 \tag{77}$$

so we have massless excitations. Thus we have 8 transverse massless modes. The directions along the brane give a gauge field A_a, while those normal to the D-brane give transverse vibrations.

The R sector gives a spacetime fermion, which gives 8 superpartners of the bosonic degrees of freedom. We have a 16 comonent spinor to start with, but the equation of motion cuts these down by half to 8.

6 Gauge fields

Now consider the closed string. From the bosonic excitations, NSNS sector, we have

$$\psi^i_{-\frac{1}{2}}\bar{\psi}^j_{-\frac{1}{2}}|0\rangle \tag{78}$$

The transverse traceless part gives a graviton, the antisymmetric part gives the B_{ij}, and the trace gives the dilaton.

In the RR sector we have

$$|0\rangle_\alpha|0\rangle_\beta \tag{79}$$

for IIB, and

$$|0\rangle_\alpha|0\rangle_{\dot{\beta}} \tag{80}$$

for IIA.

We can classify these states by inserting gamma matrices and making linear combinations. For IIB we have

$$C_0 : \quad |0\rangle_\alpha|0\rangle^\alpha \tag{81}$$

$$C_2^{\mu\nu} : \quad |0\rangle_\alpha[\gamma^\mu\gamma^\nu]^\alpha{}_\beta|0\rangle^\beta \tag{82}$$

etc.

For IIA we have

$$C_1^\mu : \quad |0\rangle_\alpha\gamma^\alpha{}_{\dot{\beta}}|0\rangle^{\dot{\beta}} \tag{83}$$

$$C_3^{\mu\nu\lambda} : \quad |0\rangle_\alpha[\gamma^\mu\gamma^\nu\gamma^\lambda]^\alpha{}_{\dot{\beta}}|0\rangle^{\dot{\beta}} \tag{84}$$

etc. This gives the RR fields for the theory.

The NSNS $B_{\mu\nu}$ field is produced by the elementary string, but the RR fields are produced by D-branes.

Closed string theory is 'complete' in the sense that all the fundamental particle states are such that they give all the background fields that the string can propagate in.

7 T-duality

The oscillator expansion of the bosonic field X^μ was

$$\begin{aligned} X^\mu(\tau,\sigma) &= [\frac{1}{2}x_0^\mu + \alpha' p_L^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_n \frac{\alpha_n}{n}e^{in(\tau+\sigma)}] \\ &= [\frac{1}{2}x_0^\mu + \alpha' p_R^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_n \frac{\bar\alpha_n}{n}e^{in(\tau-\sigma)}] \end{aligned} \tag{85}$$

We can write this as

$$X^\mu = X^\mu(\xi^+) + X^\mu(\xi^-) \tag{86}$$

Suppose we consider

$$X'^\mu = X^\mu(\xi^+) - X^\mu(\xi^-) \tag{87}$$

This will also be a solution to the equation of motion. What is the significance of this new solution?

Suppose we perform the following change of variables on the world sheet

$$\partial_a X' = \epsilon_{ab}\partial^b X \tag{88}$$

Then

$$\partial_\tau X' = \partial_\sigma X \tag{89}$$

$$\partial_\sigma X' = \partial_\tau X \tag{90}$$

where we have used that τ has negative signature, and $\epsilon_{\tau\sigma} = 1$. Thus

$$(\partial_\tau + \partial_\sigma)X' = (\partial_\tau + \partial_\sigma)X \tag{91}$$

$$(\partial_\tau - \partial_\sigma)X' = -(\partial_\tau - \partial_\sigma)X \tag{92}$$

and we achieve the change mentioned above.

7.1 Open strings

First let us see the effect of the change $X \to X'$ on open strings. Suppose the endpoint of the open string has N boundary condition

$$\partial_\sigma X(\sigma = 0) = 0 \tag{93}$$

This gives

$$\partial_\tau X'(\sigma = 0) = 0 \tag{94}$$

which gives

$$X'(\sigma = 0) = x_0 = constant \tag{95}$$

with a similar behavior at $\sigma = \pi$. Thus we get fixed endpoints, which is a D boundary condition. Thus under this change $X \to X'$ we go from N to D boundary conditions and vice versa.

7.2 Closed strings

We had

$$X^\mu = x_0^\mu + \alpha' p^\mu \tau + w^\mu \sigma + oscillators \tag{96}$$

with

$$p^\mu = p_L^\mu + p_R^\mu \tag{97}$$

$$w^\mu = \alpha'(p_L^\mu - p_R^\mu) \tag{98}$$

Under the change $X^\mu \to X'^\mu$ we get

$$p'^\mu = p_L^\mu - p_R^\mu = \frac{1}{\alpha'} w^\mu \tag{99}$$

$$w'^\mu = \alpha'(p_L^\mu + p_R^\mu) = \alpha' p^\mu \tag{100}$$

The old winding implied a distance between endpoints

$$L \equiv 2\pi R = 2\pi w^\mu = 2\pi\alpha'(p_L^\mu - p_R^\mu) \tag{101}$$

The new momentum must be correctly quantized

$$p'^\mu = \frac{1}{\alpha'} w^\mu = \frac{2\pi}{L'} = \frac{1}{R'} \tag{102}$$

Thus

$$\frac{R}{\alpha'} = \frac{1}{R'} \tag{103}$$

or

$$R' = \frac{\alpha'}{R} \tag{104}$$

This is the basic T-duality relation which relates large circles to small circles, while interchanging winding and momentum.

8 Action of T-duality on fermions

Let X^9 be the direction that is T-dualized. Then for the right movers we have taken

$$X_R'^9 = -X_R^9 \tag{105}$$

To preserve supersymmetry, we must also take

$$\psi_R'^9 = -\psi_R^9 \tag{106}$$

This means that in the R sector, when we make the combinations $\psi_R^8 \pm i\psi_R^9$, we will have

$$\psi_R'^8 \pm i\psi_R'^9 = \psi_R^8 \mp i\psi_R^9 \tag{107}$$

Thus the creation and annihilation gamma matrices have been interchanged. Thus the vacuum has been changed from $0\rangle$ to $\Gamma_{(5)}^{+}|0\rangle$. This implies a change of chirality. Thus the right vacuum has switched chirality, while the left vacuum has remained unchanged. Thus we go from IIB to IIA and vice versa.

9 The Weyl-Petersen measure

We will show that the measure

$$\frac{d^2\tau}{\tau_2^2} \tag{108}$$

is modular invariant. We have

$$\tau' = -\frac{1}{\tau} \tag{109}$$

Thus

$$\tau_1' + i\tau_2' = -\frac{1}{\tau_1 + i\tau_2} = -\frac{\tau_1 - i\tau_2}{\tau_1^2 + \tau_2^2} \equiv -\frac{\tau_1 - i\tau_2}{Q} \tag{110}$$

Thus

$$\tau_1' = -\frac{\tau_1}{Q} \tag{111}$$

$$\tau_2' = \frac{\tau_2}{Q} \tag{112}$$

Thus

$$\frac{\partial\tau_1'}{\partial\tau_1} = -\frac{1}{Q} + \frac{2\tau_1^2}{Q^2}, \quad \frac{\partial\tau_1'}{\partial\tau_2} = -\frac{2\tau_1\tau_2}{Q^2} \tag{113}$$

$$\frac{\partial\tau_2'}{\partial\tau_1'} = -\frac{2\tau_1\tau_2}{Q^2}, \quad \frac{\partial\tau_2'}{\partial\tau_2} = \frac{1}{Q} - \frac{2\tau_2^2}{Q^2} \tag{114}$$

We find that

$$\frac{\partial(\tau_1', \tau_2')}{\partial(\tau_1, \tau_2)} = \frac{1}{Q^2} \tag{115}$$

Thus

$$\frac{d\tau_1' d\tau_2'}{\tau_2'^2} = \frac{d\tau_1 d\tau_2}{\tau_2^2} \tag{116}$$

10 The overall partition function

The overall partition function will have

$$Z = \int \frac{d^2\tau}{\tau_2} V_{10}[4\pi^2\tau_2\alpha']^{-1}[|\eta|^{-2}[4\pi^2\tau_2\alpha']^{-\frac{1}{2}}]^8 Z_\psi Z_\psi^* \tag{117}$$

11 Symmetries of the IIb string

We look at the perturbative symmetries, which come from symmetries of the world sheet action. We have

$$\Omega, \quad (-1)^{F_L}, \quad (-1)^{F_R} \tag{118}$$

Each of these squares to unity. But

$$\Omega(-1)^{F_L}\Omega = (-1)^{F_R} \tag{119}$$

These make an 8 parameter group

$$1, \Omega, (-1)^{F_L}, (-1)^{F_R}, \Omega(-1)^{F_L}, \Omega(-1)^{F_R}, (-1)^{F_L}(-1)^{F_R}, \Omega(-1)^{F_L}(-1)^{F_R} \tag{120}$$

where we can either apply or not apply each of the three elements. these map to D_4, the dihedral group with 4 elements, which are the symmetries of the square. These symmetries are

$$1, R_x, R_y, R_xR_y, S, S^3, R_xS, R_yS \tag{121}$$

where R_x is reflection in the x axis, etc, and S is rotation by a right angle. Note that $R_xR_y = S^2$, and we can check that $SR_xS = R_x$.

We can make the map

$$\Omega = R_x, \quad \Omega(-1)^{F_L} = S \tag{122}$$

so that

$$(-1)^{F_L} = R_xS \tag{123}$$

Note that

$$(R_xS)^2 = R_xSR_xS = R_xR_x = 1 \tag{124}$$

12 Nonperturbative symmetries

The IIB theory also has a SL(2, Z) nonperturbative symmetry

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \tag{125}$$

Define

$$\lambda = C_0 + ie^{-\phi} \tag{126}$$

The theory has two 2-forms

$$B_{ij}, \quad C_{ij} \tag{127}$$

and a 4-form

$$C_{ijkl} \tag{128}$$

Under this symmetry we have

$$\lambda' = \frac{a\lambda + b}{c\lambda + d} \tag{129}$$

$$\begin{pmatrix} B'_{ij} \\ C'_{ij} \end{pmatrix} = M^{-1T} \begin{pmatrix} B_{ij} \\ C_{ij} \end{pmatrix} \tag{130}$$

$$C'_{ijkl} = C_{ijkl} \tag{131}$$

13 Relation between the perturbative and nonperturbative symmetries

13.1 The element R

Consider the element of SL(2, Z)

$$R = -I \tag{132}$$

Then B_{ij}, C_{ij} change sign, while λ, C_{ijkl} remain fixed. This is in fact the behavior of $(-1)^{F_L}\Omega$. Under Ω,

$$g_{ij}, \phi, C_{ij} \tag{133}$$

are fixed, while

$$C_0, B_{ij}, C_{ijkl} \tag{134}$$

change sign. To see this, note that for the NSNS fields, we interchange left and right movers, while for the RR fields, we interchange the two spinors coming from left and right. In C_0, we have the contraction

$$\eta_{\alpha\beta}\psi_L^\alpha\psi_R^\beta \tag{135}$$

where η is antisymmetric. Thus C_0 is odd. For C_{ij} we get an extra negative sign from the permutation of order of the two gamma matrices, while in C_{ijkl} the reversal of order is actually an even permutation so because of η it is odd.

Under $(-1)^{F_L}$, the NS sector is unchanged, while the R sector changes sign. Thus RR states are odd, and all C fields change sign. Thus with $(-1)^{F_L}\Omega$ we find that

$$B_{ij}, C_{ij} \tag{136}$$

change sign, while the other fields are fixed. Thus this is R.

13.2 The element S

Consider the SL(2, Z) transformation

$$S = i\sigma_2 \tag{137}$$

Then we have

$$S(-1)^{F_L}S^{-1} = \Omega \tag{138}$$

To check this, note that S changes B_{ij} to C_{ij}, which changes sign under $(-1)^{F_L}$, and then we change back to B_{ij}. Thus B_{ij} changes sign, which is a property of Ω. Similarly we can check the other elements.