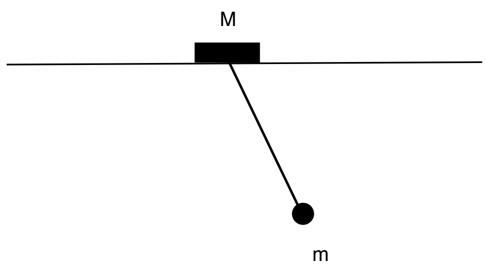
Physics 5300, Theoretical Mechanics Spring 2015

Midterm 1

Given: Wed Feb 25

Problem 1 (20 points) A mass M can slide without friction on a horizontal surface in the x direction. A mass m is suspended from M by a massless rod of length L, and can swing freely in the vertical plane.



(a) Find the Lagrangian. (10 points)

(b) Is there a cyclic coordinate? If so find the corresponding conserved momentum. (5 points)

(c) Find the Lagrange equations of motion (you do not have to solve these equations). (5 points)

Solution: (a) Let the mass M be described by the coordinate X. We have for the coordinates of the mass m

$$x = X + L\sin\theta \tag{1}$$

$$y = -L\cos\theta \tag{2}$$

Thus

$$\dot{x} = \dot{X} + L\cos\theta\dot{\theta} \tag{3}$$

$$\dot{y} = L\sin\theta\dot{\theta} \tag{4}$$

Thus

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X} + L\cos\theta\dot{\theta})^2 + \frac{1}{2}m(L\sin\theta\dot{\theta})^2 \tag{5}$$

$$V = mgy = -mgL\cos\theta \tag{6}$$

$$L = T - V = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}mL^2\dot{\theta}^2 + mL\cos\theta\dot{X}\dot{\theta} + mgL\cos\theta \tag{7}$$

(b) We see that X does not appear in the Lagrangian. The corresponding momentum is

$$p_X = (M+m)\dot{X} + mL\cos\theta\dot{\theta} \tag{8}$$

(c) The Lagrange equations are

$$X: \quad \frac{d}{dt}[(M+m)\dot{X} + mL\cos\theta\dot{\theta}] = 0 \tag{9}$$

which gives

$$(M+m)\ddot{X} + mL\cos\theta\ddot{\theta} - mL\sin\theta\dot{\theta}^2 = 0 \tag{10}$$

$$\theta: \quad \frac{d}{dt}[mL^2\dot{\theta} + mL\cos\theta\dot{X}] - [-mL\sin\theta\dot{X}\dot{\theta} - mgL\sin\theta] = 0 \tag{11}$$

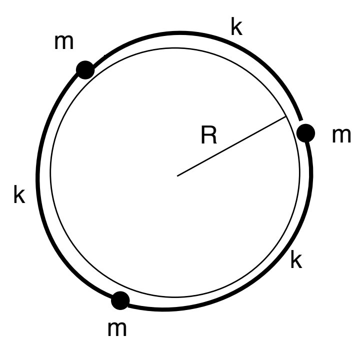
which gives

$$mL^2\ddot{\theta} + mL\cos\theta\ddot{X} - mL\sin\theta\dot{\theta}\dot{X} + mL\sin\theta\dot{X}\dot{\theta} + mgL\sin\theta = 0 \tag{12}$$

which simplifies to

$$mL^2\ddot{\theta} + mL\cos\theta\ddot{X} + mgL\sin\theta = 0 \tag{13}$$

Problem 2 (20 points) Three point masses, each of mass m, can slide on a circle of radius R. They are connected by springs of spring constant k. When the masses are evenly spaced around the circles, the springs are at their relaxed lengths.



(a) Use the angular positions $\theta_1, \theta_2, \theta_3$ as variables. Find the frequencies of small oscillations and the normal modes. (15 points)

(b) Suppose at $t = 0$ the displacements were (ϵ is a small number)

$$\theta_1 = \epsilon, \quad \theta_2 = 0, \quad \theta_3 = 0 \tag{14}$$

and $\dot{\theta}_1 = 0, \dot{\theta}_2 = 0, \dot{\theta}_3 = 0$. Find the evolution of the oscillations for later times. (5 points)

Solution: We have

$$T = \frac{1}{2} m R^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \tag{15}$$

$$V = \frac{1}{2} k R^2 [(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 + (\theta_1 - \theta_3)^2] \tag{16}$$

Thus

$$\hat{T} = m R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

$$\hat{V} = k R^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{18}$$

$$\hat{W} = \hat{T}^{-1} \hat{V} = \frac{k}{m} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{19}$$

To get the eigenvalues we solve

$$det \begin{pmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{pmatrix} = 0 \tag{20}$$

$$(2-\lambda)[(2-\lambda)^2 - 1] + [-(2-\lambda) - 1] - [1 + (2-\lambda)] = 0 \tag{21}$$

$$(2-\lambda)[3 - 4\lambda + \lambda^2] + [-3 + \lambda] - [3 - \lambda] = 0 \tag{22}$$

$$(2-\lambda)[(\lambda - 3)(\lambda - 1)] + [-3 + \lambda] - [3 - \lambda] = 0 \tag{23}$$

$$(\lambda - 3)[(2-\lambda)(\lambda - 1) + 2] = 0 \tag{24}$$

$$(\lambda - 3)[3\lambda - \lambda^2] = 0 \tag{25}$$

$$(\lambda - 3)\lambda(3 - \lambda) = 0 \tag{26}$$

Thus the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = 3 \tag{27}$$

and the frequencies are given by

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{3k}{m}}, \quad \omega_3 = \sqrt{\frac{3k}{m}} \tag{28}$$

For $\lambda = 0$, we have

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} V1 \\ V2 \\ V3 \end{pmatrix} = 0 \tag{29}$$

Setting $V_1 = 1$, we have

$$2 - V_2 - V_3 = 0, \quad V_3 = 2 - V_2 \tag{30}$$

$$-1 + 2V_2 - V3 = 0, \quad -1 + 2V_2 - 2 + V_2 = 0, \quad 3V_2 = 3, \quad V_2 = 1, \quad V_3 = 1 \tag{31}$$

Thus the eigenvector is

$$(1,1,1) \to \frac{1}{\sqrt{3}}(1,1,1) \equiv v_1 \tag{32}$$

For $\lambda = 3$, we have

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} V1 \\ V2 \\ V3 \end{pmatrix} = 3 \begin{pmatrix} V1 \\ V2 \\ V3 \end{pmatrix} \tag{33}$$

Setting $V_1 = 1$, we have

$$2 - V_2 - V_3 = 3, \quad V_3 = -1 - V_2 \tag{34}$$

$$-1 + 2V_2 - V3 = 3V_2, \quad -1 + 2V_2 + 1 + V_2 = 3V_2, \quad 0 = 0 \tag{35}$$

so this equation has no independent information.

$$-1 - V_2 + 2V_3 = 3V_3, \quad -1 - V_2 - 2 - 2V_2 = -3 - 3V_2, \quad 0 = 0 \tag{36}$$

Thus the only information we have is $V_3 = -1 - V_2$. We can take $V_2 = 1, V_3 = -2$, to get

$$(1,1,-2) \to \frac{1}{\sqrt{6}}(1,1,-2) \equiv v_2 \tag{37}$$

or $V_2 = -1, V_3 = 0$, to get

$$(1,-1,0) \to \frac{1}{\sqrt{2}}(1,-1,0) \equiv v_3 \tag{38}$$

(c) Let us write

$$w = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \tag{39}$$

The evolution is

$$w = (a + bt)v_1 + c_2 v_1 \cos(\omega_2 t + \phi_2) + c_3 v_2 \cos(\omega_3 t + \phi_3) \tag{40}$$

Then the time derivative is

$$\dot{w} = bv_1 - c_2 v_2 \sin(\omega_2 t + \phi_2) - c_3 v_3 \sin(\omega_3 t + \phi_3) \tag{41}$$

At $t = 0$

$$\dot{w}(t = 0) = bv_3 - c_2 v_2 \sin(\phi_2) - c_3 v_3 \sin(\phi_3) \tag{42}$$

Since all the $\dot{\theta}_i$ are zero, we can take $b = 0, \phi_2 = 0, \phi_3 = 0$. Thus

$$w = a + c_2 v_2 \cos(\omega_2 t) + c_3 v_3 \cos(\omega_3 t) \tag{43}$$

At $t = 0$, we have $w = \epsilon(1, 0, 0)$. The dot products are

$$w \cdot v_1 = \epsilon \frac{1}{\sqrt{3}}, \quad w \cdot v_2 = \epsilon \frac{1}{\sqrt{6}}, \quad w \cdot v_3 = \epsilon \frac{1}{\sqrt{2}} \tag{44}$$

Thus we have

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = (w \cdot v_1) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (w \cdot v_2) \frac{1}{\sqrt{6}} \cos(\sqrt{\frac{3k}{m}} t) \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + (w \cdot v_3) \frac{1}{\sqrt{2}} \cos(\sqrt{\frac{3k}{m}} t) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{45}$$

which is

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \epsilon \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \epsilon \frac{1}{6} \cos(\sqrt{\frac{3k}{m}} t) \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \epsilon \frac{1}{2} \cos(\sqrt{\frac{3k}{m}} t) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{46}$$