Physics 5300, Theoretical Mechanics Spring 2015

Midterm 2

Given: Tue March 23

Problem 1 (20 points) Consider a free particle with Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 \tag{1}$$

(a) Find the canonical momentum p, and the Hamiltonian H. (2 points)

$$p = m\dot{x}, \quad H = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m} \tag{2}$$

(b) Consider the motion of this particle from the initial position $(t, x) = (0, 0)$ to the final position $(t, x) = (t_f, x_f)$. Find the action $S(t_f, x_f)$ for this motion. (6 points)

The path is

$$x = \frac{x_f}{t_f}t \tag{3}$$

Thus

$$S = \int_{t=0}^{t=t_f} dt \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(\frac{x_f}{t_f})^2 t_f = \frac{1}{2}m\frac{x_f^2}{t_f} \tag{4}$$

(c) Find $\frac{\partial S(t_f,x_f)}{\partial x_f}$ and show that $\frac{\partial S(t_f,x_f)}{\partial x_f} = p_f$. [The subscript f on any quantity means that it is evaluated at $t = t_f$]. (6 points)

$$\frac{\partial S(t_f, x_f)}{\partial x_f} = m\frac{x_f}{t_f} = m\dot{x} = p_f \tag{5}$$

(d) Compute $\frac{\partial S(t_f,x_f)}{\partial t_f}$ and show that $\frac{\partial S(t_f,x_f)}{\partial t_f} + H(t_f) = 0$. (6 points)

$$\frac{\partial S(t_f, x_f)}{\partial t_f} = -\frac{1}{2}m\frac{x_f^2}{t_f^2} = -\frac{1}{2}m\dot{x}_f^2 = -\frac{p_f^2}{2m} = -H \tag{6}$$

Thus

$$\frac{\partial S(t_f, x_f)}{\partial t_f} + H_f = 0 \tag{7}$$

Problem 2 (20 points) A system has the Lagrangian

$$L = \frac{1}{2}\dot{q}^2 + t^2 q^2 \dot{q} \tag{8}$$

(a) Find the canonical momentum p and the Hamiltonian H. (5 points)

$$p = \dot{q} + q^2 t^2 \tag{9}$$

$$H = (\dot{q}^2 + q^2 t^2 \dot{q}) - \frac{1}{2}\dot{q}^2 - q^2 \dot{q} t^2 = \frac{1}{2}\dot{q}^2 \tag{10}$$

$$\dot{q} = p - q^2 t^2 \tag{11}$$

$$H = \frac{1}{2}(p - q^2 t^2)^2 \tag{12}$$

(b) We wish to use a new coordinate

$$Q = -\frac{p}{2tq} \tag{13}$$

Find a generating function of type $F(q, Q)$ which will give such a transformation, and find the new conjugate momentum $P(q, p, t)$. (15 points)

$$p = F_{,q}, \quad -2tqQ = F_{,q}, \quad F = -tq^2 Q + g(Q) = -tq^2 Q \tag{14}$$

$$P = -F_{,Q} = tq^2 \tag{15}$$

$$\{Q, P\} = -(-)\frac{1}{2tq}(2tq) = 1 \tag{16}$$