1. Some algebraic identities

We have encountered the determinant of the metric

$$g \equiv \det[g_{ab}] \tag{1.1}$$

For any matrix M_{ab} the change in the determinant can be related to the change in the entries of the matrix as follows

$$\delta[\det M] = (\det M) M^{-1}_{ba} \delta M_{ab} \tag{1.2}$$

Thus

$$\delta g = g \ g^{ab} \delta g_{ab} \tag{1.3}$$

(recall that $g^{ab} = g^{ba}$). Further, from

$$g^{ac} g_{cb} = \delta^a_b \tag{1.4}$$

we get

$$\delta g^{ac} g_{cb} + g^{ac} \delta g_{cb} = 0 \tag{1.5}$$

which gives

$$\delta g^{ab} = -g^{ac} g^{bd} \delta g_{cd} \tag{1.6}$$

Thus we can also write (1.3) as

$$\delta g = -g \ g_{ab} \delta g^{ab} \tag{1.7}$$

We will need the following corollary of these results

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-)\delta g = \frac{1}{2\sqrt{-g}}(-g) g^{ab} \delta g_{ab} = \frac{1}{2}\sqrt{-g} g^{ab} \delta g_{ab} \tag{1.8}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{ab} \delta g^{ab} \tag{1.9}$$

2. Integration by parts

Suppose we have d-dimensional flat space equipped with Cartesian coordinates. Then we have the 'divergence theorem' which we obtain by doing an 'integration by parts'

$$\int d^d x \partial_i V^i = \text{surface term} \tag{2.1}$$

In particular if the vector V^i vanishes on the boundary of the region of integration then we get

$$\int d^d x \partial_i V^i = 0 \tag{2.2}$$

Let us see how this result extends to curved space, where we must use general curvilinear coordinates. We have already seen that the volume element in curvilinear coordinates is $\sqrt{-g} d^d \xi$. Thus we integrate a scalar function $S(\xi)$ as

$$\int d^d \xi \sqrt{-g} S(\xi) \tag{2.3}$$

Note that it only makes sense to integrate scalar functions; integrating other tensors will give a result that depends on the choice of coordinate system. The expression (2.3) for integration of a scalar has a simple intepretation: We cut up the spacetime into small cells, the factor $d^d \xi \sqrt{-g}$ gives the volume of a cell, $S(\xi)$ gives the value of the function on the cell, and the integration sign denotes the summation over all cells. Thus the integration of a scalar function is a geometric process, yielding a result that is independent of coordinate choice.

A particular choice of scalar function is $S(\xi) = V^a_{;a}$, where V^a is a vector field. Thus we can define the integral

$$\int d^d \xi \sqrt{-g} V^a{}_{;a} \tag{2.4}$$

This differs from the LHS of (2.2) in two ways: We have a covariant derivative of V instead of an ordinary partial derivative, and we have the factor $\sqrt{-g}$. It is thus not obvious that we can convert (2.4) to a surface term. But we will see that these two differences cancel each other out, and we do get a result similar to (2.2). To see this we note the following identity

$$V^a{}_{;a} = \frac{1}{\sqrt{-g}} (V^a \sqrt{-g})_{,a} \tag{2.5}$$

To prove this identity note that the LHS expands as

$$V^a{}_{;a} = V^a{}_{,a} + \Gamma^a_{ba} V^b = V^a{}_{,a} + \frac{1}{2} g^{af} (g_{fb,a} + g_{fa,b} - g_{ba,f}) V^b \tag{2.6}$$

Note that g^{af} is symmetric in a, f while the combination $g_{fb,a} - g_{ba,f}$ is antisymmetric in a, f. Thus these terms yield zero, and we get

$$V^a{}_{;a} = V^a{}_{,a} + \frac{1}{2} g^{af} g_{fa,b} V^b \tag{2.7}$$

The RHS of (2.5) gives

$$\begin{aligned} V^a{}_{,a} + \frac{1}{\sqrt{-g}}(\sqrt{-g})_{,a} V^a &= V^a{}_{,a} + \frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta\xi^a} V^a \\ &= V^a{}_{,a} + \frac{1}{\sqrt{-g}} \frac{1}{2}\sqrt{-g} g^{bc} \frac{\delta g_{bc}}{\delta\xi^a} V^a \\ &= V^a{}_{,a} + \frac{1}{\sqrt{-g}} \frac{1}{2}\sqrt{-g} g^{bc} g_{bc,a} V^a \end{aligned} \tag{2.8}$$

where in the second step we have used (1.8). Comparing (2.7) and (2.8) we establish (2.5).

Returning to (2.4) we find

$$\int d^d\xi \sqrt{-g} V^a{}_{;a} = \int d^d\xi \sqrt{-g} [\frac{1}{\sqrt{-g}} (V^a \sqrt{-g})_{,a}] = \int d^d\xi (V^a \sqrt{-g})_{,a} \tag{2.9}$$

Thus we see that we get a simple integration by parts as in (2.1), with an ordinary partial derivative and an ordinary integration over coordinates ξ. The usual integration by parts then gives a surface term, and in particular if V^a vanishes on the boundary of the region of integration then

$$\int d^d\xi \sqrt{-g} V^a{}_{;a} = 0 \tag{2.10}$$