Physics 5300, Theoretical Mechanics   Spring 2015

Quiz 7

Given: Friday Feb 27

Problem 1   Consider the Atwood machine, with the pulley being a uniform disc with a mass M and a radius R. The two masses connected by a massless string are m_1 and m_2. Use the vertical position y of the mass m_1 as a generalised coordinate. [Note that the moment of inertia of a uniform disc about its central axis is $\frac{1}{2}MR^2$.]

(a) Write the Lagrangian. (3 points)
(b) Find the generalized momentum p. (2 points)
(c) Find the Hamiltonian. (2 points)
(d) Solve Hamilton's equations to get $\ddot{y}$. (3 points)

Solution:

(a) We have

$$T = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\frac{\dot{y}^2}{R^2} = \frac{1}{2}(m_1 + m_2 + \frac{1}{2}M)\dot{y}^2 \tag{1}$$

$$V = m_1gy - m_2gy = (m_1 - m_2)gy \tag{2}$$

Thus

$$L = T - V = \frac{1}{2}(m_1 + m_2 + \frac{1}{2}M)\dot{y}^2 - (m_1 - m_2)gy \tag{3}$$

(b)

$$p = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2 + \frac{1}{2}M)\dot{y} \tag{4}$$

(c)

$$H = p\dot{y} - L = (m_1+m_2+\frac{1}{2}M)\dot{y}^2 - [\frac{1}{2}(m_1+m_2+\frac{1}{2}M)\dot{y}^2 - (m_1-m_2)gy] = \frac{1}{2}(m_1+m_2+\frac{1}{2}M)\dot{y}^2 + (m_1-m_2)gy \tag{5}$$

We write this as

$$H = \frac{p^2}{(m_1 + m_2 + \frac{1}{2}M)} + (m_1 - m_2)gy \tag{6}$$

(d)

$$\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m_1 + m_2 + \frac{1}{2}M} \tag{7}$$

$$\dot{p} = -\frac{\partial H}{\partial y} = (m_2 - m_1)g \tag{8}$$

Differentiating the first equation gives

$$\ddot{y} = \frac{\dot{p}}{m_1 + m_2 + \frac{1}{2}M} = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{1}{2}M} \tag{9}$$