Physics 5300, Theoretical Mechanics Spring 2015

Quiz 5 solutions

Given: Friday Feb 13

Problem 1 A bead of mass m is threaded on a frictionless circular wire hoop of radius R and mass m (same mass). The hoop is suspended at the point A and is free to swing in its own vertical plane. Using angles ϕ_1, ϕ_2 as generalized coordinates, solve for the normal frequencies of small oscillations. [The KE of the hoop is $\frac{1}{2}I\omega^2$, where $I = 2mR^2$.]

Solution: : We have for the KE of the hoop

$$T_{loop} = \frac{1}{2}I(\dot{\phi}_1)^2 = mR^2\dot{\phi}_1^2 \tag{1}$$

The position of the centrer of the hoop is

$$x' = R\sin\phi_1, \quad y' = -R\cos\phi_1 \tag{2}$$

The position of the bead compared to the center is

$$x'' = R\sin\phi_2, \quad y = -R\cos\phi_2 \tag{3}$$

Thus the position of the bead is

$$x = x' + x'' = R\sin\phi_1 + R\sin\phi_2 \tag{4}$$

$$y = y' + y'' = -R\cos\phi_1 - R\cos\phi_2 \tag{5}$$

Thus

$$\dot{x} = R\cos\phi_1\dot{\phi}_1 + R\cos\phi_2\dot{\phi}_2 \approx R(\dot{\phi}_1 + \dot{\phi}_2) \tag{6}$$

Here we have noted that we want only linear terms in the variables $\phi_1, \phi_1, \dot{\phi}_1, \dot{\phi}_2$, so we have set $\cos\phi_1 = 1, \cos\phi_2 = 1$. We also have

$$\dot{y} = R\sin\phi_1\dot{\phi}_1 + R\sin\phi_2\dot{\phi}_2 \approx R(\phi_1\dot{\phi}_1 + \phi_2\dot{\phi}_2) \tag{7}$$

This is already quadratic in the variables, so will become quartic inthe variables when we compute $\dot{y}^2$, and so will not contribute to T in the problem of small oscillations. Thus

$$T_{bead} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mR^2(\dot{\phi}_1 + \dot{\phi}_2)^2 \tag{8}$$

Thus

$$T = mR^2\dot{\phi}_1^2 + \frac{1}{2}mR^2(\dot{\phi}_1 + \dot{\phi}_2)^2 \tag{9}$$

The center of the loop is at a vertical position $y' = -R\cos\phi_1$, so

$$V_{loop} = -mgR\cos\phi_1 = -mgR(1 - \frac{1}{2}\phi_1^2) \to \frac{1}{2}mgR\phi_1^2 \tag{10}$$

where in the last step we have dropped an irrelavant constant. The bead is at the vertical position

$$y = -R\cos\phi_1 - R\cos\phi_2 = -R(1 - \frac{1}{2}\phi_1^2 + 1 - \frac{1}{2}\phi_2^2) \tag{11}$$

Thus dropping an irrelevant constant,

$$V_{bead} = \frac{1}{2}mgR(\phi_1^2 + \phi_2^2) \tag{12}$$

$$V = \frac{1}{2}mgR\phi_1^2 + \frac{1}{2}mgR(\phi_1^2 + \phi_2^2) \tag{13}$$

Thus

$$L = T - V = mR^2\dot\phi_1^2 + \frac{1}{2}mR^2(\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2) - [\frac{1}{2}mgR\phi_1^2 + \frac{1}{2}mgR(\phi_1^2 + \phi_2^2)] \tag{14}$$

$$L = \frac{3}{2}mR^2\dot\phi_1^2 + \frac{1}{2}mR^2\dot\phi_2^2 + mR^2\dot\phi_1\dot\phi_2 - [mgR\phi_1^2 + \frac{1}{2}mgR\phi_2^2] \tag{15}$$

We get

$$\hat{T} = \begin{pmatrix} 3mR^2 & mR^2 \\ mR^2 & mR^2 \end{pmatrix} = mR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \tag{16}$$

$$\hat{V} = \begin{pmatrix} 2mgR & 0 \\ 0 & mgR \end{pmatrix} = mgR \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{17}$$

Thus

$$\hat{T}^{-1} = \frac{1}{mR^2}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \tag{18}$$

$$\hat{T}^{-1}\hat{V} = \frac{g}{2R}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} \tag{19}$$

To get the eigenvalues λ' of $\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$ we have the equation

$$(2-\lambda')(3-\lambda') - 2 = 0, \quad \lambda'^2 - 5\lambda' + 4 = 0, \quad (\lambda'-1)(\lambda'-4) = 0 \tag{20}$$

Thus

$$\lambda' = 1, \quad \lambda' = 4 \tag{21}$$

and the eigenvalues of the problem are

$$\lambda = \omega^2 = \frac{g}{2R}, \quad \frac{2g}{R} \tag{22}$$

Thus the normal mode frequencies are

$$\omega = \sqrt{\frac{g}{2R}}, \quad \sqrt{\frac{2g}{R}} \tag{23}$$