Physics 5300, Theoretical Mechanics Spring 2015

Assignment 5 solutions

The problems numbers below are from Classical Mechanics, John R. Taylor, University Science Books (2005).

Problem 1 Taylor 10.10

Solution: (a)

$$I = \frac{M}{L}\int_{x=0}^{L} dx x^2 = \frac{M}{L}\frac{L^3}{3} = \frac{ML^2}{3} \tag{1}$$

(b)

$$I = 2\frac{M}{L}\int_{x=0}^{\frac{L}{2}} x^2 dx = \frac{2M}{L}\frac{L^3}{24} = \frac{ML^2}{12} \tag{2}$$

Problem 2 Taylor 10.12

Solution: We first compute I for rotation about an axis parallel to the z axis, passing through one corner. We will then use the parallel axis theorem to get I around an axis passing through the centroid.

Let the length of the prism be L. The area of each side is $2a$. The height is

$$h = \sqrt{4a^2 - a^2} = \sqrt{3}a \tag{3}$$

The area is

$$A = \frac{1}{2}(2a)\sqrt{3}a = \sqrt{3}a^2 \tag{4}$$

The volume is then

$$V = \sqrt{3}a^2 L \tag{5}$$

and the density is

$$\rho = \frac{M}{V} = \frac{M}{\sqrt{3}a^2 L} \tag{6}$$

Let the y axis be along the perpendicular of the triangle, with $y = 0$ being the vertex of the triangle. The range of y is

$$0 \le y \le \sqrt{3}a \tag{7}$$

At any given value of y, the range of x is

$$-\frac{1}{\sqrt{3}}y \leq x \leq \frac{1}{\sqrt{3}}y \tag{8}$$

The distance squared from the z axis is $x^2 + y^2$. Thus

$$I = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy \int_{x=0}^{\frac{1}{\sqrt{3}}y} dx(x^2 + y^2) \tag{9}$$

The first part is

$$I_1 = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy \int_{x=0}^{\frac{1}{\sqrt{3}}y} dx x^2 = 2\rho L \int_{y=0}^{\sqrt{3}a} dy \frac{x^3}{3}\Big|_{x=0}^{x=\frac{1}{\sqrt{3}}y} \tag{10}$$

$$= 2\rho L \int_{y=0}^{\sqrt{3}a} dy \frac{y^3}{9\sqrt{3}} = 2\rho L \frac{1}{9\sqrt{3}} \frac{y^4}{4}\Big|_{y=0}^{\sqrt{3}a} = 2\rho L \frac{1}{9\sqrt{3}} \frac{9a^4}{4} = \frac{1}{2\sqrt{3}}\rho L a^4 = \frac{1}{2\sqrt{3}} \frac{M}{\sqrt{3}a^2 L} L a^4 = \frac{1}{6} M a^2 \tag{11}$$

The second part is

$$I_2 = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy y^2 \int_{x=0}^{\frac{1}{\sqrt{3}}y} dx = 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy y^2 x\Big|_{x=0}^{x=\frac{1}{\sqrt{3}}y} \tag{12}$$

$$= 2\rho \int_{z=0}^{L} dz \int_{y=0}^{\sqrt{3}a} dy y^3 \frac{1}{\sqrt{3}} = \frac{2\rho L}{\sqrt{3}} \frac{y^4}{4}\Big|_{y=0}^{\sqrt{3}a} = \frac{2\rho L}{\sqrt{3}} \frac{9a^4}{4} = \frac{3\sqrt{3}}{2}\rho L a^4 = \frac{3\sqrt{3}}{2} \frac{M}{\sqrt{3}a^2 L} L a^4 = \frac{3}{2} M a^2 \tag{13}$$

The centroid is at a distance

$$\frac{2}{3}\sqrt{3}a = \frac{2}{\sqrt{3}}a \tag{14}$$

from the vertex. Thus we need to subtract an amount

$$M\frac{4}{3}a^2 \tag{15}$$

from the sum of the above two terms. This gives

$$I = \frac{3}{2}Ma^2 + \frac{1}{6}Ma^2 - \frac{4}{3}Ma^2 = \frac{1}{3}Ma^2 \tag{16}$$

The products of inertia will vanish by symmetry, if we compute them around the midpoints of the prism.

Problem 3 Taylor 10.15

Solution: (a) Let the axis of rotation be the z axis, and the cube extend from 0 to a on each axis. The density is

$$\rho = \frac{M}{a^3} \tag{17}$$

We get

$$I = \rho \int_{z=0}^{a} dz \int_{y=0}^{a} dy \int_{x=0}^{a} dx (x^2 + y^2) \tag{18}$$

We have

$$\rho \int_{z=0}^{a} dz \int_{y=0}^{a} dy \int_{x=0}^{a} dx x^2 = \rho \int_{z=0}^{a} dz \int_{y=0}^{a} dy \frac{a^3}{3} = \rho a^2 \frac{a^3}{3} = \rho \frac{a^5}{3} = \frac{Ma^2}{3} \tag{19}$$

The other integral gives the same result, so we get

$$I = \frac{2Ma^2}{3} \tag{20}$$

(b) The height of the center of mass is $\frac{a}{\sqrt{2}}$ above the table. At the end, the center of mass is at a height $\frac{a}{2}$. Thus the PE lost is

$$Mga(\frac{1}{\sqrt{2}} - \frac{1}{2}) = Mga\frac{\sqrt{2} - 1}{2} \tag{21}$$

The KE gained is

$$\frac{1}{2} I \omega^2 \tag{22}$$

Thus we get

$$\frac{1}{2} \frac{2Ma^2}{3} \omega^2 = Mga\frac{\sqrt{2} - 1}{2} \tag{23}$$

$$\omega^2 = \frac{g}{a} \frac{3}{2} (\sqrt{2} - 1) \tag{24}$$

Problem 4 Taylor 10.21

Solution: If we have a diagonal term like $i = x, j = x$ then we get

$$I_{xx} = \int \rho(x^2 + y^2 + z^2 - x^2) = \int \rho(y^2 + z^2) \tag{25}$$

which is correct. If we have an off diagonal term, then we get

$$I_{xy} = \int \rho(-xy) \tag{26}$$

which is correct as well.

Problem 5 Taylor 10.35

Solution: We have

$$I_{xx} = \sum_i m_i(y^2+z^2) = 2m[a^2+a^2] + 3m[a^2+a^2] = 10ma^2 \tag{27}$$

$$I_{yy} = \sum_i m_i(x^2+z^2) = m[a^2] + 2m[a^2] + 3m[a^2] = 6ma^2 \tag{28}$$

$$I_{zz} = \sum_i m_i(x^2+y^2) = m[a^2] + 2m[a^2] + 3m[a^2] = 6ma^2 \tag{29}$$

$$I_{xy} = -\sum_i m_i xy = m[0] = 0 \tag{30}$$

$$I_{yz} = -\sum_i m_i yz = -m[0] - 2m[a^2] - 3m[-a^2] = ma^2 \tag{31}$$

$$I_{xz} = -\sum_i m_i xz = -m[0] - 2m[0] = 0 \tag{32}$$

Thus

$$I = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} \tag{33}$$

The eigenvalues are found from the equation (keeping apart a factor of ma^2)

$$Det \begin{pmatrix} 10-\lambda & 0 & 0 \\ 0 & 6-\lambda & 1 \\ 0 & 1 & 6-\lambda \end{pmatrix} = 0 \tag{34}$$

$$(10-\lambda)[(6-\lambda)^2 - 1] = 0 \tag{35}$$

One solution is

$$\lambda = 10 \tag{36}$$

The other solutions are given by

$$(6-\lambda)^2 = 1, \quad 6-\lambda = \pm 1, \quad \lambda = 5, \quad \lambda = 7 \tag{37}$$

Thus the moments of inertia are

$$I_1 = 10ma^2, \quad I_2 = 5ma^2, \quad I_3 = 7ma^2 \tag{38}$$

The eigenvectors are given by

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = 10 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{39}$$

Let $V_x = 1$. Then we get

$$6V_y + V_z = 10V_y, \quad -4V_y + V_z = 0, \quad V_y = \frac{V_z}{4} \tag{40}$$

We also get

$$V_y + 6V_z = 10V_z, \quad V_y - 4V_z = 0 \tag{41}$$

Thus we get the eigenvector

$$(1, 0, 0) \tag{42}$$

For $\lambda = 5$ we get

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = 5 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{43}$$

Thus

$$10V_x = 5V_x, \quad V_x = 0 \tag{44}$$

$$6V_y + V_z = 5V_y, \quad V_y = -V_z \tag{45}$$

$$V_y + 6V_z = 5V_z, \quad V_y = -V_z \tag{46}$$

Thus we can take the eigenvector

$$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \tag{47}$$

For $\lambda = 7$ we get

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = 7 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{48}$$

Thus

$$10V_x = 7V_x, \quad V_x = 0 \tag{49}$$

$$6V_y + V_z = 7V_y, \quad V_y = V_z \tag{50}$$

$$V_y + 6V_z = 7V_z, \quad V_y = V_z \tag{51}$$

Thus we can take the eigenvector

$$(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \tag{52}$$

Problem 6 Taylor 10.40

Solution: (a) We have

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \tag{53}$$

Multiplying by $L_1 = I_1\omega_1$ gives

$$L_1 I_1\dot{\omega}_1 - I_1(I_2 - I_3)\omega_2\omega_3\omega_1 = 0 \tag{54}$$

We have two similar equations for the other components. Adding them, the second terms in these equations are seen to add to zero. The first terms add to

$$L_1(I_1\dot{\omega}_1) + L_2(I_2\dot{\omega}_2) + L_3(I_3\dot{\omega}_3) = 0 \tag{55}$$

This is

$$I_1\dot{I}_1 + I_2\dot{I}_2 + I_3\dot{I}_3 = 0 \tag{56}$$

$$\frac{d}{dt}[\frac{1}{2}(I_1^2 I_2^2 + I_3^2)] = 0 \tag{57}$$

which is

$$\frac{d}{dt}L^2 = 0 \tag{58}$$

so the magnitude of the angular momentum remains unchanged.

(b) We have

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \tag{59}$$

Multiplying by ω_1 gives

$$I_1\dot{\omega}_1\omega_1 - (I_2 - I_3)\omega_2\omega_3\omega_1 = 0 \tag{60}$$

When we add the three equations, the second terms cancel. The first terms give

$$I_1\frac{1}{2}\frac{d}{dt}(\omega_1^2) + I_2\frac{1}{2}\frac{d}{dt}(\omega_2^2) + I_3\frac{1}{2}\frac{d}{dt}(\omega_3^2) = \frac{d}{dt}T = 0 \tag{61}$$