Physics 5300, Theoretical Mechanics Spring 2015

Assignment 6 solutions

The problems numbers below are from Classical Mechanics, John R. Taylor, University Science Books (2005).

Problem 1 Taylor 13.1

Solution:

$$L = \frac{1}{2}m\dot{x}^2 \tag{1}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{2}$$

$$H = p\dot{x} - L = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m} \tag{3}$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{4}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = 0 \tag{5}$$

$$p = m\dot{x} = p_0, \quad \dot{x} = \frac{p_0}{m}, \quad x = \frac{p_0}{m}t + x_0 \tag{6}$$

Problem 2 Taylor 13.2

Solution: Using x oriented downwards, we have

$$L = \frac{1}{2}m\dot{x}^2 + mgx \tag{7}$$

$$p = m\dot{x} \tag{8}$$

$$H = \frac{p^2}{2m} - mgx \tag{9}$$

$$\dot{x} = \frac{p}{m} \tag{10}$$

$$\dot{p} = mg \tag{11}$$

Problem 3 Taylor 13.5

Solution:

$$T = \frac{1}{2}m[\dot{z}^2 + R^2\dot{\phi}^2] = \frac{1}{2}m[c^2\dot{\phi}^2 + R^2\dot{\phi}^2] = \frac{1}{2}m[c^2 + R^2]\dot{\phi}^2 \tag{12}$$

$$V = mgz = mgc\phi \tag{13}$$

$$L = T - V = \frac{1}{2}m[c^2 + R^2]\dot{\phi}^2 - mgc\phi \tag{14}$$

$$p = m[c^2 + R^2]\dot{\phi} \tag{15}$$

$$H = p\dot{\phi} - L = \frac{1}{2}m[c^2 + R^2]\dot{\phi}^2 + mgc\phi = \frac{p^2}{2m[c^2 + R^2]} + mgc\phi \tag{16}$$

$$\dot{\phi} = \frac{\partial H}{\partial p} = \frac{p}{m[c^2 + R^2]} \tag{17}$$

$$\dot{p} = -\frac{\partial H}{\partial \phi} = -mgc \tag{18}$$

$$m[c^2 + R^2]\ddot{\phi} = -mgc, \quad \ddot{\phi} = -\frac{gc}{[c^2 + R^2]}, \quad \ddot{z} = -\frac{gc^2}{[c^2 + R^2]} \tag{19}$$

Problem 4 Taylor 13.9

Solution:

$$L = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2] - mgy \tag{20}$$

$$p_x = m\dot{x}, \quad p_y = m\dot{y} \tag{21}$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy \tag{22}$$

$$\dot{x} = \frac{p_x}{m} \tag{23}$$

$$\dot{y} = \frac{p_y}{m} \tag{24}$$

$$\dot{p}_x = 0 \tag{25}$$

$$\dot{p}_y = -mg \tag{26}$$

Problem 5 Taylor 13.26

Solution: We have

$$F = -kx^3 = -\frac{dV}{dx}, \quad V = \frac{kx^4}{4} \tag{27}$$

where we can set an arbitrary additive constant in V to zero.

$$L = \frac{1}{2}m\dot{x}^2 - \frac{kx^4}{4} \tag{28}$$

$$p = m\dot{x} \tag{29}$$

$$H = \frac{p^2}{2m} + \frac{kx^4}{4} \tag{30}$$

$$\dot{x} = p \tag{31}$$

$$\dot{p} = -kx^3 \tag{32}$$

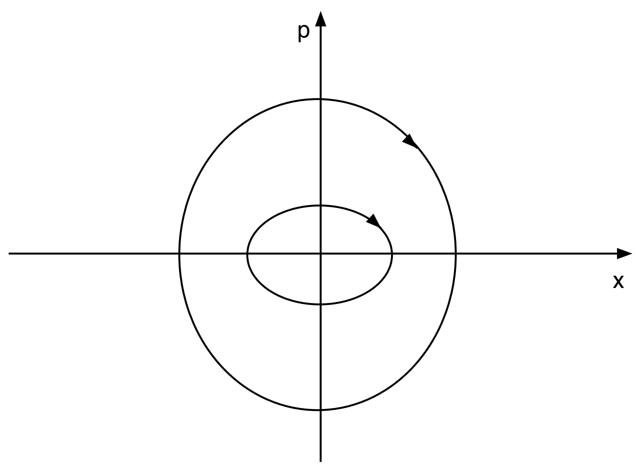

Figure 1: Phase space orbits.

Problem 6 Taylor 13.28

Solution:

$$F = kx, \quad V = -\frac{1}{2}kx^2 \tag{33}$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{34}$$

$$p = m\dot{x} \tag{35}$$

$$H = \frac{p^2}{2m} - \frac{1}{2}kx^2 \tag{36}$$

$$\dot{x} = \frac{p}{m} \tag{37}$$

$$\dot{p} = kx \tag{38}$$

For

$$\frac{p^2}{2m} - \frac{1}{2}kx^2 = E < 0 \tag{39}$$

we have

$$x^2 = \frac{2}{k}[\frac{p^2}{2m} + |E|] \tag{40}$$

Since $p \geq 0$, we have

$$|x| \geq \sqrt{\frac{2|E|}{k}} \tag{41}$$

The orbits come from $|x| = \infty$ and return to $|x| = \infty$.

For

$$\frac{p^2}{2m} - \frac{1}{2}kx^2 = E > 0 \tag{42}$$

we have

$$p^2 = 2m[\frac{1}{2}kx^2 + |E|] \tag{43}$$

Thus

$$|p| \geq \sqrt{2mE} \tag{44}$$

Thus a particle travelling to the right keeps travelling to the right, while one travelling to the left keeps travelling to the left.

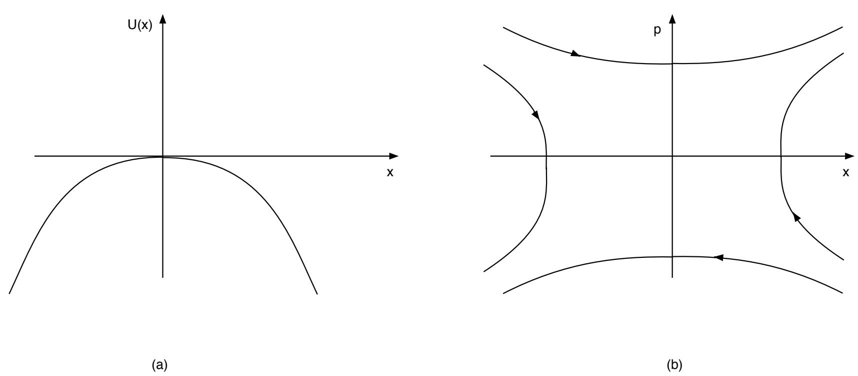

Figure 2: (a) The potential (b) Phase space orbits.