Number (NOT NAME): ________________________

Physics 7501 Quantum Mechanics Fall 2018

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Midterm 2

Given: Wednesday, Nov 7, 2018 *Time: 55 minutes* *Total marks: 40*

Problem 1: (15 points) Consider the potential barrier

$$\begin{aligned} V &= 0, \quad x < 0 \\ V &= V_0 > 0, \quad 0 < x < a \\ V &= 0, \quad x > a \end{aligned} \tag{1}$$

A particle of mass m is incident on this barrier from the left (i.e., from the side $x < 0$). The particle has an energy $E < V_0$. Assume that a is very large. Find an approximate expression for the probability that the particle tunnels through to the region $x > a$.

Solution: We have for the amplitude of tunneling

$$T \sim e^{-\int_{x=a}^{x=a} dx \sqrt{\frac{2m(V_0-E)}{\hbar^2}} dx} = e^{-\sqrt{\frac{2m(V_0-E)}{\hbar^2}} a} \tag{2}$$

The tunneling probability is then

$$P \sim |T|^2 \sim e^{-2\sqrt{\frac{2m(V_0-E)}{\hbar^2}} a} \tag{3}$$

Problem 2: (15 points) Consider the Hamiltonian for a harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \tag{4}$$

The creation and annihilation operators are

$$A = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} + i \frac{1}{\sqrt{2m\omega\hbar}} \hat{p} \tag{5}$$

$$A^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \hat{x} - i \frac{1}{\sqrt{2m\omega\hbar}} \hat{p} \tag{6}$$

(a) Find the ground state wavefunction (upto an arbitrary normalization constant) using the relation $\hat{A}|0\rangle = 0$. (6 points)

(b) Normalize this wavefunction. [You can use that $\int_{z=-\infty}^{\infty} dz \quad e^{-\alpha z^2} = \left(\sqrt{\frac{\pi}{\alpha}}\right)$] (3 points)

(c) Find the normalized wavefunction for the first excited state $|1\rangle$; i.e., the wavefunction for the state with energy $\frac{3}{2}\hbar\omega$. (6 points)

Solution: (a) We use

$$\hat{A}|0\rangle = 0 \tag{7}$$

We had

$$A = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega\hbar}}p \tag{8}$$

Thus

$$[\sqrt{\frac{m\omega}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega\hbar}}(-i\hbar\partial_x)]\psi(x) = 0 \tag{9}$$

$$[\sqrt{\frac{m\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2m\omega}}\partial_x]\psi(x) = 0 \tag{10}$$

$$\partial_x\psi = -\frac{m\omega}{\hbar}x\psi \tag{11}$$

We try

$$\psi = e^{-ax^2} \tag{12}$$

$$\partial_x\psi = -2ax\psi \tag{13}$$

Thus

$$2a = \frac{m\omega}{\hbar} \tag{14}$$

$$a = \frac{m\omega}{2\hbar} \tag{15}$$

Thus

$$\psi_0(x) = Ce^{-\frac{m\omega}{2\hbar}x^2} \tag{16}$$

(b) We have

$$\int_{x=-\infty}^{\infty} dx|\psi(x)|^2 = \int_{x=-\infty}^{\infty} dx|C|^2e^{-\frac{m\omega}{\hbar}x^2} = |C|^2\sqrt{\frac{\pi\hbar}{m\omega}} \tag{17}$$

Thus

$$C = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \tag{18}$$

(c) We have

$$|1\rangle = \hat{A}^\dagger|0\rangle \tag{19}$$

Thus we have

$$A^\dagger|0\rangle = [\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{1}{\sqrt{2m\omega\hbar}}\hat{p}]\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} \tag{20}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} [\sqrt{\frac{m\omega}{2\hbar}}x + \frac{1}{\sqrt{2m\omega\hbar}}\frac{m\omega}{\hbar}\hbar] e^{-\frac{m\omega}{2\hbar}x^2} \tag{21}$$

$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{\frac{m\omega}{2\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2} \tag{22}$$

Problem 3: (10 points) Given that $[\hat{A}, \hat{A}^\dagger] = 1$, find the commutator

$$[\hat{A}, (\hat{A}^\dagger)^n] \tag{23}$$

Solution: We have

$$[\hat{A}, (\hat{A}^\dagger)^n] = \hat{A}(\hat{A}^\dagger)^n - (\hat{A}^\dagger)^n \hat{A} \tag{24}$$

$$\hat{A}(\hat{A}^\dagger)^n = \hat{A}^\dagger \hat{A}(\hat{A}^\dagger)^{n-1} + (\hat{A}^\dagger)^{n-1} \tag{25}$$

$$\hat{A}^\dagger \hat{A}(\hat{A}^\dagger)^{n-1} = \hat{A}^\dagger \hat{A}^\dagger \hat{A}(\hat{A}^\dagger)^{n-2} + (\hat{A}^\dagger)^{n-1} \tag{26}$$

Thus we will get

$$\hat{A}(\hat{A}^\dagger)^n = n(\hat{A}^\dagger)^{n-1} + (\hat{A}^\dagger)^n \hat{A} \tag{27}$$

Thus

$$[\hat{A}, (\hat{A}^\dagger)^n] = n(\hat{A}^\dagger)^{n-1} \tag{28}$$