Physics 5300, Theoretical Mechanics Spring 2015

Assignment 7 solutions

The problems numbers below are from Classical Mechanics, John R. Taylor, University Science Books (2005).

Problem 1 Taylor 13.35

Solution: The volume of phase space will be conserved. The z part is unchanged, so the volume of the $x-y$ part will be unchanged. This volume is proportional to

$$R_0^2 \Delta p_\perp^2 \tag{1}$$

at the start, since we have a 2-dimensional space. After the focusing we have

$$R^2 (\Delta p_{\perp,new})^2 = R_0^2 \Delta p_\perp^2 \tag{2}$$

Thus we get

$$\Delta p_{\perp,new} = \Delta p_\perp \frac{R_0}{R} \tag{3}$$

Problem 2 Taylor 13.36

Solution: The components of the vector field are

$$V_{q_i} = \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{4}$$

$$V_{p_i} = \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{5}$$

Thus

$$\vec{\nabla} \cdot \vec{V} = \sum_i \left(\frac{\partial}{\partial q_i} V_{q_i} + \frac{\partial}{\partial p_i} V_{p_i} \right) = \sum_i \left(\frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0 \tag{6}$$

Thus the volume of any element of phase space is conserved along the phase space flow.

Problem 3 Taylor 12.21

Solution: The value of the fixed point is $x^* \approx 0.739$.

We have

$$f'(x^*) = \sin x^* \approx .67 \tag{7}$$

Thus we see that

$$|f'(x^*)| < 1 \tag{8}$$

and so the fixed point should be stable.

Problem 4 Taylor 12.22

Solution: (a) the fixed points are given by

$$x^2 = x, \quad x(x-1) = 0 \tag{9}$$

which gives

$$x = 0, \quad x = 1 \tag{10}$$

(b) We have

$$f'(x) = 2x \tag{11}$$

Thus

$$x = 0: \quad f'(x) = 0, \quad |f'(x)| < 1 \tag{12}$$

Thus $x = 0$ is stable.

$$x = 1: \quad f'(x) = 2, \quad |f'(x)| > 1 \tag{13}$$

So this fixed point is unstable.

If $-1 < x < 1$, then $|x^2| < |x|$, and we come closer to $x = 0$.

(c) If $x| > 1$, Then $|x^2| > |x|$, and we go towards larger $|x|$. In fact

$$|f(x) - x| = |x^2 - x| = |x||x-1| > |x-1| \tag{14}$$

where in the last step we have used the fact that $|x| > 1$. Thus if we start with any value of $|x-1|$ greater than zero, then at the next step of iteration the increase in the value of x is at least equal to this number $|x-1|$. Thus the increases at successive steps are not decreasing in size (in fact they are increasing), so the sequence of points cannot converge to a finite value of x. Thus we reach $x = \infty$ as the number of iterations tends to infinity.

Problem 5 Taylor 12.27

Solution: The Logistic map is

$$f(x) = rx(1-x) \tag{15}$$

Thus

$$f(f(x)) = r[rx(1-x)][1-rx(1-x)] = r^2x(1-x)(1-rx+rx^2) \tag{16}$$

Thus

$$x-f(f(x)) = x-r^2x(1-x)(1-rx+rx^2) = x[1-r^2(1-x)(1-rx+rx^2)] = x[rx-(r-1)][r^2x^2-r^2x-rx+r+1] \tag{17}$$

Thus two roots are $x = 0, x = \frac{r-1}{r}$. We knew these had to be roots since they were solutions of the single iterate $f(x) = x$. The other two roots are obtained by solving

$$rx-(r-1)][r^2x^2-r^2x-rx+r+1 = 0 \tag{18}$$

which gives

$$x = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r} \tag{19}$$

(b) We have $r > 0$ so $(r+1) > 0$. If $r < 3$ the $r-3 < 0$, and the quantity under the square root is negative. This gives complex roots for x, so there are no new fixed points of the double map.

(c) If $r > 3$ then the quantity under the square root is positive and we get real solutions. For $r = 3.2$ we have

$$x = .513, \quad x = .799 \tag{20}$$

Problem 6 Taylor 12.28

Solution: (a)

$$g'(x_a) = f'(x_a)f'(x_b) \tag{21}$$

We have

$$f'(x) = r(1-x) - rx \tag{22}$$

$$f'(x_a) = -1 + \sqrt{(r+1)(r-3)} \tag{23}$$

$$f'(x_b) = -1 - \sqrt{(r+1)(r-3)} \tag{24}$$

$$g'(x_a) = f'(x_a)f'(x_b) = 4 - r(r-2) \tag{25}$$

(b) The condition for stability is met for $g'(x_a)| < 1$. We get

$$g'(x_a) = 1 \tag{26}$$

for

$$r = 3 \tag{27}$$

(We get another root $r = -1$, but since $r > 0$, we ignore it.) We get

$$g'(x_a) = -1 \tag{28}$$

for

$$r = 1 + \sqrt{6} \tag{29}$$

(We get another root $r = 1 - \sqrt{6} < 0$, but since $r > 0$ we ignore it.) Thus we get stability for

$$3 < r < 1 + \sqrt{6} \tag{30}$$