Physics 5300, Theoretical Mechanics Spring 2015

Assignment 3 solutions

Given: Wed, Jan 28,   Due Tue Feb 3

The problems numbers below are from Classical Mechanics, John R. Taylor, University Science Books (2005).

Problem 1 Taylor 7.1

Solution: We have

$$L = T - V \tag{1}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{2}$$

$$V = mgz \tag{3}$$

Thus

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \tag{4}$$

The Lagrange equations are

$$x: \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0, \quad m\ddot{x} = 0 \tag{5}$$

$$y: \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y} = 0, \quad m\ddot{y} = 0 \tag{6}$$

$$z: \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{z}}) - \frac{\partial L}{\partial z} = 0, \quad m\ddot{z} + mg = 0 \tag{7}$$

These are as expected since the x and y directons feel no gorce, while the z moton gives $m\ddot{z} = -mg$ so the projectile feels a downward force mg.

Problem 2 Taylor 7.2

Solution: The force $F = -kx$ comes from a potential

$$V = \frac{1}{2}kx^2 \tag{8}$$

since we have

$$F = -\frac{d}{dx}V \tag{9}$$

Then we get

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{10}$$

The Lagrange equation is

$$x: \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0, \quad m\ddot{x} + kx = 0 \tag{11}$$

The solution is

$$x = A\cos(\sqrt{\frac{k}{m}}t + \phi) \tag{12}$$

Problem 3 Taylor 7.3

Solution: We have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) \tag{13}$$

The Lagrange equations are

$$x: \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0, \quad m\ddot{x} + kx = 0 \tag{14}$$

$$y: \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y} = 0, \quad m\ddot{y} + ky = 0 \tag{15}$$

Problem 4 Taylor 7.10

Solution: The coordinates are ρ, ϕ. The half angle is α, and the cone points down. Consider any $z < 0$. Then we have

$$\tan\alpha = \frac{\rho}{(-z)} \tag{16}$$

Thus

$$z = -\frac{\rho}{\tan\alpha} \tag{17}$$

We also have

$$x = \rho\cos\phi \tag{18}$$

$$y = \rho\sin\phi \tag{19}$$

Problem 5 Taylor 7.17

Solution:
We have

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}I\omega^2 \tag{20}$$

$$V = m_1gy_1 + m_2gy_2 \tag{21}$$

We have the constraint

$$y_1 + y_2 = Y = constant \tag{22}$$

We solve this constraint. Thus

$$\dot{y}_1 = \dot{y}_2 \tag{23}$$

We also have

$$\omega = \frac{v}{R} = \frac{\dot{y}_1}{R} \tag{24}$$

where R is the radius of the pulley. Thus we have

$$T = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})\dot{y}_1^2 \tag{25}$$

$$V = (m_1 - m_2)gy_1 + m_2gY \tag{26}$$

$$L = T - V = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})\dot{y}_1^2 - (m_1 - m_2)gy_1 - m_2gY \tag{27}$$

The Lagranges equation is

$$y_1: \quad \frac{d}{dt}(\frac{\partial L}{\partial \dot{y}_1}) - \frac{\partial L}{\partial y_1} = 0 \tag{28}$$

$$(m_1 + m_2 + \frac{I}{R^2})\ddot{y}_1 - [-(m_1 - m_2)g] = 0 \tag{29}$$

$$\ddot{y}_1 = -\frac{(m_1 - m_2)g}{(m_1 + m_2 + \frac{I}{R^2})} \tag{30}$$

The solution is

$$y_1 = y_1^0 + v_1^0t - \frac{1}{2}\frac{(m_1 - m_2)g}{(m_1 + m_2 + \frac{I}{R^2})}t^2 \tag{31}$$

Problem 6 Taylor 7.20

Solution: We have

$$\dot{z} = \lambda\dot{\phi} \tag{32}$$

Thus

$$T = \frac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2) = \frac{1}{2}m(\dot{z}^2 + \frac{R^2}{\lambda^2}\dot{z}^2) = \frac{1}{2}m(1 + \frac{R^2}{\lambda^2})\dot{z}^2 \tag{33}$$

$$V = mgz \tag{34}$$

Thus

$$L = \frac{1}{2}m(1 + \frac{R^2}{\lambda^2})\dot{z}^2 - mgz \tag{35}$$

The Lagrange equation is

$$z: \quad m(1 + \frac{R^2}{\lambda^2})\ddot{z} - (-mg) = 0 \tag{36}$$

$$\ddot{z} = -\frac{g}{(1 + \frac{R^2}{\lambda^2})} \tag{37}$$

In the limit $R \to 0$ we get

$$\ddot{z} = -g \tag{38}$$

which makes sense since in this case we just have a vertical wire, and the bead will slide straight down with acceleration $-g$.