Number (NOT NAME): ____________________

Physics 7501 Quantum Mechanics Fall 2018

.

Midterm 1

Given: Friday, Oct 5, 2018 *Time: 55 minutes* *Total marks: 40*

Problem 1: (15 points) Consider a particle of mass m moving in the potential

$$\begin{aligned} V &= V_0, \quad x < 0 \qquad (V_0 > 0) \\ V &= 0, \quad x > 0 \end{aligned} \tag{1}$$

(a) Suppose a wave with energy $E > V_0$ is incident on the step from the left (i.e., from the side $x < 0$). Find the incident, reflected and transmitted waves. (10 points)

(b) What is the probability that the incident particles get transmitted to the side $x > 0$, and what is the probability that they get reflected back to the side $x < 0$? (5 points)

Solution: (a) On the left we have

$$e^{ikx} + Be^{-ikx} \tag{2}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \tag{3}$$

On the right we have

$$Ce^{ik'x} \tag{4}$$

$$k' = \sqrt{\frac{2mE}{\hbar^2}} \tag{5}$$

Matching gives

$$1 + B = C \tag{6}$$

$$(ik)[1 - B] = ik'C \tag{7}$$

$$1 - B = \frac{k'}{k}C \tag{8}$$

$$2 = (1 + \frac{k'}{k})C, \quad C = \frac{2}{1 + \frac{k'}{k}} = \frac{2k}{k + k'} \tag{9}$$

$$2B = (1 - \frac{k'}{k})C, \quad B = \frac{k - k'}{2k}\frac{2k}{k + k'} = \frac{k - k'}{k + k'} \tag{10}$$

(b) The current from the incoming wave is

$$J = \frac{\hbar}{2im}[(e^{ikx})^* \partial_x (e^{ikx}) - \partial_x (e^{ikx})^* e^{ikx}] = \frac{\hbar}{2im}(2ik) = \frac{\hbar k}{m} \tag{11}$$

The reflected current is

$$-|B|^2 \frac{\hbar k}{m} = -(\frac{k-k'}{k+k'})^2 \frac{\hbar k}{m} \tag{12}$$

Thus the probability of reflection is

$$P_{reflect} = (\frac{k-k'}{k+k'})^2 \tag{13}$$

The transmitted current is

$$|C|^2 \frac{\hbar k'}{2m} \tag{14}$$

Thus the probability of transmission is

$$P_{transmission} = (\frac{2k}{k+k'})^2 \frac{k'}{k} = \frac{4kk'}{(k+k')^2} \tag{15}$$

Problem 2: (15 points) Consider a delta-function potential

$$V(x) = -Q\delta(x) \tag{16}$$

with $Q > 0$. Find the bound states in this potential.

Solution: On the left we take

$$e^{qx} \tag{17}$$

On the right we take

$$Ae^{-qx} \tag{18}$$

Matching of values gives

$$B = 1 \tag{19}$$

Matching derivatives gives

$$-\frac{\hbar^2}{2m}[\partial u_{x\to 0^+} - \partial u_{x\to 0^-}] - Qu = 0 \tag{20}$$

$$[\partial u_{x\to 0^+} - \partial u_{x\to 0^-}] = -\frac{2mQ}{\hbar^2} \tag{21}$$

Thus

$$(-q) - q = -\frac{2mQ}{\hbar^2} \tag{22}$$

$$q = \frac{mQ}{\hbar^2} \tag{23}$$

$$E = -\frac{\hbar^2}{2m}q^2 = -\frac{\hbar^2}{2m}(\frac{mQ}{\hbar^2})^2 = -\frac{mQ^2}{2\hbar^2} \tag{24}$$

There is only one eigenvalue.

Problem 3: (10 points) Consider the wavefunction

$$\psi(x) = Ae^{-ax^2} \tag{25}$$

where $a > 0$.

(a) Find A.

(b) Find $\langle p^2 \rangle$.

[You can use that $\int_{z=-\infty}^{\infty} dz \quad z^{2n}e^{-\alpha z^2} = \left(-\frac{\partial}{\partial\alpha}\right)^n \left(\sqrt{\frac{\pi}{\alpha}}\right)$]

Solution: (a) We have

$$|A|^2 \int_{-\infty}^{\infty} dxe^{-2ax^2} = |A|^2\sqrt{\frac{\pi}{2a}} = 1 \tag{26}$$

which gives

$$A = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \tag{27}$$

(b) We have

$$\langle p^2 \rangle = |A|^2 \int_{-\infty}^{\infty} dxe^{-ax^2}[-\hbar^2\partial_x^2]e^{-ax^2} \tag{28}$$

$$\langle p^2 \rangle = -|A|^2\hbar^2 \int_{-\infty}^{\infty} dxe^{-ax^2}[(4a^2x^2) - 2a)e^{-ax^2} \tag{29}$$

$$\langle p^2 \rangle = -4|A|^2\hbar^2a^2 \int_{-\infty}^{\infty} dxx^2e^{-2ax^2} + 2|A|^2\hbar^2a \int_{-\infty}^{\infty} dxe^{-2ax^2} \tag{30}$$

$$\langle p^2 \rangle = -2\hbar^2|A|^2a^2\sqrt{\frac{\pi}{(2a)^3}} + 2\hbar^2|A|^2a\sqrt{\frac{\pi}{2a}} = \hbar^2|A|^2\sqrt{\frac{\pi a}{2}}(-1+2) = \hbar^2\sqrt{\frac{\pi a}{2}}\sqrt{\frac{2a}{\pi}} \tag{31}$$

$$\langle p^2 \rangle = \hbar^2 a \tag{32}$$