Name: ________________________

Physics 7502 Quantum Mechanics Spring 2019

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Quiz 3

Given: Monday, Feb 10, 2019 *Total points: 10* *Time: 12.00-12.25 am*

Problem 1: Consider a system with spin 1, having states $|l_1, m_1\rangle = |1,1\rangle, |1,0\rangle, |1,-1\rangle$. Consider another system with spin $\frac{1}{2}$, having states $|l_2, m_2\rangle = |\frac{1}{2},\frac{1}{2}\rangle, |\frac{1}{2},-\frac{1}{2}\rangle$. The goal of this problem is to decompose the full set of states into representations of the rotation group $|l_T, m_T\rangle$.

(a) Find all states of the representation with $l = \frac{3}{2}$. Give normalized states. (5 points)

(b) Find all states of the representation with $l = \frac{1}{2}$. Give normalized states. (5 points)

[You can use that the angular momentum commutation relations are

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \tag{1}$$

We define $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$, and have the normalized states $|l, m\rangle$. We have

$$\hat{L}_+|l,m\rangle = c_+(l,m)|l,m+1\rangle, \quad \hat{L}_-|l,m\rangle = c_-(l,m)|l,m-1\rangle \tag{2}$$

$$c_+(l,m) = \hbar\sqrt{(l-m)(l+m+1)}, \quad c_-(l,m) = \hbar\sqrt{(l+m)(l-m+1)} \tag{3}$$

$$\hat{L}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle, \quad \hat{L}_z|l,m\rangle = \hbar m|l,m\rangle \tag{4}$$

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Solution: (a) We have

$$|\frac{3}{2},\frac{3}{2}\rangle = |1,1\rangle|\frac{1}{2},\frac{1}{2}\rangle \tag{5}$$

$$L_-|\frac{3}{2},\frac{3}{2}\rangle = c_-(\frac{3}{2},\frac{3}{2})|\frac{3}{2},\frac{1}{2}\rangle = \sqrt{3}\hbar|\frac{3}{2},\frac{1}{2}\rangle \tag{6}$$

On the other hand

$$L_- = L_1^{(1)} + L_-^{(2)} \tag{7}$$

Thus

$$L_-|1,1\rangle|\frac{1}{2},\frac{1}{2}\rangle = \hbar\sqrt{2}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle + \hbar|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{8}$$

Thus

$$|\frac{3}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{9}$$

Similarly,

$$L_-|\frac{3}{2},\frac{1}{2}\rangle = c_-(\frac{3}{2},\frac{1}{2})|\frac{3}{2},-\frac{1}{2}\rangle = 2\hbar|\frac{3}{2},-\frac{1}{2}\rangle \tag{10}$$

On the other hand

$$\begin{aligned}(L_-^{(1)}+L_-^{(2)}) & \left(\sqrt{\frac{2}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle\right) \\ &= \sqrt{\frac{2}{3}}\hbar\sqrt{2}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\hbar|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}\hbar\sqrt{2}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle \\ &= 2\hbar\frac{1}{\sqrt{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle + 2\hbar\sqrt{\frac{2}{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle\end{aligned} \tag{11}$$

Thus

$$|\frac{3}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{12}$$

Finally,

$$L_-|\frac{3}{2},-\frac{1}{2}\rangle = c_-(\frac{3}{2},-\frac{1}{2})|\frac{3}{2},-\frac{3}{2}\rangle = \sqrt{3}\hbar|\frac{3}{2},-\frac{3}{2}\rangle \tag{13}$$

On the other hand

$$\begin{aligned}(L_-^{(1)}+L_-^{(2)}) & \left(\frac{1}{\sqrt{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle\right) \\ &= \sqrt{\frac{1}{3}}\hbar|1,-1\rangle|\frac{1}{2},-\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}\hbar\sqrt{2}|1,-1\rangle|\frac{1}{2},-\frac{1}{2}\rangle \\ &= \sqrt{3}\hbar|1,-1\rangle|\frac{1}{2},-\frac{1}{2}\rangle\end{aligned} \tag{14}$$

Thus

$$|\frac{3}{2},-\frac{3}{2}\rangle = |1,-1\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{15}$$

(b) We have to take a state that is orthogonal to

$$|\frac{3}{2},\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle + \frac{1}{\sqrt{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{16}$$

This state is

$$\sqrt{\frac{1}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{17}$$

This is already normalized, so we can take

$$|\frac{1}{2},\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{18}$$

We have

$$L_-|\frac{1}{2},\frac{1}{2}\rangle = \hbar|\frac{1}{2},-\frac{1}{2}\rangle \tag{19}$$

On the other hand

$$\begin{aligned}(L_-^{(1)} + L_-^{(2)}) & \left(\sqrt{\frac{1}{3}}|1,0\rangle|\frac{1}{2},\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|1,1\rangle|\frac{1}{2},-\frac{1}{2}\rangle\right) \\ = & \sqrt{\frac{1}{3}}\hbar\sqrt{2}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}\hbar|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}\hbar\sqrt{2}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle \\ = & \hbar\sqrt{\frac{2}{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle - \hbar\frac{1}{\sqrt{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle\end{aligned} \tag{20}$$

Thus

$$|\frac{1}{2},-\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle - \frac{1}{\sqrt{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle \tag{21}$$