Physics 5300, Theoretical Mechanics   Spring 2015

Assignment 2 solutions

Given: Tue, Jan 20,   Due Tue Jan 27

The problems numbers below are from Classical Mechanics, John R. Taylor, University Science Books (2005).

Problem 1   Taylor 6.1

Solution: Consider a sphere of radius R. The distance between points (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ is given by

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Consider the path to be given by a function $\phi(\theta)$. Then we have

$$ds = R\sqrt{(d\theta^2 + \sin^2\theta d\phi^2)} = R\sqrt{1 + \sin^2\theta\phi'^2}\, d\theta$$

Thus the total length is

$$L = \int_i^f ds = R\int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta\phi'^2}\, d\theta$$

Problem 2   Taylor 6.2

Solution: The cylinder has all its points at $\rho = R$. Thus the points of interest are given by (R, ϕ, z) and $(R, \phi + d\phi, z + dz)$. The distance is given by

$$ds^2 = R^2 d\phi^2 + dz^2$$

Writing the path as $\phi(z)$, we get,

$$ds = \sqrt{R^2\phi'^2 + 1}\, dz$$

Thus the total length is

$$L = \int_i^f ds = \int_{\theta_1}^{\theta_2} \sqrt{R^2\phi'^2 + 1}\, dz$$

Problem 3   Taylor 6.7

Solution: The quantity to be minimized is

$$L = \int_i^f ds = \int_{z_1}^{z_2} \sqrt{R^2\phi'^2 + 1} dz$$

The equation is then

$$\frac{d}{dz}[\frac{R\phi'}{\sqrt{\phi'^2 + 1}}] = 0$$

Thus

$$\frac{R\phi'}{\sqrt{R^2\phi'^2 + 1}} = const$$

Thus

$$\phi' = const$$

Thus

$$\phi = az + b$$

Laid out on a flat $y - z$ plane, the y coordinate is $R\phi$ and the z coordinate is z itself. Thus we expect a straight line in the $y - z$ plane, and this is what we have obtained from the variational equation.

Problem 4   Taylor 6.9

Solution: We wish to miniize

$$L = \int_O^P (y'^2 + yy' + y^2) dx$$

The variational equation is

$$\frac{d}{dx}[(2y' + y)] - (y' + 2y) = 0$$

This is

$$2y'' - 2y = 0$$

or

$$y'' = y$$

The solution has the form

$$y = Ae^x + Be^{-x}$$

At the origin O we have

$$0 = A + B$$

which implies $A = -B$. At P we have

$$1 = Ae + Be^{-1}$$

Thus

$$B(-e + e^{-1}) = 1, \quad B = \frac{1}{-e + e^{-1}} = -\frac{e}{e^2 - 1}$$

Thus we get

$$y = \frac{e}{e^2 - 1}(e^x - e^{-x})$$

Problem 5 Taylor 6.11

Solution: The quantity to be minimized has the form $\int_i^f L dx$, where

$$L = \sqrt{x}\sqrt{1 + y'^2}$$

The variational equation is

$$\frac{d}{dx}[\frac{\sqrt{x}\, y'}{\sqrt{1 + y'^2}}] = 0$$

Thus

$$\frac{\sqrt{x}\, y'}{\sqrt{1 + y'^2}} = C$$

$$\frac{xy'^2}{1 + y'^2} = C^2$$

$$xy'^2 = C^2(1 + y'^2)$$

$$y'^2(x - C^2) = C^2$$

$$y' = \frac{C}{\sqrt{x - C^2}}$$

$$y = 2C\sqrt{x - C^2} + D$$

Problem 6 Taylor 6.12

Solution: The quantity to be minimized has the form $\int_i^f L dx$, where

$$L = x\sqrt{1 - y'^2}$$

The variational equation is

$$\frac{d}{dx}[\frac{x\,y'}{\sqrt{1-y'^2}}]=0$$

Thus

$$\frac{x\,y'}{\sqrt{1-y'^2}}=C$$

$$\frac{x^2y'^2}{1-y'^2}=C^2$$

$$x^2y'^2=C^2(1-y'^2)$$

$$y'^2(x^2+C^2)=C^2$$

$$y'=\frac{C}{\sqrt{x^2+C^2}}$$

$$y=\int\frac{C}{\sqrt{x^2+C^2}}dx$$

To solve this equation, we write

$$x=C\sinh\mu$$

Then

$$dx=C\cosh\mu\,d\mu$$

$$y=\int C\cosh\mu\,d\mu\frac{C}{C\cosh\mu}=C\int d\mu=C\mu+D=C\sinh^{-1}\frac{x}{C}+D$$