# Relativistic Electrodynamics

Notes (I will try to update if typos are found)

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## 1 Dot products

The Pythagorean theorem says that distances are given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{1}$$

With time as a fourth direction, we find

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2 \tag{2}$$

We can rewrite this by defining two kinds of objects. One is a vector with components

$$(\Delta x, \Delta y, \Delta z, \Delta t) \tag{3}$$

It does not matter at the end whether we write this as a row or a column vector, since we will write all matrix index summations explicitly. All we care about is that it has 4 elements. We write these components with a superscript, i.e. an 'up' index:

$$\Delta x^\mu, \quad \mu = 1, 2, 3, 4 \tag{4}$$

We define another 4-component object

$$\Delta x_\mu : \quad (\Delta x, \Delta y, \Delta z, -c^2 \Delta t) \tag{5}$$

Thus the set with index written as a subscript differs from the case with the superscript in that the last term is multiplied by $-c^2$.

This looks like a lot of notation, but it makes our task of computing dot products easier. Each dot product will involve one quantity with an 'up' index and one quantity with a 'down' index. Thus we will write

$$(\Delta s)^2 = \sum_{\mu=1}^{4} \Delta x_\mu \Delta x^\mu \tag{6}$$

One can check that this reproduces (2). The advantage of setting up all this notation is that now we do not see factors of $c$ in the expression for the dot product. We will try to use up and down indices carefully so that we never have to write $c$ explicitly if at all possible. To summarize, our general rule is the following. If we have a vector with up indices

$$V^\mu = (V^1, V^2, V^3, V^4) \tag{7}$$

then we get a version with lower indices as follows

$$V_\mu = (V^1, V^2, V^3, -c^2 V^4) \tag{8}$$

Conversely, if we have a vector with indices down, like

$$W_\mu = (W_1, W_2, W_3, W_4) \tag{9}$$

then we get a version with up indices as follows

$$W^\mu = (W_1, W_2, W_3, -\frac{1}{c^2} W_4) \tag{10}$$

Thus multiplying the last component by $-c^2$ make an up index go down, while dividing by $-c^2$ makes a down index go up.

## 2 Joining quantities into 4-vectors

Many of the familiar quantities can now be expressed in this new notation. The current and charge density make a 4-vector with an up index

$$J^\mu = (J_x, J_y, J_z, \rho) \tag{11}$$

On the right hand side it does not matter whether we write the indices $x, y, z$ up or down, since these just stand for the three components of the usual current, and when we were dealing with normal 3-component vectors there is no notion of up or down indices. The up and down difference affects only the fourth component.

The derivatives can be grouped as

$$(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}) \tag{12}$$

Our convention is that the basic coordinates $(x, y, z, t)$ have an up index. In taking a derivative, we see that these coordinates are in the denominator, so the 4-vector of derivatives is a quantity with indices down.

Now we recall the continuity equation:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \tag{13}$$

In full this is

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \tag{14}$$

We see that this is a natural dot product of the derivative operation with the current vector

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} J^\mu = 0 \tag{15}$$

Note that as expected, the $J^\mu$ is a quantity with an up index, and the derivatives $\frac{\partial}{\partial x^\mu}$ are quantities with a lower index. so we have a natural dot product. (All dot products should involve a summation sign, an up index, and a down index.) The goal of writing (15) is that it is a lot simpler than (13).

## 3 The gauge potential

We have one scalar potential $V$, and three components of a vector potential $\vec{A} = A_x, A_y, A_z$. It turns out that these can be grouped into a 4-vector, which has indices *down*. Its components are

$$A_\mu = (A_x, A_y, A_z, A_t) \tag{16}$$

where the fourth component is

$$A_t = -V \tag{17}$$

How do we know that there should be a negative sign here? We have to get the electric and magnetic fields out of these potentials. Recall the expressions

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \tag{18}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{19}$$

Our goal is to get these electric and magnetic fields from the potentials in a natural way. But as it stands, the equation for $\vec{E}$ looks quite different from the equation for $\vec{B}$. We will now see that in 4-component notation, these equations are actually completely similar.

Note that

$$B_x = \partial_y A_z - \partial_z A_y \tag{20}$$

Thus $B_x$ can be called the $y - z$ component of the curl. Similarly we have

$$B_y = \partial_z A_x - \partial_x A_z \tag{21}$$

$$B_z = \partial_x A_y - \partial_y A_x \tag{22}$$

Now that we have a fourth variable $t$, we should ask about similar things we can make with $t$. If we make the $t - x$ component of the curl we find

$$\partial_x A_t - \partial_t A_x = -\partial_x V - \partial_t A_x \tag{23}$$

But we see from (18) that this is just $E_x$. Thus we have

$$\partial_x A_t - \partial_t A_x = -\partial_x V - \partial_t A_x = E_x \tag{24}$$

$$\partial_y A_t - \partial_t A_y = -\partial_y V - \partial_t A_y = E_y \tag{25}$$

$$\partial_z A_t - \partial_t A_z = -\partial_z V - \partial_t A_z = E_z \tag{26}$$

Thus we can now define a new quantity with *two* indices, both of which are written as 'down' indices. This quantity is called $F_{\mu\nu}$, and is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{27}$$

We then find

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \tag{28}$$

# 4 Maxwell's equations

In this section we will look at Maxwell's equations and see that they take a nice form in the 4-vector language.

## 4.1 The Gauss law equation $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Let us look at the first equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{29}$$

In full, this is

$$\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z = \frac{\rho}{\epsilon_0} \tag{30}$$

In terms of the $F_{\mu\nu}$ quantity that we have defined, this is

$$\frac{\partial}{\partial x}F_{xt} + \frac{\partial}{\partial x}F_{yt} + \frac{\partial}{\partial x}F_{zt} = \frac{\rho}{\epsilon_0} \tag{31}$$

Let us make two more changes. First, since $F_{tt} = 0$, we can add a term involving $F_{tt}$ to the LHS to make it look like a sum over all of $x, y, z, t$

$$\frac{\partial}{\partial x}F_{xt} + \frac{\partial}{\partial x}F_{yt} + \frac{\partial}{\partial x}F_{zt} - \frac{1}{c^2}\frac{\partial}{\partial t}F_{tt} = \frac{\rho}{\epsilon_0} \tag{32}$$

Let us see why we added this last term in this particular way. Note that on the LHS we now have a sum of terms involving in turn $x, y, z, t$. We would like to write this sum as a dot product in the usual way. We see that the dot product involves the derivative operator index and the first index of $F$. Recall that a dot product must involve one up and one down index. Recall also that the derivative operator has a down index. But both indices of $F$ are also down. So we must write the equation a little differently, so that $F$ appears in the equation with the first index as an 'up' index. In the three terms on the LHS of (31) the first index is 1,2 and 3 respectively, so we can think of this index as 'up' if we want. The fourth term which we added is zero anyway, but as written now it is

$$-\frac{1}{c^2}\frac{\partial}{\partial t}F_{tt} = \frac{\partial}{\partial t}F^t{}_t \tag{33}$$

Thus the equation becomes

$$\sum_{\mu=1}^{4}\frac{\partial}{\partial x^\mu}F^\mu{}_t = \frac{\rho}{\epsilon_0} \tag{34}$$

The next thing we do is write the RHS as

$$\frac{\rho}{\epsilon_0} = \mu_0 c^2 \rho \tag{35}$$

Recall from (11) that $\rho = J^t$, the fourth component of the current $J^\mu$. In (34) on the LHS we have an index $t$ that has not been summed over, but that index $t$ is *down*. Thus on the RHS we should also write something with a 'down' index. We note that from our rule for indices

$$-c^2 J^t = J_t \tag{36}$$

Thus we can write the RHS as $-\mu_0 J_t$, and the full equation becomes

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_t = -\mu_0 J_t \tag{37}$$

### 4.2 The equation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Given the equation (37), it is natural to expect the equations

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_x = -\mu_0 J_x \tag{38}$$

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_y = -\mu_0 J_y \tag{39}$$

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_z = -\mu_0 J_z \tag{40}$$

What are these equations? Let us look at the first one, eq. (38). This is

$$\partial_x F^x{}_x + \partial_y F^y{}_x + \partial_z F^z{}_x + \partial_t F^t{}_x = -\mu_0 J_x \tag{41}$$

To make sense of these terms, let us write them in terms of $F$ with both indices down, which is the way we had defined $F$. For the first term on the LHS, the up index is $x$, and as we have seen before, there is no change if we write it as a down index. Similarly for the next two terms. Thus we have

$$\partial_x F_{xx} + \partial_y F_{yx} + \partial_z F_{zx} + \partial_t F^t{}_x = -\mu_0 J_x \tag{42}$$

Now note that $F^t{}_x = -\frac{1}{c^2} F_{tx}$. Thus we have

$$\partial_x F_{xx} + \partial_y F_{yx} + \partial_z F_{zx} - \frac{1}{c^2}\partial_t F_{tx} = -\mu_0 J_x \tag{43}$$

Now note that $F_{xx} = 0$. Next, note that

$$\partial_y F_{yx} + \partial_z F_{zx} = -\partial_y B_z + \partial_z B_y = -(\vec{\nabla} \times \vec{B})_x \tag{44}$$

So our equation is

$$-(\vec{\nabla} \times \vec{B})_x - \frac{1}{c^2}\partial_t F_{tx} = -\mu_0 J_x \tag{45}$$

Finally, note that $F_{tx} = -E_x$, and the equation is

$$-(\vec{\nabla} \times \vec{B})_x + \frac{1}{c^2}\partial_t E_x = -\mu_0 J_x \tag{46}$$

which is just one component of the last Maxwell equation:

$$(\vec{\nabla} \times \vec{B})_x = \mu_0 J_x + \frac{1}{c^2}\frac{\partial E_x}{\partial t} \tag{47}$$

Similarly, (39),(40) give the other components of the last Maxwell equation, i.e. we get all three components of

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{48}$$

### 4.3 The equation $\vec{\nabla} \cdot \vec{B} = 0$

This equation is

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \tag{49}$$

In terms of $F$, this is

$$\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} = 0 \tag{50}$$

Note that $F_{zx} = -F_{xz}$, and we have written the terms so that each comes with a positive sign. In each term the indices $x, y, z$ appear. How does the second term differ from the first? The indices $x, y, z$ have been permuted. If we interchange $xy$, then the order $xyz$ goes to $yxz$. If we further permute $xz$, we get $yzx$. This is the order in the second term, so it comes from *two* simple permutations put together. We call this an *even* permutation since an even number, 2, of simple permutations were needed. By contrast, if we wanted to just go to the order $yxz$ then one permutation would be needed, and we call $yxz$ an *odd* permutation of the starting set $xyz$. Note that there are 6 permutations of $xyz$ in all. Note that

$$\partial_x F_{yz} = -\partial_x F_{zy} \tag{51}$$

etc. Thus we can write (50) as

$$-\partial_x F_{zy} - \partial_y F_{xz} - \partial_z F_{yx} = 0 \tag{52}$$

or, adding (50) and (52), we can write the same equation as

$$\partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} - \partial_x F_{zy} - \partial_y F_{xz} - \partial_z F_{yx} = 0 \tag{53}$$

Now we have a simple structure. We have taken three indices $, y, z$. We have written the terms with all 6 possible permutations. The even permutations come with a positive sign, and the odd terms come with a negative sign. The equation just says that the sum of all these terms is zero.

All this may look like a lot of work for rewriting the simple equation $\vec{\nabla} \cdot \vec{B} = 0$. What is the advantage? We will now see that the next Maxwell equation, which looks quite different on the face of it, really has the same structure!

### 4.4 The equation $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$

In the above equation we permuted $xyz$. But we have 4 indices now to choose from since $t$ is also there, so we can choose some other set of three indices. Let us choose $x, y, t$. Then the analog of (50) is

$$\partial_x F_{yt} + \partial_y F_{tx} + \partial_t F_{xy} = 0 \tag{54}$$

This is

$$\partial_x E_y - \partial_y E_x + \partial_t B_z = 0 \tag{55}$$

which is

$$(\vec{\nabla} \times \vec{E})_z = -\frac{\partial}{\partial t} B_z \tag{56}$$

This is just the $z$ component of the Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B} \tag{57}$$

We get the other components by taking other sets of indices $xzt$ or $yzt$.

### 4.5 Summary of Maxwell equations

We see that the two Maxwell's equations with 'source' can be unified into one form:

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x^\mu} F^\mu{}_\nu = -\mu_0 J_\nu \tag{58}$$

where we can set the index $\nu$ to 1,2,3 or 4 to get different equations.

We also see that the two 'source free' equations are unified into just one type of equation, which we can write as

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{59}$$

Here we can choose $\mu, \nu, \lambda$ as any three different indices from the set $x, y, z, t$. Different choices give different equations. (Question: What happens if we choose two of the indices to be the same; e.g. $\mu = x,\ \nu = x,\ \lambda = y$?)

# 5 Lorentz transformations

First consider motion in only one dimension $x$. Suppose one observer uses coordinates $x, t$. Suppose another observer is moving in the positive $x$ direction with constant velocity $v$. The coordinates for the moving observer will be denoted $x', t'$.

In Newtonian mechanics, time is unchanged, so

$$t' = t \tag{60}$$

while

$$x' = x - vt \tag{61}$$

Thus we can write

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{62}$$

In relativistic mechanics we have a more symmetric transformation

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\frac{\gamma v}{c^2} & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{63}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Note that for $v \ll c$, the relativistic case reduce to the Newtonian case. With motion in the $x$ direction, the $y, z$ coordinates are not affected. Thus we can write the full Lorentz transformation

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{64}$$

## 5.1 Tensors

We are now ready to learn all about tensors. Recall that we had written the position coordinates (and their differences) with an 'up' index. Thus we have

$$x^\mu = (x, y, z, t) \tag{65}$$

In the new frame we have

$$x'^\mu = (x', y', z', t') \tag{66}$$

We have the Lorentz transformation matrix as a $4 \times 4$ matrix. We call this matrix $\Lambda$. Any matrix has two indices, with the first denoting row number and the second column number. In our present notation, we will write the first index 'up' and the second 'down'. Thus we write

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \tag{67}$$

We can now write (64) as

$$x'^\mu = \sum_{\nu=1}^{4} \Lambda^\mu{}_\nu x^\nu \tag{68}$$

Note that the index which is summed over appears once as an up index and once as a down index, just like in any dot product. The other index, $\mu$, is up on the left and so up on the right, as it should be.

What is all this notation good for? The key point comes now:

*Any vector with an up index will change the same way when we go to a moving frame as any other vector with an up index.*

Thus take the 4-vector for the current $J^\mu$. If we go to a moving frame, the new components will be

$$J'^\mu = \sum_{\nu=1}^{4} \Lambda^\mu{}_\nu J^\nu \tag{69}$$

Thus suppose in our initial frame we have only a charge density $\rho$, and no current density. Thus we have

$$J^\mu = (0, 0, 0, \rho) \tag{70}$$

Now we go to a frame moving to the right with a velocity $v$. What will we see? First, the charge will appear to be moving in the negative $x$ direction, so we should see a negative current $J_x$. But we should also see a length contraction, which will make the charge distribution appear 'compressed'; thus the charge density will appear *higher* than in the original frame.

How can we get all these effects quantitatively? All we have to do is use (69). This will give

$$J'^1 = -\gamma v \rho, \quad J'^2 = 0, \quad J'^3 = 0, \quad J'^4 = \gamma \rho \tag{71}$$

Thus we see that the charge density in the moving frame is indeed higher

$$J'^4 = \rho' = \gamma \rho = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \rho \tag{72}$$

while we have also found a current in the negative $x$ direction

$$J'^1 = -\gamma v \rho = -v\rho' \tag{73}$$

Thus we see that the current in the moving frame is indeed the charge density in the moving frame times the velocity of that charge in the moving frame, so our old equation $\vec{J} = \rho\vec{v}$ is still true.

### 5.2 Transformation of fields

So much for the transformations of vectors with an 'up' index. What about vectors with a 'down' index? When we go to a moving frame, the components change again, but the change is slightly different. To illustrate the rule, take the vector $A_\mu$ given in (16). Then in the moving frame we will have components given by $A'_\mu$, with

$$A'_\mu = \sum_{\nu=1}^{4} \tilde{\Lambda}_\mu{}^\nu A_\nu \tag{74}$$

Where

$$\tilde{\Lambda}_\mu{}^\nu = \begin{pmatrix} \gamma & 0 & 0 & \frac{\gamma v}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} \tag{75}$$

One thing which can be checked is that if we have a dot product

$$\sum_{\mu=1}^{4} W_\mu V^\mu \tag{76}$$

then it does not change if we change $V^\mu$ by its rule for vectors with an up index and change $W_\mu$ by its rule for vectors with a down index:

$$\sum_{\mu=1}^{4} W'_\mu V'^\mu = \sum_{\mu=1}^{4} W_\mu V^\mu \tag{77}$$

Checking this takes a little work, but is worth doing.

Now we can say what happens to $F_{\mu\nu}$, which has all the electric and magnetic fields in it. There are two 'down' indices. First look at one index, and ignore the other one. Thinking of this as a single down vector index, write the change expected in the moving frame. Now do the same for the other index. The result is

$$F'_{\mu\lambda} = \sum_{\nu=1}^{4}\sum_{\kappa=1}^{4} \tilde{\Lambda}_\mu{}^\nu \tilde{\Lambda}_\lambda{}^\kappa F_{\nu\kappa} \tag{78}$$

Let us now apply this to various cases.

*Example 1:* Suppose we have only $E_x$ in the initial reference frame. We can ask what is $E'_x$ in the moving frame. To do this, we have to first note that

$$E_x = F_{xt}, \quad E'_x = F'_{xt} \tag{79}$$

Then we write

$$F'_{xt} = \sum_{\nu=1}^{4}\sum_{\kappa=1}^{4} \tilde{\Lambda}_x{}^\nu \tilde{\Lambda}_t{}^\kappa F_{\nu\kappa} \tag{80}$$

Note that if we have only $E_x$ nonzero, then the nonzero terms of $F_{\mu\nu}$ are

$$F_{xt} = E_x, \quad F_{tx} = -E_x \tag{81}$$

Then in the summation we find the following nonzero terms

$$F'_{xt} = \tilde{\Lambda}_x{}^x \tilde{\Lambda}_t{}^t F_{xt} + \tilde{\Lambda}_x{}^t \tilde{\Lambda}_t{}^x F_{tx} \tag{82}$$

Note that $F_{tx} = -F_{xt}$. Thus we have

$$F'_{xt} = [\gamma^2 - \frac{\gamma^2 v^2}{c^2}]F_{xt} = \gamma^2(1 - \frac{v^2}{c^2})F_{xt} = F_{xt} \tag{83}$$

We thus see that

$$E'_x = E_x \tag{84}$$

so that $E_x$ is not changed by a boost along the $x$ direction. This fact was not something that we could have seen right away without some calculation.

*Example 2:* Suppose we have $B_y \neq 0$, all other components zero. Now we boost in the $x$ direction as before. What are the fields in the moving frame?

One field that becomes nonzero now is $E'_z$. To check this and find its value, first note that

$$B_y = F_{zx}, \quad E'_z = F'_{zt} \tag{85}$$

Thus we want to know if $F'_{zt}$ is nonzero if we started with $F_{zx}$ nonzero. We find only one nonzero term in (78)

$$F'_{zt} = \tilde{\Lambda}_z{}^z \tilde{\Lambda}_t{}^x F_{zx} = \gamma v F_{zx} \tag{86}$$

Thus we find that

$$E'_z = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} B_y \tag{87}$$

## 6 Sample problems

*Problem 1:* Suppose in the rest frame we have

$$E_x = \alpha, \quad B_y = \beta \tag{88}$$

Now we go to a frame moving with velocity $\vec{v} = v\hat{y}$.

(a) Write the matrices $\Lambda, \tilde{\Lambda}$ for this case.

(b) Find all the nonzero components of $\vec{E}, \vec{B}$ in the new frame.

*Problem 2:* In the rest frame we have an infinite wire in the $x$ direction carrying a current $I$ and a charge per unit length $\lambda$. Now we go to a frame moving with velocity $v$ in the positive $x$ direction.

(a) Find the electric and magnetic fields in the original frame at the point $(x_0, 0, z_0)$

(b) Find all the nonzero components of the fields at this point in the moving frame.