Name: ______________________

Physics 7501 Quantum Mechanics Fall 2018

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Quiz 2

Given: Monday, Sep 10, 2018 *Time: 20 minutes*

[**Note:** You are given that

$$\int_{-\infty}^{\infty} dz e^{-az^2+bz} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \tag{1}$$

In part (b) you can use any formulae that you have learnt. But in parts (a), (c), you cannot use the expression for the answer from memory; you have to show all derivations using the above integral.]

Consider the wavefunction at time $t = 0$ defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx} \tag{2}$$

with

$$\tilde{f}(k) = C e^{-\alpha(k-k_0)^2} \tag{3}$$

(a) By doing the relevant integral find $f(x)$ at $t = 0$. (8 points)

(b) Suppose that

$$\omega(k) = \beta k^2 \tag{4}$$

Find the group velocity for the wavefunction described by the choice (3). (5 points)

(c) By doing the relevant integral, find $f(x,t)$ for an arbitrary time t. (7 points)

Solution: (a)

$$f(x) = \frac{1}{\sqrt{2\pi}} C \int_{k=-\infty}^{\infty} dk e^{i(k-k_0)x} e^{ik_0x} e^{-\alpha(k-k_0)^2} = \frac{1}{\sqrt{2\pi}} C e^{ik_0x} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{x^2}{4\alpha}} \tag{5}$$

(b) We have

$$v_g = \frac{d\omega}{dk}(k_0) = 2\beta k|_{k=k_0} = 2\beta k_0 \tag{6}$$

(c)

$$e^{ikx} \to e^{i(kx-\omega(k)t)} \tag{7}$$

$$\omega(k) = \omega(k_0) + \frac{d\omega}{dk}(k_0)(k-k_0) + \frac{1}{2}\frac{d^2\omega}{dk^2}(k_0)(k-k_0)^2 = \beta k_0^2 + 2\beta k_0(k-k_0) + \beta(k-k_0)^2 \tag{8}$$

$$f(x,t) = \frac{1}{\sqrt{2\pi}} C \int_{k=-\infty}^{\infty} dk e^{i(k-k_0)x} e^{ik_0 x} e^{-\alpha(k-k_0)^2} e^{-i[\beta k_0^2 + 2\beta k_0(k-k_0) + \beta(k-k_0)^2]t} \tag{9}$$

$$= \frac{1}{\sqrt{2\pi}} C e^{ik_0 x} e^{-i\beta k_0^2 t} \int_{k=-\infty}^{\infty} dk e^{i(k-k_0)x} e^{-\alpha(k-k_0)^2} e^{-i[2\beta k_0(k-k_0) + \beta(k-k_0)^2]t} \tag{10}$$

This is a Gaussian integral

$$\int_{-\infty}^{\infty} dz e^{-az^2+bz} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \tag{11}$$

with

$$a = \alpha + i\beta t \tag{12}$$

$$b = i[x - 2\beta k_0 t] \tag{13}$$

Thus we get

$$f(x,t) = \frac{1}{\sqrt{2\pi}} C e^{ik_0 x} e^{-i\beta k_0^2 t} \sqrt{\frac{\pi}{\alpha + i\beta t}} e^{-\frac{(x-2\beta k_0 t)^2}{4(\alpha+i\beta t)}} \tag{14}$$