Name: ______________________

Physics 7501 Quantum Mechanics Fall 2018

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Quiz 3

Given: Monday, Sep 24, 2018 *Time: 20 minutes*

Consider a square well with $V = 0$ in the range $0 < x < a$ and $V = \infty$ elsewhere. The normalized eigenfunctions are

$$u_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \tag{1}$$

in the range $0 < x < a$, and zero elsewhere.

(a) Using the eigenvalue equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}u_n(x) + V(x)u_n(x) = E_n u_n(x) \tag{2}$$

find the eigenvalues E_n for these eigenfunctions. (Do not just write down the answer from memory). (5 points)

(b) Suppose that at time $t = 0$ the wavefunction is

$$\psi(x) \;\; = \;\; A\left(\sin\frac{\pi x}{a} + \sin\frac{3\pi x}{a}\right), \quad 0 < x < a \tag{3}$$

and zero elsewhere. Find the coefficients A_n in the expansion

$$\psi(x) = \sum_n A_n u_n(x) \tag{4}$$

(You can do this by inspection if you wish; you do not have to all the needed integrals.) (5 points)

(c) Find $\psi(x,t)$ at $t = \frac{4ma^2}{9\pi\hbar}$ (5 points)

(d) Consider a wavefunction

$$\chi(x) = f(x) + ig(x) \tag{5}$$

where $f(x)$ and $g(x)$ are real functions. Find the probability current $J(x)$ for the wavefunction $\chi(x)$. (5 points)

Solution: (a) We have $V = 0$ in this region. Applying the other part of the Hamiltonian, we get

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\sin\frac{n\pi x}{a} = -\frac{\hbar^2}{2m}[-\frac{n^2\pi^2}{a^2}]\sin\frac{n\pi x}{a} = \frac{\hbar^2\pi^2}{2ma^2}n^2\sin\frac{n\pi x}{a} \tag{6}$$

Thus

$$E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2 \tag{7}$$

(b) We have

$$\psi(x) = A\left(\sin\frac{\pi x}{a} + \sin\frac{3\pi x}{a}\right) \tag{8}$$

Since the average of $\sin^2 z$ over a half period is $\frac{1}{2}$, and the length of the x interval is a, we get

$$|A|^2\frac{a}{2} + A|^2\frac{a}{2} = 1, \quad |A| = \frac{1}{\sqrt{a}} \tag{9}$$

and we get

$$\psi(x) = \frac{1}{\sqrt{a}}\left(\sin\frac{\pi x}{a} + \sin\frac{3\pi x}{a}\right) \tag{10}$$

We write

$$\psi(x) = A_1u_1(x) + A_3u_3(x) \tag{11}$$

$$= A\sqrt{\frac{a}{2}}\sqrt{\frac{a}{2}}\sin\frac{\pi x}{a} + A\sqrt{\frac{a}{2}}\sqrt{\frac{a}{2}}\sin\frac{3\pi x}{a} \tag{12}$$

Thus we have

$$A_1 = A\sqrt{\frac{a}{2}} = \frac{1}{\sqrt{2}}, \qquad A_3 = A\sqrt{\frac{a}{2}} = \frac{1}{\sqrt{2}} \tag{13}$$

(c) We have

$$\psi(t) = \frac{1}{\sqrt{a}}\left(\sin\frac{\pi x}{a}e^{-i\frac{E_1}{\hbar}t} + \sin\frac{3\pi x}{a}e^{-i\frac{E_3}{\hbar}t}\right) = \frac{1}{\sqrt{a}}\left(\sin\frac{\pi x}{a}e^{-i\frac{2\pi}{9}} + \sin\frac{3\pi x}{a}\right) \tag{14}$$

(d)

$$J = \frac{\hbar}{2im}[(f-ig)(f'+ig') - (f'-ig')(f+ig)] = \frac{\hbar}{2m}[fg' - gf'] \tag{15}$$