Physics 5300, Theoretical Mechanics   Spring 2015

Quiz 6 solutions (corrected)

Given: Friday Feb 20

Problem 1   A rigid body consists of three equal masses m fastened at positions $(a, 0, 0), (0, a, 2a), (0, 2a, a)$. Find the inertia tensor I. (b) Find the principal moments and a set of orthoginal principal axes.

Solution: We have

$$I_{xx} = \sum_i m_i(y^2 + z^2) = m[a^2 + 4a^2 + 4a^2 + a^2] = 10ma^2 \tag{1}$$

$$I_{yy} = \sum_i m_i(x^2 + z^2) = m[a^2 + 4a^2 + a^2] = 6ma^2 \tag{2}$$

$$I_{zz} = \sum_i m_i(x^2 + y^2) = m[a^2 + a^2 + 4a^2] = 6ma^2 \tag{3}$$

$$I_{xy} = -\sum_i m_i xy = m[0] = 0 \tag{4}$$

$$I_{yz} = -\sum_i m_i yz = m[2a^2 + 2a^2] = -4a^2 \tag{5}$$

$$I_{xz} = -\sum_i m_i xz = m[0] = 0 \tag{6}$$

Thus

$$I = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix} \tag{7}$$

The eigenvalues are found from the equation (keeping apart a factor of ma^2)

$$Det \begin{pmatrix} 10-\lambda & 0 & 0 \\ 0 & 6-\lambda & -4 \\ 0 & -4 & 6-\lambda \end{pmatrix} = 0 \tag{8}$$

$$(10-\lambda)[(6-\lambda)^2 - 16] = 0 \tag{9}$$

One solution is

$$\lambda = 10 \tag{10}$$

The other solutions are given by

$$(6-\lambda)^2 = 16, \quad 6-\lambda = \pm 4, \quad \lambda = 2, \;\; \lambda = 10 \tag{11}$$

Thus the moments of inertia are

$$I_1 = 10ma^2, \quad I_2 = 2ma^2, \quad I_3 = 10ma^2 \tag{12}$$

The eigenvectors are given by

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = 10 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{13}$$

Let $V_x = 1$. Then we get

$$6V_y - 4V_z = 1 = 10V_y, \quad -4V_y - 4V_z = 0, \quad V_y = -V_z \tag{14}$$

We also get

$$-4V_y + 6V_z = 10V_z, \quad -4V_y - 4V_z = 0, \quad V_y = -V_z \tag{15}$$

Thus we get the family of eigenvectors

$$(1, q, -q) \tag{16}$$

for any q. Thus we can take as eigenvectors

$$(1,0,0), \quad (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \tag{17}$$

The other eigenvector satisfies

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = 2 \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{18}$$

Thus

$$10V_x = 2V_x, \quad V_x = 0 \tag{19}$$

$$6V_y - 4V_z = 2V_y, \quad 4V_y - 4V_z = 0, \;\; V_z = V_y \tag{20}$$

$$-4V_y + 6V_z = 2V_z, \quad -4V_y + 4V_z = 0, \quad V_z = V_y \tag{21}$$

Thus we can take the eigenvector

$$(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \tag{22}$$

Thus for the direction with $I = 10ma^2$, we can take as a basis

$$(1,0,0), \quad (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \tag{23}$$

and for $I = 2ma^2$ we get

$$(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \tag{24}$$