Physics 5300, Theoretical Mechanics Spring 2015

Assignment 1

Given: Tue, Jan 13, Due Tue Jan 20

The problems numbers below are from Classical Mechanics, John R. Taylor, University Science Books (2005).

Problem 1 Taylor 5.5

Solution: We start with the form

$$I: \quad x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} \tag{1}$$

We have

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t) \tag{2}$$

$$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t) \tag{3}$$

Thus we get

$$\begin{aligned} x(t) &= C_1[cos(\omega t) + i\sin(\omega t)] + C_2[cos(\omega t) - i\sin(\omega t)] \\ &= (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t) \\ &\equiv B_1\cos(\omega t) + B_2\sin(\omega t) \end{aligned} \tag{4}$$

Thus we have

$$II: \quad B_1 = (C_1 + C_2), \quad B_2 = i(C_1 - C_2) \tag{5}$$

Now consider the expression

$$III: \quad x(t) = A\cos(\omega t - \delta) \tag{6}$$

This gives

$$x(t) = A\cos(\omega t)\cos\delta + A\sin(\omega t)\sin\delta \tag{7}$$

Comparing to II, we have

$$A\cos\delta = B_1, \quad A\sin\delta = B_2 \tag{8}$$

Thus we have

$$A^2(\cos^2\delta + \sin^2\delta) = A^2 = B_1^2 + B_2^2 \tag{9}$$

which gives

$$A = \sqrt{B_1^2 + B_2^2} \tag{10}$$

and

$$\tan\delta = \frac{B_2}{B_1}, \quad \delta = \tan^{-1}\frac{B_2}{B_1} \tag{11}$$

Now consider the expression

$$IV: \quad x(t) = Re[Ce^{i\omega t}] \tag{12}$$

We write

$$C = |C|e^{i\phi} \tag{13}$$

Then we have

$$x(t) = Re[|C|e^{i(\omega t+\phi)}] = |C|Re[e^{i(\omega t+\phi)}] = |C|\cos(\omega t+\phi) \tag{14}$$

Comparing to III, we have

$$|C| = A, \quad \phi = -\delta \tag{15}$$

Now start with

$$x(t) = Re[Ce^{i\omega t}] = \frac{1}{2}[Ce^{i\omega t} + C^* e^{-i\omega t}] \tag{16}$$

This is of the form I with

$$C_1 = \frac{1}{2}C, \quad C_2 = \frac{1}{2}C^* \tag{17}$$

Problem 2 Taylor 5.7

Solution: (a) We have

$$x(t) = B_1\cos(\omega t) + B_2\sin(\omega t) \tag{18}$$

$$\dot{x}(t) = -B_1\omega\sin(\omega t) + B_2\omega\cos(\omega t) \tag{19}$$

Thus

$$x(0) = x_0 = B_1, \quad \dot{x}(0) = v_0 = B_2\omega \tag{20}$$

Thus

$$B_1 = x_0, \quad B_2 = \frac{v_0}{\omega} \tag{21}$$

(b) We have

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{.5}} = 10 \tag{22}$$

$$B_1 = x_0 = 3 \tag{23}$$

$$B_2 = \frac{v_0}{\omega} = \frac{50}{10} = 5 \tag{24}$$

(c) We have

$$x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t) = 3\cos(\omega t) + 5\sin(\omega t) \tag{25}$$

To get $x(t) = 0$ we need

$$3\cos(\omega t) + 5\sin(\omega t) = 0, \quad \tan(\omega t) = -\frac{3}{5} \tag{26}$$

We have

$$-\tan^{-1}\frac{3}{5} = -.54, -.54 + \pi = 2.6, \ldots \tag{27}$$

Thus the earliest positive time when $x(t) = 0$ is

$$t = \frac{2.6}{\omega} = \frac{2.6}{10} = .26\,s \tag{28}$$

Similarly, we have

$$\dot{x}(t) = -B_1\omega \sin(\omega t) + B_2\omega \cos(\omega t) = -30\sin(\omega t) + 50\cos(\omega t) \tag{29}$$

We get $\dot{x}(t) = 0$ when

$$\tan(\omega t) = \frac{5}{3} \tag{30}$$

This gives

$$\omega t = 1.03 \tag{31}$$

and therefore

$$t = \frac{1.03}{\omega} = .103 \tag{32}$$

Problem 3   Taylor 5.22

Solution: (a) The equation for oscillations is

$$\ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = 0 \tag{33}$$

The solution is

$$x(t) = C_1 e^{-\omega_0 t} + C_2 t e^{-\omega_0 t} \tag{34}$$

We have

$$x(0) = 0 \tag{35}$$

which gives

$$C_1 e^{-\omega_0 t} + C_2 t e^{-\omega_0 t} C_1 = C_1 = 0 \tag{36}$$

The velocity is

$$\dot{x} = -\omega_0 C_1 e^{-\omega_0 t} - C_2\omega_0 t e^{-\omega_0 t} + C_2 e^{-\omega_0 t} \tag{37}$$

Setting this to v_0 at $t = 0$ we get

$$C_2 = v_0 \tag{38}$$

Thus we have

$$x(t) = v_0 t e^{-\omega_0 t} \tag{39}$$

(b) Now we have

$$x(0) = C_1 = x_0 \tag{40}$$

$$\dot{x}(0) = -\omega_0 C_1 + C_2 = 0, \quad C_2 = \omega_0 C_1 \tag{41}$$

Thus

$$x(t) = x_0 e^{-\omega_0 t} + \omega_0 x_0 t e^{-\omega_0 t} \tag{42}$$

At $t = \frac{2\pi}{\omega_0}$, we have

$$x = x_0 e^{-2\pi} + 2\pi x_0 e^{-2\pi} = x_0(1 + 2\pi)e^{-2\pi} \tag{43}$$

Problem 4 Taylor 5.33

Solution: We have

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)] \tag{44}$$

Thus

$$\dot{x}(t) = -A\omega \sin(\omega t - \delta) - \beta e^{-\beta t}[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)] + e^{-\beta t}[-B_1 \omega_1 \sin(\omega_1 t) + B_2 \omega_1 \cos(\omega_1 t)] \tag{45}$$

Thus we have

$$x_0 = A\cos(\delta) + B_1 \tag{46}$$

$$v_0 = -A\omega \sin(\delta) - \beta B_1 + B_2 \omega_1 \tag{47}$$

Thus

$$B_1 = x_0 - A\cos\delta \tag{48}$$

$$B_2 = \frac{1}{\omega_1}[v_0 + A\omega \sin\delta + \beta(x_0 - A\cos\delta)] \tag{49}$$

Problem 5 Taylor 5.41

Solution: We have

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \tag{50}$$

The peak of A^2 occurs at $\omega = \omega_0$, which gives

$$A^2 = \frac{f_0^2}{4\beta^2\omega^2} \approx \frac{f_0^2}{4\beta^2\omega_0^2} \tag{51}$$

Thus A^2 has half this maximum value when

$$\frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \approx \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega_0^2} = \frac{f_0^2}{8\beta^2\omega_0^2} \tag{52}$$

This gives

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega_0^2 = 8\beta^2\omega_0^2 \tag{53}$$

$$(\omega_0^2 - \omega^2)^2 = 4\beta^2\omega_0^2 \tag{54}$$

$$\omega_0^2 - \omega^2 = \pm 2\beta\omega_0 \tag{55}$$

We can write this as

$$(\omega + \omega_0)(\omega - \omega_0) \approx 2\omega_0(\omega - \omega_0) = \pm 2\beta\omega_0 \tag{56}$$

which gives

$$\omega - \omega_0 = \pm\beta \tag{57}$$

Thus $\omega = \omega_0 \pm \beta$, and the full width at half-maximum is

$$(\omega_0 + \beta) - (\omega_0 - \beta) = 2\beta \tag{58}$$

Problem 6 Taylor 5.47

Solution: (i) We have

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)] \tag{59}$$

Thus

$$\cos(n\omega t)\cos(m\omega t) = \frac{1}{2}[\cos((n+m)\omega t) + \cos((n-m)\omega t)] \tag{60}$$

We get

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \cos(n\omega t)\cos(m\omega t) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \frac{1}{2}[\cos((n+m)\omega t) + \cos((n-m)\omega t)] \tag{61}$$

If $n = 0, m = 0$, then we get

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \frac{1}{2}[\cos((n+m)\omega t) + \cos((n-m)\omega t)] = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt = \tau \tag{62}$$

If $n = m \neq 0$, then we have

$$\begin{aligned}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \cos(n\omega t)\cos(m\omega t) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \frac{1}{2}[\cos((2n)\omega t) + 1] \\ &= \frac{1}{2}[\frac{1}{(2n\omega)}\sin((2n)\omega t)]|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} + \frac{1}{2}\tau \\ &= \tau\end{aligned} \tag{63}$$

If $n \neq m$, then $n - m \neq 0$, and $n + m \neq 0$. (The latter statement is true because n, m are positive numbers.) Thus we get

$$\begin{aligned}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \cos(n\omega t)\cos(m\omega t) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \frac{1}{2}[\cos((n+m)\omega t) + \cos((n-m)\omega t)] \\ &= \frac{1}{2}[\frac{1}{(m+n)\omega}\sin((n+m)\omega t) + \frac{1}{(m-n)\omega}\sin((n-m)\omega t)]|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\ &= 0\end{aligned} \tag{64}$$

where in the last step we have used the fact that

$$\omega\frac{\tau}{2} = \pi \tag{65}$$

(ii) We have

$$\sin A \sin B = \frac{1}{2}[-\cos(A+B) + \cos(A-B)] \tag{66}$$

Thus

$$\sin(n\omega t)\sin(m\omega t) = \frac{1}{2}[-\cos((n+m)\omega t) + \cos((n-m)\omega t)] \tag{67}$$

We get

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \sin(n\omega t)\sin(m\omega t) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \frac{1}{2}[-\cos((n+m)\omega t) + \cos((n-m)\omega t)] \tag{68}$$

If either $n = 0$ or $m = 0$, then the integrand vanishes. Thus we must have $n > 0, m > 0$. If $n \neq m$ then we get

$$\begin{aligned}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \sin(n\omega t)\sin(m\omega t) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \frac{1}{2}[-\cos((n+m)\omega t) + \cos((n-m)\omega t)] \\ &= \frac{1}{2}[-\frac{1}{(m+n)\omega}\sin((n+m)\omega t) + \frac{1}{(m-n)\omega}\sin((n-m)\omega t)]|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\ &= 0\end{aligned} \tag{69}$$

where in the last step we have used the fact that

$$\omega\frac{\tau}{2} = \pi \tag{70}$$

If $n = m$, then we get

$$\begin{aligned}
\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \sin(n\omega t)\sin(n\omega t) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt\frac{1}{2}[-\cos(2n\omega t) + 1] \\
&= \frac{1}{2}[-\frac{1}{(2n)\omega}\sin((2n)\omega t)]|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} + \frac{1}{2}\tau \\
&= \frac{\tau}{2}
\end{aligned} \tag{71}$$

(iii) We have

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \tag{72}$$

Thus

$$\cos(n\omega t)\sin(m\omega t) = \frac{1}{2}[\sin((n+m)\omega t) - \sin((n-m)\omega t)] \tag{73}$$

We get

$$\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \cos(n\omega t)\sin(m\omega t) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt\frac{1}{2}[\sin((n+m)\omega t) - \sin((n-m)\omega t)] \tag{74}$$

If either $m = 0$, then the integrand vanishes. Thus we must have $m > 0$. If $n \neq m$ then we get

$$\begin{aligned}
\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \cos(n\omega t)\sin(m\omega t) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt\frac{1}{2}[\sin((n+m)\omega t) - \sin((n-m)\omega t)] \\
&= \frac{1}{2}[-\frac{1}{(m+n)\omega}\cos((n+m)\omega t) + \frac{1}{(m-n)\omega}\cos((n-m)\omega t)]|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= 0
\end{aligned} \tag{75}$$

where in the last step we have used the fact that for any number p

$$\cos(p\omega\frac{\tau}{2}) = \cos(p\omega(-\frac{\tau}{2})) \tag{76}$$

If $m = n$, then we have

$$\begin{aligned}
\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \cos(n\omega t) \sin(n\omega t) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dt \frac{1}{2}[\sin((2n)\omega t) - 0] \\
&= \frac{1}{2}[-\frac{1}{(2n)\omega} \cos((2n)\omega t)]|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= 0
\end{aligned} \tag{77}$$